ABSTRACTS

of the international conference

MATHEMATICAL ANALYSIS AND ITS APPLICATIONS IN MODERN MATHEMATICAL PHYSICS

PART I

Samarkand
September 23-24, 2022

MINISTRY OF HIGHER AND SECONDARY SPECIAL
EDUCATION OF THE REPUBLIC OF UZBEKISTAN

SAMARKAND STATE UNIVERSITY NAMED AFTER SH.RASHIDOV
MATHEMATICS INSTITUTE OF THE ACADEMY OF SCIENCE OF UZBEKISTAN

INTERNATIONAL CONFERENCE

MATHEMATICAL ANALYSIS AND ITS APPLICATIONS IN MODERN MATHEMATICAL PHYSICS

September 23-24, 2022; Samarkand, Uzbekistan

PART I

SAMARKAND – 2022

UDK 51:517

Mathematical analysis and its applications in modern mathematical physics: international scientific conference (September 23-24, 2022 y. Samarkand).

Chief Editor Lakaev S.N. – Samarkand, 2022 y.

The collection of abstracts of the reports of the international scientific conference "Mathematical analysis and its applications in modern mathematical physics" contains scientific reports on the following areas: Mathematical analysis, differential equations, mathematical physics, algebra, geometry, numerical mathematics, mathematical modeling, probability theory, mathematical statistics, mathematical engineering and information technologies .

It is intended for specialists in the field of physical and mathematical sciences and information technologies, teachers, doctoral students and undergraduates of universities.

2. *Shoyimardonov S.K.* Neimark-Sacker bifurcation and stability analysis in a discrete phytoplankton-zooplankton system with Holling type II functional response, arXiv:2207.01961 [math.DS]. P.1.16.

Threshold analysis for the family of generalized Friedrichs models

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Operators known as generalized Friedrichs model [1] appear in a series of problems in analysis, mathematical physics and probability theory. In the present note we discuss the threshold analysis for the family of generalized Friedrichs models corresponding to a system of quasi-particles, where their number is finite, but not fixed.

Let $\mathbb{C}$ be the field of complex numