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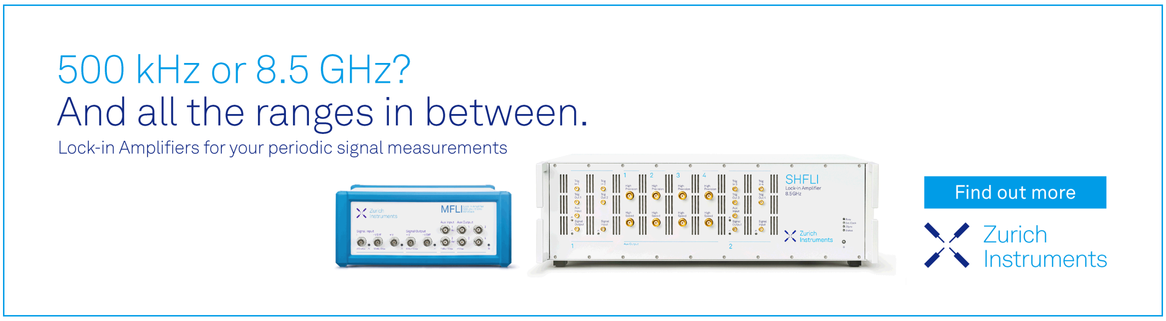

Essential Spectrum of a Family of 3×3 Operator Matrices: Location of the Branches

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Abstract. In the article we have considered a family $\mathscr{A}(K)$, $K \in \mathbf{T}^{\rm d} := (-\pi;\pi]^{\rm d}$ of operator matrices of order three. They arise in the spectral analysis problem of the so called lattice truncated spin-boson Hamiltonian with at most two bosons. The position and structure of two-particle as well three-particle branches (subsets) of $\sigma_{ess}(\mathscr{A}(K))$ are investigated.

INTRODUCTION

There is an important quantum-mechanical model so called he spin-boson model which depicts the interaction between a photon field and a two-level atom. We suggest to [1] and [2] for the best reviews respectively from mathematical and physical outcomes. Regardless of whether the fundamental space is a dD torus or the dD euclidian space $\mathbf{R}^{\rm d}$, the total spin-boson model is an endless operator framework in Fock space with a finite N of bosons for which comprehensive results are exceedingly difficult to get in. We discuss the projection to the truncated Fock space with a finite N of bosons as one approach. The truncated standard spin-boson model was fully investigated [3] in for tiny values of the parameter α for $N = 1,2$. The case $N = 3$ was assumed in [4]. There was proof of the existence of constructed wave operators as well as its asymptotic completeness. When N is an arbitrary it was studied in [5] and [6]. For sufficiently tiny coupling constants,the spectral properties of the truncated spin-boson matrix A_N was learned using a Mourre type estimate. The discrete spectrum of the truncated spin-boson matrix with two photons in $\mathbf{R}^{\rm d}$ is finite for all values of coupling $\alpha > 0$, according to [7].

In [8] a lattice matrix $\mathscr{B}_2$ – so-called truncated lattice spin-boson matrix with at most two photons are analyzed. The position of the $\sigma_{\rm ess}(\mathscr{B}_2)$ is depicted. The finiteness of the number of eigenvalues below the bottom of any coupling constant's essential spectrum is established. Considering a general lattice matrix and estimating the essential spectrum's left boundary the results are achieved.

Because of boundedness and self-adjointness the spectrum of the lattice matrix $\mathscr{B}_2$ is more intricate than the continuous case. The two-particle and three-particle branches of the $\sigma_{\rm ess}(\mathscr{B}_2)$ in the continuous case are made up of semi interval $[\kappa,\infty)$ with quantity $\kappa < 0$. As a result (see [3]), finding the eigenvalues in the case of at most 1 photon suffices to elucidate the $\sigma_{\rm ess}(\mathscr{B}_2)$ of this matrix, and the approach employed to carry out is not difficult. The two-particle and three-particle branches of the essential spectrum in the lattice scenario are made up of finite-length intervals that may or may not intersect. We obtain a natural question: Are there eigenvalues located between the branches, and if so, how many are there? As a result, analyzing the essential spectrum as well as the $\mathscr{B}_2$ conditions are important.

In [9] the essential spectrum of the matrix $\mathscr{B}_2$ is investigated in detail with respect to the dimension $\rm d \in \mathbf{N}$ and the values of coupling $\alpha > 0$.

In the following article we have considered the matrix family $\mathscr{A}(K)$, $K \in \mathbf{T}^{\rm d}$ (3×3 operator matrices), related the system of particles where the number of particles isn't conserved. For the analysis of the lattice truncated spin-boson matrix with two bosons, the matrix family $\mathscr{A}(K)$ is required. Obtained matrix of order 6 is unitary equivalent to a diagonal matrix of order 2 with two copies of the case of $\mathscr{A}(K)$ on the diagonal, as shown in see [8]. As a result, the the set $\sigma_{\rm ess}(\mathscr{A}(K))$ and finiteness of $\sigma_{disc}(\mathscr{A}(K))$ are determined by spectral information on the matrix family $\mathscr{A}(K)$.

It is easy to see that matrix family $\mathscr{A}(K)$ has almost same spectral properties of the three lattice particle Hamiltonian $H(K)$, known us as lattice analog of the standard three-particle Schrödinger operator, arising in lattice field theory [10], [11] and solid state physics models (see [12] – [13]).

The three-particle discrete Schrödinger operator $H(K)$, $K \in \mathbf{T}^3$ is discussed in [14, 15]. The finiteness of the number of eigenvalues of $H(K)$ is proven for all sufficiently small nonzero values of K, and the limit relation

$$\lim_{|K|\to 0} \frac{N(K)}{|\log|K||} = U_0 \, (0 < U_0 < \infty) \tag{1}$$

is given for the number $N(K)$ of negative eigenvalues of $H(K)$.

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The spectral properties of $\mathscr{A}(K_0)$ for a fixed K_0 were studied in [16, 17, 18, 19, 20, 21, 22, 23, 24], see also the references therein. In [25], [26] was founded a finite set $\Lambda \subset \mathbf{T}^3$ to demonstrate the existence of an infinitely many discrete eigenvalues of the matrix family $\mathscr{A}(K)$ for all $K \in \Lambda$, when the associated Friedrichs model has a virtual level at 0. In addition, it is shown that if for the generalized Friedrichs model the number 0 is an eigenvalue or the number 0 is the regular type point for positive definite Friedrichs model, the matrix $\mathscr{A}(K)$ have finitely many negative discrete eigenvalues for every $K \in \Lambda$.

However, with regard to the spectral parameter K, the asymptotic formula of the form (1) was not established. It is critical to determine the structure and location of the matrix family $\mathscr{A}(K)$ in order to reach this type of result. To this purpose, the geometric position of two-particle as well three-particle branches of the $\sigma_{ess}(\mathscr{A}(K))$ is investigated in this paper.

The article is dealt with as follows: an introduction to the whole investigation is given in Section 1. In Section 2, the matrix family $\mathscr{A}(K)$, $K \in \mathbf{T}^{\mathrm{d}}$ are described as the family of self-adjoint bounded linear operators in the direct sum of zero-particle subspace, one-particle subspace and two-particle subspace of the bosonic Fock space and the most important aims of the work are pointed. In Section 3, we studied the so called channel operator $\mathscr{A}_{\mathrm{ch}}(K)$ relative to $\mathscr{A}(K)$ and analyzed its spectrum using a family of generalized Friedrichs model. In the text Section a more detailed data on the position of $\sigma_{\mathrm{ess}}(\mathscr{A}(K))$ and its branches is given.

FAMILY OF OPERATOR MATRICES OF ORDER 3 AND ITS RELATION WITH THE LATTICE SPIN-BOSON MATRIX

First of all we will determine some setting, they are useful within this work. As $\mathbf{T}^{\mathrm{d}}$ we denote the dD torus. Channel 1 – $\mathscr{H}_0 := \mathbf{C}$, channel 2 – $\mathscr{H}_1 := L^2(\mathbf{T}^{\mathrm{d}})$ and channel 3 – $\mathscr{H}_2 := L^2_{\mathrm{sym}}((\mathbf{T}^{\mathrm{d}})^2)$ is a subspace of $L^2((\mathbf{T}^{\mathrm{d}})^2)$ containing all symmetric functions. The direct sum of these three channels, that is, the spaces $\mathscr{H}_0$, $\mathscr{H}_1$ and $\mathscr{H}_2$ will be denoted by $\mathscr{H}$, i.e., $\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1 \oplus \mathscr{H}_2$. Usually $\mathscr{H}_0$ is called zero-particle subspace, $\mathscr{H}_1$ is called one-particle subspace and $\mathscr{H}_2$ is called two-particle subspace of the bosonic (or symmetric) Fock space $\mathscr{F}_{\mathrm{s}}(L^2(\mathbf{T}^{\mathrm{d}}))$ with respect to $L^2(\mathbf{T}^{\mathrm{d}})$. The components F of $\mathscr{H}$ have a form $F = (F_0, F_1, F_2)$ with $F_i \in \mathscr{H}_i$, $i = 0,1,2$ and for $F = (F_0, F_1, F_2), G = (G_0, G_1, G_2) \in \mathscr{H}$ we have the equality

$$(F,G) := F_0\overline{G_0} + \int_{\mathbf{T}^{\mathrm{d}}} F_1(x)\overline{G_1(x)}dx + \int_{(\mathbf{T}^{\mathrm{d}})^2} F_2(x,y)\overline{G_2(x,y)}dxdy.$$

It is well known that each linear bounded operator always can be presented as a 3×3 operator matrix, if its domain is decomposed into three components [27].

In this work a family

$$\mathscr{A}(K) := \begin{pmatrix} A_{00}(K) & A_{01} & 0 \\ A_{01}^* & A_{11}(K) & A_{12} \\ 0 & A_{12}^* & A_{22}(K) \end{pmatrix}, \quad K \in \mathbf{T}^{\mathrm{d}} \tag{2}$$

is considered in $\mathscr{H}$. Here for $n = 0,1,2$ the diagonal elements $A_{nn}(K) : \mathscr{H}_n \to \mathscr{H}_n$ and for $n < m$, $n,m = 0,1,2$ the off-diagonal elements $A_{nm} : \mathscr{H}_m \to \mathscr{H}_n$ are defined as

$$A_{00}(K)g_0 = u_0(K)g_0, \quad A_{01}g_1 = \int_{\mathbf{T}^{\mathrm{d}}} \varphi_0(s)g_1(s)ds,$$
$$(A_{11}(K)g_1)(x) = u_1(K;x)g_1(x), \quad (A_{12}g_2)(x) = \int_{\mathbf{T}^{\mathrm{d}}} \varphi_1(s)g_2(x,s)ds,$$
$$(A_{22}(K)g_2)(x,y) = u_2(K;x,y)g_2(x,y), \quad g_i \in \mathscr{H}_k, \quad k = 0,1,2.$$

Throughout the article it is assumed that the functions with real-values $u_0(\cdot)$, $\varphi_k(\cdot)$, $k = 0,1$; $u_1(\cdot;\cdot)$ and $u_2(\cdot;\cdot,\cdot)$ are continuous on $\mathbf{T}^{\mathrm{d}}$; $(\mathbf{T}^{\mathrm{d}})^2$ and $(\mathbf{T}^{\mathrm{d}})^3$, respectively. Moreover, for $K \in \mathbf{T}^{\mathrm{d}}$ the assertion $u_2(K;x,y) = u_2(K;x,y)$ is valid for all $x,y \in \mathbf{T}^{\mathrm{d}}$.

Then the boundedness and self-adjointness of $\mathscr{A}(K)$ in $\mathscr{H}$ can be shown easily.

After direct calculations we get

$$A_{01}^* : \mathscr{H}_0 \to \mathscr{H}_1, \quad (A_{01}^*g_0)(x) = \varphi_0(x)g_0, \quad g_0 \in \mathscr{H}_0;$$
$$A_{12}^* : \mathscr{H}_1 \to \mathscr{H}_2, \quad (A_{12}^*g_1)(x,y) = \frac{1}{2}(\varphi_1(x)g_1(y) + \varphi_1(y)g_1(x)), \quad g_1 \in \mathscr{H}_1.$$

These operators have a wide application in quantum mechanics.

Let us now study the relation of $\mathscr{A}(K)$ to the lattice truncated spin boson matrix $\mathscr{B}_2$ with at most two photons. We remember that the operator $\mathscr{B}_2$ acts in $\mathbf{C}^2 \otimes \mathscr{H}$:

$$\mathscr{B}_2 := \begin{pmatrix} \mathscr{B}_{00} & \mathscr{B}_{01} & 0 \\ \mathscr{B}_{01}^* & \mathscr{B}_{11} & \mathscr{B}_{12} \\ 0 & \mathscr{B}_{12}^* & \mathscr{B}_{22} \end{pmatrix},$$

with elements

$$\mathscr{B}_{00} g_0^{(\mathrm{s})} = \varepsilon \mathrm{s} g_0^{(\mathrm{s})}, \quad \mathscr{B}_{01} g_1^{(\mathrm{s})} = \alpha \int_{\mathbf{T}^{\mathrm{d}}} v(t) g_1^{(-\mathrm{s})}(t) dt,$$

$$(\mathscr{B}_{11} g_1^{(\mathrm{s})})(x) = (\varepsilon \mathrm{s} + w(x)) g_1^{(\mathrm{s})}(x), \quad (\mathscr{B}_{12} g_2^{(\mathrm{s})})(x) = \alpha \int_{\mathbf{T}^{\mathrm{d}}} v(t) g_2^{(-\mathrm{s})}(x,t) dt,$$

$$(\mathscr{B}_{22} g_2^{(\mathrm{s})})(x,y) = (\varepsilon \mathrm{s} + w(x) + w(y)) g_2^{(\mathrm{s})}(x,y), \quad g = \{g_0^{(\mathrm{s})}, g_1^{(\mathrm{s})}, g_2^{(\mathrm{s})}; \mathrm{s} = \pm\} \in \mathbf{C}^2 \otimes \mathscr{H}.$$

Here $\varepsilon > 0$; $w(\cdot)$ (dispersion function) is an analytic on $\mathbf{T}^{\mathrm{d}}$ with real values; the function $v(\cdot)$ is an analytic on $\mathbf{T}^{\mathrm{d}}$ with real values; $\alpha > 0$ (coupling constant) is a real number.

To learn the spectrum of $\mathscr{B}_2$, we determine two (with discrete parameter) matrices $\mathscr{B}_2^{(\mathrm{s})}$, $\mathrm{s} = \pm$, in $\mathscr{H}$ as

$$\mathscr{B}_2^{(\mathrm{s})} := \begin{pmatrix} \widehat{\mathscr{B}}_{00}^{(\mathrm{s})} & \widehat{\mathscr{B}}_{01} & 0 \\ \widehat{\mathscr{B}}_{01}^* & \widehat{\mathscr{B}}_{11}^{(\mathrm{s})} & \widehat{\mathscr{B}}_{12} \\ 0 & \widehat{\mathscr{B}}_{12}^* & \widehat{\mathscr{B}}_{22}^{(\mathrm{s})} \end{pmatrix}$$

with the entries

$$\widehat{\mathscr{B}}_{00}^{(\mathrm{s})} g_0 = \varepsilon \mathrm{s} g_0, \quad \widehat{\mathscr{B}}_{01} g_1 = \alpha \int_{\mathbf{T}^{\mathrm{d}}} v(t) g_1(t) dt,$$

$$(\widehat{\mathscr{B}}_{11}^{(\mathrm{s})} g_1)(x) = (-\varepsilon \mathrm{s} + w(x)) g_1(x), \quad (\widehat{\mathscr{B}}_{12} g_2)(x) = \alpha \int_{\mathbf{T}^{\mathrm{d}}} v(t) g_2(x,t) dt,$$

$$(\widehat{\mathscr{B}}_{22}^{(\mathrm{s})} g_2)(x,y) = (\varepsilon \mathrm{s} + w(x) + w(x)) g_2(x,y), \quad (g_0, g_1, g_2) \in \mathscr{H}.$$

The following relations between subsets of the spectrum of $\mathscr{B}_2$ and $\mathscr{B}_2^{(\mathrm{s})}$ are proven in [8]:

The equality $\sigma(\mathscr{B}_2) = \sigma(\mathscr{B}_2^{(+)}) \cup \sigma(\mathscr{B}_2^{(-)})$ holds. Moreover,

$$\sigma_{\mathrm{ess}}(\mathscr{B}_2) = \sigma_{\mathrm{ess}}(\mathscr{B}_2^{(+)}) \cup \sigma_{\mathrm{ess}}(\mathscr{B}_2^{(-)}), \quad \sigma_{\mathrm{p}}(\mathscr{B}_2) = \sigma_{\mathrm{p}}(\mathscr{B}_2^{(+)}) \cup \sigma_{\mathrm{p}}(\mathscr{B}_2^{(-)}).$$

Since the subset of $\sigma_{\mathrm{disc}}(\mathscr{B}_2^{(\mathrm{s})})$ can be lie inside of $\sigma_{\mathrm{ess}}(\mathscr{B}_2)$ we obtain the result

$$\sigma_{\mathrm{disc}}(\mathscr{B}_2) \subseteq \sigma_{\mathrm{disc}}(\mathscr{B}_2^{(+)}) \cup \sigma_{\mathrm{disc}}(\mathscr{B}_2^{(-)}).$$

It is easy to check that if

$$w_0(K_0^{(\mathrm{s})}) = \varepsilon \mathrm{s}, \quad w_1(K_0^{(\mathrm{s})}; p) = -\varepsilon \mathrm{s} + w(p), \quad w_2(K_0^{(\mathrm{s})}; p, q) = \varepsilon \mathrm{s} + w(p) + w(q)$$

for some $K_0^{(\mathrm{s})} \in \mathbf{T}^{\mathrm{d}}$ and $v_i(p) = \alpha v(p)$, $i = 0, 1$, then $\mathscr{A}(K_0^{(\mathrm{s})}) = \mathscr{B}_2^{(\mathrm{s})}$. Therefore, using a connection between the operators $\mathscr{B}_2$ and $\mathscr{A}(K_0^{(\mathrm{s})})$, $\mathrm{s} = \pm$, the results for $\mathscr{B}_2$ can be obtained by considering a more general family of operator matrices $\mathscr{A}(K)$, $K \in \mathbf{T}^{\mathrm{d}}$.

A FAMILY OF CHANNEL OPERATORS AND ITS SPECTRUM

In the following we consider a family of channel operators $\mathscr{A}_{\mathrm{ch}}(K)$, $K \in \mathbf{T}^{\mathrm{d}}$ related with $\mathscr{A}(K)$ and learn its spectrum.

We construct the operator $\mathscr{A}_{\mathrm{ch}}(K)$ in $L^2(\mathbf{T}^{\mathrm{d}}) \oplus L^2((\mathbf{T}^{\mathrm{d}})^2)$ using the rule

$$\mathscr{A}_{\mathrm{ch}}(K) := \begin{pmatrix} A_{11}(K) & \frac{1}{\sqrt{2}} A_{12} \\ \frac{1}{\sqrt{2}} A_{12}^* & A_{22}(K) \end{pmatrix}, \quad K \in \mathbf{T}^{\mathrm{d}}. \tag{3}$$

With respect to the domain in the considered case the operator A_{12}^* is acting as

$$A_{12}^* : L^2(\mathbf{T}^{\mathrm{d}}) \to L^2((\mathbf{T}^{\mathrm{d}})^2), \quad (A_{12}^* g_1)(x,y) = \varphi_1(q) g_1(p), \quad g_1 \in L^2(\mathbf{T}^{\mathrm{d}}).$$

The boundedness and self-adjointness of $\mathscr{A}_{\mathrm{ch}}(K)$ in $L^2(\mathbf{T}^{\mathrm{d}}) \oplus L^2((\mathbf{T}^{\mathrm{d}})^2)$ easily follows from the definition. For a bounded function $\gamma(\cdot)$ on $\mathbf{T}^{\mathrm{d}}$ we consider multiplication operator U_γ by

$$U_\gamma \begin{pmatrix} f_1(x) \\ f_2(x,y) \end{pmatrix} = \begin{pmatrix} \gamma(x) f_1(x) \\ \gamma(x) f_2(x,y) \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in L^2(\mathbf{T}^{\mathrm{d}}) \oplus L^2((\mathbf{T}^{\mathrm{d}})^2).$$

One can easily check that $\mathscr{A}_{\mathrm{ch}}(K) U_\gamma = U_\gamma \mathscr{A}_{\mathrm{ch}}(K)$. By this reason the assertion

$$L^2(\mathbf{T}^{\mathrm{d}}) \oplus L^2((\mathbf{T}^{\mathrm{d}})^2) = \int_{\mathbf{T}^{\mathrm{d}}} \oplus (\mathscr{H}_0 \oplus \mathscr{H}_1) dk \tag{4}$$

yields the following decomposition

$$\mathscr{A}_{\mathrm{ch}}(K) = \int_{\mathbf{T}^{\mathrm{d}}} \oplus \widehat{\mathscr{A}}(K,k) dk. \tag{5}$$

Here the fibered operators $\widehat{\mathscr{A}}(K,k)$ are defined in $\mathscr{H}_0 \oplus \mathscr{H}_1$ as operator matrices 2×2

$$\widehat{\mathscr{A}}(K,k) := \begin{pmatrix} \widehat{A}_{00}(K,k) & \widehat{A}_{01} \\ \widehat{A}_{01}^* & \widehat{A}_{11}(K,k) \end{pmatrix}, \tag{6}$$

with components

$$\widehat{A}_{00}(K,k) g_0 = u_1(K,k) g_0, \ \widehat{A}_{01} g_1 = \frac{1}{\sqrt{2}} \int_{\mathbf{T}^{\mathrm{d}}} \varphi_1(t) g_1(t) dt,$$
$$(\widehat{A}_{11}(K,k) g_1)(y) = u_2(K;k,y) g_1(y), \quad g_l \in \mathscr{H}_l, \quad l = 0,1.$$

In this case

$$\widehat{A}_{01}^* : \mathscr{H}_0 \to \mathscr{H}_1, \quad (\widehat{A}_{01}^* g_0)(x) = \frac{1}{\sqrt{2}} \varphi_1(x) g_0, \quad g_0 \in \mathscr{H}_0.$$

Note that in the direct integral expansion (4) the identical layers appear. Applying theorem about the spectrum of decomposable operators [13] we obtain the relation

$$\sigma(\mathscr{A}_{\mathrm{ch}}(K)) = \bigcup_{k \in \mathbf{T}^{\mathrm{d}}} \sigma(\widehat{\mathscr{A}}(K,k)). \tag{7}$$

Thus, learning the spectrum of a channel operator $\mathscr{A}_{\mathrm{ch}}(K)$ is reduced to learning the spectrum of a family of generalized Friedrichs models, which is simple than $\widehat{\mathscr{A}}(K,k)$ and easy to study. No we start to study the spectrum of $\widehat{\mathscr{A}}(K,k)$.

Let

$$\widehat{\mathscr{A}_0}(K,k) := \begin{pmatrix} 0 & 0 \\ 0 & \widehat{A}_{11}(K,k) \end{pmatrix}.$$

Then for the operator $\widehat{\mathscr{A}}(K,k) - \widehat{\mathscr{A}_0}(K,k)$ we have $(\widehat{\mathscr{A}}(K,k) - \widehat{\mathscr{A}_0}(K,k))^* = \widehat{\mathscr{A}}(K,k) - \widehat{\mathscr{A}_0}(K,k)$ and $\mathrm{rank}(\widehat{\mathscr{A}}(K,k) - \widehat{\mathscr{A}_0}(K,k)) = 2$. Taking into account these facts and Weyl's theorem we obtain

$$\sigma_{\mathrm{ess}}(\widehat{\mathscr{A}}(K,k)) = [E_{\min}(K,k); E_{\max}(K,k)],$$

here

$$E_{\min}(K,k) := \min_{q\in\mathbf{T}^{\mathrm{d}}} w_2(K;k,q) \quad \text{and} \quad E_{\max}(K,k) := \max_{q\in\mathbf{T}^{\mathrm{d}}} w_2(K;k,q).$$

We determine the Fredholm determinant $\Delta_K(k;\cdot)$ of $\widehat{\mathscr{A}}(K,k))$ in $\mathbf{C}\setminus[E_{\min}(K,k);E_{\max}(K,k)]$ by

$$\Delta_K(k;z) := w_1(K;k) - z - \frac{1}{2}I_K(k;z), \quad I_K(k;z) := \int_{\mathbf{T}^{\mathrm{d}}} \frac{\varphi_1^2(s)ds}{w_2(K;k,s)-z}.$$

We have the following lemma [17].

Lemma 1. The quantity $z(K,k)\in\mathbf{C}\setminus[E_{\min}(K,k);E_{\max}(K,k)]$ is an discrete eigenvalue of $\widehat{\mathscr{A}}(K,k)$ iff $\Delta_K(k;z(K,k)) = 0$.

By Lemma 1 the discrete spectrum of $\widehat{\mathscr{A}}(K,k)$ satisfies the equality

$$\sigma_{\mathrm{disc}}(\widehat{\mathscr{A}}(K,k)) = \{z\in\mathbf{C}\setminus[E_{\min}(K,k);E_{\max}(K,k)] : \Delta_K(k;z)=0\}.$$

In the following we formulate lemma about the eigenvalues of $\widehat{\mathscr{A}}(K,k)$.

Lemma 2. The matrix $\widehat{\mathscr{A}}(K,k)$ hasn't more than 1 simple discrete eigenvalue located on the left hand side (respectively right hand side) of $E_{\min}(K,k)$ (respectively $E_{\max}(K,k)$).

The proof of Lemma 2 is an elementary.

Introduce the following notations

$$m_K := \min_{p,q\in\mathbf{T}^{\mathrm{d}}} w_2(K;p,q), \quad M_K := \max_{p,q\in\mathbf{T}^{\mathrm{d}}} w_2(K;p,q),$$
$$\Lambda_K := \bigcup_{k\in\mathbf{T}^{\mathrm{d}}} \sigma_{\mathrm{disc}}(\widehat{\mathscr{A}}(K,k)), \quad \Sigma_K := [m_K;M_K]\cup\Lambda_K.$$

Definition of the set Λ_K and the equality

$$\bigcup_{k\in\mathbf{T}^{\mathrm{d}}} [E_{\min}(K,k);E_{\max}(K,k)] = [m_K;M_K]$$

imply equality

$$\bigcup_{k\in\mathbf{T}^{\mathrm{d}}} \sigma(\widehat{\mathscr{A}}(K,k)) = \Sigma_K. \tag{8}$$

Now, the equalities (7) and (8) imply that the spectrum of the matrix $\mathscr{A}_{\mathrm{ch}}(K)$ is a purely essential and the relation $\sigma(\mathscr{A}_{\mathrm{ch}}(K)) = \Sigma_K$ holds for its spectrum.

POSITION OF SUBSETS OF THE ESSENTIAL SPECTRUM

Main results of the work:

Theorem 1. The equality $\sigma_{\mathrm{ess}}(\mathscr{A}(K)) = \sigma(\mathscr{A}_{\mathrm{ch}}(K))$ is valid. In addition, the set $\sigma(\mathscr{A}_{\mathrm{ch}}(K))$ consists of at most 3 segments.

It is remarkable that the channel operator $\mathscr{A}_{\mathrm{ch}}(K)$ defined as above has a simpler structure than the operator $\mathscr{A}(K)$, therefore, Theorem 1 plays an key role in other investigation of the spectrum of $\mathscr{A}(K)$.

Let us introduce the following notations:

$$E_{\min}^{(l)}(K) := \min\{\Lambda_K\cap(-\infty;m_K]\}, \quad E_{\max}^{(l)}(K) := \max\{\Lambda_K\cap(-\infty;m_K]\},$$
$$E_{\min}^{(r)}(K) := \min\{\Lambda_K\cap[M_K;+\infty)\}, \quad E_{\max}^{(r)}(K) := \max\{\Lambda_K\cap[M_K;+\infty)\},$$
$$\sigma_{\mathrm{two}}^{(l)}(K) := [E_{\min}^{(l)}(K);E_{\max}^{(l)}(K)], \quad \sigma_{\mathrm{two}}^{(r)}(K) := [E_{\min}^{(r)}(K);E_{\max}^{(r)}(K)].$$

Since the function $I_K(k;\cdot)$ is increasing in the intervals $(-\infty;m_K)$ and $(M_K;+\infty)$ for any fixed $K,k\in\mathbf{T}^{\mathrm{d}}$, by the Lebesgue theorem a finite or infinite limits

$$\lim_{z\to m_K-0} I_K(k;z)=I_K(k;m_K) \quad \text{and} \quad \lim_{z\to M_K+0} I_K(k;z)=I_K(k;M_K)$$

are exist.

Let $I_{K_0}(k_0;m_{K_0})=+\infty$ for some $K_0,k_0\in\mathbf{T}^{\mathrm{d}}$. Then

$$\lim_{z\to m_{K_0}-0}\Delta_{K_0}(k_0;z)=\Delta_{K_0}(k_0;m_{K_0})=-\infty,$$

hence from the equality

$$\lim_{z\to-\infty}\Delta_{K_0}(k_0;z)=+\infty$$

and Lemma 1 it follows that there exists an unique eigenvalue $z(K_0,k_0)$ in $(-\infty;m_{K_0})$. Using the definitions of Λ_K and $E^{(l)}_{\min}(K)$ we obtain that $E^{(l)}_{\min}(K_0)<m_{K_0}$. Analogously, if $I_{K_1}(k_1;m_{K_1})=-\infty$ for some $K_1,k_1\in\mathbf{T}^{\mathrm{d}}$, then $E^{(r)}_{\max}(K_0)>m_{K_0}$.

Next we suppose that for all $K,k\in\mathbf{T}^{\mathrm{d}}$ there exist finite integrals $I_K(k;m_K)$ and $I_K(k;M_K)$, that is, $|I_K(k;m_K)|<\infty$ and $|I_K(k;M_K)|<\infty$. In this case the functions $\Delta_K(\cdot;m_K)$ and $\Delta_K(\cdot;M_K)$ are continuous on $\mathbf{T}^{\mathrm{d}}$.

In the following three theorems the location of $\sigma_{\mathrm{ess}}(\mathscr{A}(K))$ and its structure can be exactly described.

Theorem 2. *Let $K\in\mathbf{T}^{\mathrm{d}}$ be a fixed and $\min\limits_{k\in\mathbf{T}^{\mathrm{d}}}\Delta_K(k;m_K)\geq 0$. Then*

$$\sigma_{\mathrm{ess}}(\mathscr{A}(K))=\begin{cases}[m_K;M_K],\ \text{if}\ \max\limits_{k\in\mathbf{T}^{\mathrm{d}}}\Delta_K(k;M_K)\leq 0;\\ \left[m_K;E^{(r)}_{\max}(K)\right],\ \text{if}\ \min\limits_{k\in\mathbf{T}^{\mathrm{d}}}\Delta_K(k;M_K)\leq 0\ \text{and}\ \max\limits_{k\in\mathbf{T}^{\mathrm{d}}}\Delta_K(k;M_K)>0;\\ [m_K;M_K]\cup\sigma^{(r)}_{\mathrm{two}}(K),\ \text{if}\ \min\limits_{k\in\mathbf{T}^{\mathrm{d}}}\Delta_K(k;M_K)>0;\end{cases}$$

moreover $E^{(l)}_{\min}(K)=m_K$.

Theorem 3. *Assume $K\in\mathbf{T}^{\mathrm{d}}$ and*

$$\min_{k\in\mathbf{T}^{\mathrm{d}}}\Delta_K(k\,m_K)<0,\quad \max_{k\in\mathbf{T}^{\mathrm{d}}}\Delta_K(k\,m_K)\geq 0.$$

Then for the $\sigma_{\mathrm{ess}}(\mathscr{A}(K))$ we have

$$\sigma_{\mathrm{ess}}(\mathscr{A}(K))=\begin{cases}\left[E^{(l)}_{\min}(K);M_K\right],\ \text{if}\ \max\limits_{k\in\mathbf{T}^{\mathrm{d}}}\Delta_K(k,M_K)\leq 0;\\ \left[E^{(l)}_{\min}(K);E^{(r)}_{\max}(K)\right],\ \text{if}\ \min\limits_{k\in\mathbf{T}^{\mathrm{d}}}\Delta_K(k;M_K)\leq 0,\ \max\limits_{k\in\mathbf{T}^{\mathrm{d}}}\Delta_K(k;M_k)>0;\\ \left[E^{(l)}_{\min}(K);M_K\right]\cup\sigma^{(r)}_{\mathrm{two}}(K),\ \text{if}\ \min\limits_{k\in\mathbf{T}^{\mathrm{d}}}\Delta_K(k;M_K)>0.\end{cases}$$

Moreover $E^{(l)}_{\min}(K)<m_K$.

Theorem 4. *Let $K\in\mathbf{T}^{\mathrm{d}}$ be a fixed and $\max\limits_{k\in\mathbf{T}^{\mathrm{d}}}\Delta_K(k\,m_K)<0$. Then the essential spectrum of $\mathscr{A}(K)$ has the following structure*

$$\sigma_{\mathrm{ess}}(\mathscr{A}(K))=\begin{cases}\sigma^{(l)}_{\mathrm{two}}(K)\cup[m_K;M_K],\ \text{if}\ \max\limits_{k\in\mathbf{T}^{\mathrm{d}}}\Delta_K(k;M_K)\leq 0;\\ \sigma^{(l)}_{\mathrm{two}}(K)\cup\left[m_K;E^{(r)}_{\max}(K)\right],\ \text{if}\ \min\limits_{k\in\mathbf{T}^{\mathrm{d}}}\Delta_K(k;M_K)\leq 0,\ \max\limits_{k\in\mathbf{T}^{\mathrm{d}}}\Delta_K(k;M_K)>0;\\ \sigma^{(l)}_{\mathrm{two}}(K)\cup[m_K;M_K]\cup\sigma^{(r)}_{\mathrm{two}}(K),\ \text{if}\ \min\limits_{k\in\mathbf{T}^{\mathrm{d}}}\Delta_K(k;M_K)>0.\end{cases}$$

Moreover, $E^{(l)}_{\max}(K)<m_K$.

Remark. *We notice that in the first assertions of Theorems* $2-4$ *we have* $E^{(r)}_{\max}(K) = M_K$*; in the second assertions* $E^{(r)}_{\max}(K) > M_K$*; in the third assertions* $E^{(r)}_{\min}(K) > M_K$*.*

Proof of Theorem 4. Assume $K \in \mathbf{T}^{\mathrm{d}}$ and $\max\limits_{k\in\mathbf{T}^{\mathrm{d}}} \Delta_K(k\, m_K) < 0$.

Since $\mathbf{T}^{\mathrm{d}}$ is a compact set from continuity of $\Delta_K(\cdot;m_K)$ on $\mathbf{T}^{\mathrm{d}}$, for all $k \in \mathbf{T}^{\mathrm{d}}$ we have the inequality

$$\Delta_K(k;m_K) < 0.$$

From the continuity and monotonicity of $\Delta_K(k;\cdot)$ on $(-\infty;m_K]$ and from

$$\lim_{z\to-\infty} \Delta_K(k;z) = +\infty$$

we conclude that there exist an unique point $z^{(l)}_K(k) \in (-\infty;m_K)$ with $\Delta_K(k;z^{(l)}_K(k) = 0$. Hence by Lemma 1 the point $z^{(l)}_K(k$ is the eigenvalue of $\widehat{\mathscr{A}}(K,k)$ in $(-\infty;m_K)$. By assumptions $z^{(l)}_K : k \in \mathbf{T}^{\mathrm{d}} \to z^{(l)}_K(k$ is a continuous on $\mathbf{T}^{\mathrm{d}}$ with real value. Therefore, $\mathrm{Im}\, \mathrm{z}^{(1)}_{\mathrm{K}}$ as subset of $(-\infty;m_K)$ is closed and connected, that is, $\mathrm{Im}\, \mathrm{z}^{(1)}_{\mathrm{K}} = [E^{(l)}_{\min}(K);E^{(l)}_{\max}(K)]$ and $E^{(l)}_{\max}(K) < m_K$.

Let now $\max\limits_{k\in\mathbf{T}^{\mathrm{d}}} \Delta_K(k;M_K) \leq 0$. Since $\mathbf{T}^{\mathrm{d}}$ is a compact set from the continuity of $\Delta_K(\cdot;M_K)$ on $\mathbf{T}^{\mathrm{d}}$ we have $\Delta_K(k;M_K) \leq 0$ for all $k \in \mathbf{T}^{\mathrm{d}}$. One has $\lim\limits_{z\to+\infty} \Delta_K(k;z) = -\infty$. Taking into account the monotonicity of $\Delta_K(k;\cdot)$ on $(M_K;+\infty)$ we get that the function $\Delta_K(k;\cdot)$ hasn't zeros bigger than M_K. Then by the Lemma 1 the matrix $\widehat{\mathscr{A}}(K,k)$ hasn't discrete eigenvalues bigger than M_K. Hence, Theorem 1 implies that $\max \sigma_{\mathrm{ess}}(\mathscr{A}(K)) = M_K$, i.e.

$$\sigma_{\mathrm{ess}}(\mathscr{A}(K)) = [E^{(l)}_{\min}(K);E^{(l)}_{\max}(K)] \cup [m_K;M_K] \quad \text{with} \quad E^{(l)}_{\max}(K) < m_K.$$

Let us now suppose that

$$\min_{k\in\mathbf{T}^{\mathrm{d}}} \Delta_K(k;M_K) \leq 0 \quad \text{and} \quad \max_{k\in\mathbf{T}^{\mathrm{d}}} \Delta_K(k;M_K) > 0.$$

Introduce the notation:

$$D_K :\equiv \{k \in \mathbf{T}^{\mathrm{d}} : \Delta_K(k;M_K) > 0\}.$$

Since $\mathbf{T}^{\mathrm{d}}$ is compact by the continuity of $\Delta_K(\cdot;m_K)$ on $\mathbf{T}^{\mathrm{d}}$ we get that there are $p^{(1)}_K, p^{(2)}_K \in \mathbf{T}^{\mathrm{d}}$ such that the inequalities

$$\min_{k\in\mathbf{T}^{\mathrm{d}}} \Delta_K(k;M_K) = \Delta_K(p^{(1)}_K;M_K) \leq 0,$$
$$\max_{k\in\mathbf{T}^{\mathrm{d}}} \Delta_K(k;M_K) = \Delta_K(p^{(2)}_K;M_K) > 0$$

are valid. It means voidness and openness of D_K with $D_K \neq \mathbf{T}^{\mathrm{d}}$.

From continuity and monotonicity of $\Delta_K(k;\cdot)$ on $[M_K;+\infty)$ and from $\lim\limits_{z\to+\infty} \Delta_K(k;z) = -\infty$ we imply that there is a unique quantity $z^{(r)}_K(k) \in (M_K;+\infty)$ so that $\Delta_K(k;z^{(r)}_K(k)) = 0$ for any $k \in D_K$. By Lemma 1 the point $z^{(r)}_K(k)$ is the unique discrete eigenvalue of the matrix $\widehat{\mathscr{A}}(K,k)$ lying on r.h.s. of M_K.

For $z > M_K$ and $k \in \mathbf{T}^{\mathrm{d}} \setminus D_K$ one have

$$\Delta_K(k;z) < \Delta_K(k;M_K) \leq 0.$$

Hence by Lemma 1 for each $k \in \mathbf{T}^{\mathrm{d}} \setminus D_K$ the operator $\widehat{\mathscr{A}}(K,k)$ hasn't discrete eigenvalues bigger than M_K.

By the continuity of $v_1(\cdot)$, $w_1(\cdot;\cdot)$ and $w_2(\cdot;\cdot,\cdot)$ on its domain, we get the continuity of $z^{(r)}_K : k \in D_K \to z^{(r)}_K(k)$ on D_K.

From the boundedness of $\widehat{\mathscr{A}}(K,k)$ and from compactness of $\mathbf{T}^{\mathrm{d}}$ we get that there is $C_K > 0$ with $\sup\limits_{k\in\mathbf{T}^{\mathrm{d}}} \|\widehat{\mathscr{A}}(K,k)\| \leq C_K$ and we receive

$$\sigma(\widehat{\mathscr{A}}(K,k)) \subset [-C_K;C_K]. \qquad (9)$$

For any $k \in \partial D_K = \{k \in \mathbf{T}^{\mathrm{d}} : \Delta_K(k;M_K) = 0\}$ there exist $\{k_n(K)\} \subset D_K$ such that $k_n(K) \to k(K)$ as $n \to \infty$. Set $z_K^{(n)} := z_K^{(r)}(k_n(K))$. Then for any $\{k_n(K)\} \in D_K$ the inequality $z_K^{(n)} > M_K$ holds and from (9) we get $\{z_K^{(n)}\} \subset [M_K;C_K]$. Suppose $z_K^{(n)} \to z_K^{(0)}$ as $n \to \infty$ for some $z_K^{(0)} \in [M_K;C_K]$.

From the continuity of the function $\Delta_K(\cdot\,;\cdot)$ in $\mathbf{T}^{\mathrm{d}} \times [M_K;+\infty)$ and $k_n(K) \to k(K)$ and $z_K^{(n)} \to z_K^{(0)}$ as $n \to \infty$ it follows that

$$0 = \lim_{n \to +\infty} \Delta_K(k_n(K);z_K^{(n)}) = \Delta_K(k;z_K^{(0)}).$$

By the monotonicity of $\Delta_K(k;\cdot)$ on $[M_K;+\infty)$ and by $k(K) \in \partial D_K$ we see that $\Delta_K(k;z_K^{(0)}) = 0$ if and only if $z_K^{(0)} = M_K$.

For any $k \in \partial D_K$ we define

$$z_K(k) = \lim_{k' \to k,\, k' \in D_K} z_K(k') = M_K.$$

From the continuity of $z_K(\cdot)$ on $D_K \cup \partial D_K$ and $z_K(k) = M_K$ for all $k \in \partial D_K$ we conclude that

$$\mathrm{Im} z_K(\cdot) = [M_K;E_{\max}^{(r)}(K)], \quad E_{\max}^{(r)}(K) > M_K.$$

Then by Theorem 1 we get

$$\sigma_{\mathrm{ess}}(\mathscr{A}(K)) = [E_{\min}^{(l)}(K);E_{\max}^{(l)}(K)] \cup [m_K;E_{\max}^{(r)}(K)] \quad \text{with} \quad E_{\max}^{(r)}(K) > M_K.$$

Finally, let $\min_{k \in \mathbf{T}^{\mathrm{d}}} \Delta_K(k;M_K) > 0$. Similarly to the case $\max_{k \in \mathbf{T}^{\mathrm{d}}} \Delta_K(k\, m_K) < 0$, one can show that matrix $\widehat{\mathscr{A}}(K,k)$ have an unique discrete eigenvalue $z_K^{(r)}(k)$ in $(M_K;+\infty)$ and

$$\mathrm{Im}\, z_K^{(r)} = [E_{\min}^{(r)}(K);E_{\max}^{(r)}(K)] \quad \text{and} \quad E_{\min}^{(r)}(K) > M_K.$$

Therefore

$$\sigma_{\mathrm{ess}}(\mathscr{A}(K)) = [E_{\min}^{(l)}(K);E_{\max}^{(l)}(K)] \cup [m_K;M_K] \cup [E_{\min}^{(r)}(K);E_{\max}^{(r)}(K)].$$

Here $E_{\max}^{(l)}(K) < m_K$ and $E_{\min}^{(r)}(K) > M_K$.

We finish the proof of Theorem 4.

Sketch of the proof of Theorem 2. Let $K \in \mathbf{T}^{\mathrm{d}}$ be a fixed and $\min_{k \in \mathbf{T}^{\mathrm{d}}} \Delta_K(k;m_K) \geq 0$. Then for $z < m_K$ we receive

$$\Delta_K(k;z) > \Delta_K(k;m_K) \geq 0.$$

By Lemma 1 it means that the matrix $\widehat{\mathscr{A}}(K,k)$ hasn't eigenvalues smaller than m_K. Determination of Λ_K implies

$$\Lambda_K \cap (-\infty;M_K] = [m_K;M_K].$$

The rest of the proof is like the proof of Theorem 4.

Sketch of the proof of Theorem 3. Assume that $K \in \mathbf{T}^{\mathrm{d}}$ is a fixed and

$$\min_{k \in \mathbf{T}^{\mathrm{d}}} \Delta_K(k\, m_K) < 0, \quad \max_{k \in \mathbf{T}^{\mathrm{d}}} \Delta_K(k\, m_K) \geq 0.$$

No likely the proof of the second assertion of Theorem 4 we get

$$\Lambda_K \cap (-\infty;M_K] = [E_{\min}^{(l)}(K);M_K] \quad \text{with} \quad E_{\min}^{(l)}(K) < m_K.$$

The rest of the proof runs as the proof of Theorem 4.

CONCLUSION

In the present paper the family $\mathscr{A}(K)$, $K \in \mathbf{T}^{\mathrm{d}} := (-\pi;\pi]^{\mathrm{d}}$ of the 3×3 block operator matrices is considered. Such matrices arise in the spectral analysis problem of the so called lattice truncated spin-boson Hamiltonian with at most two bosons. Exact relation between this family and the lattice spin-boson model is indicated. The corresponding channel operator is constructed and applying theorem about the spectrum of decomposable operators its spectrum is described. The position and structure of two-particle as well three-particle branches (subsets) of $\sigma_{ess}(\mathscr{A}(K))$ are investigated. In our analysis the key role is played the existence conditions of the eigenvalues of the generalized Friedrichs model.

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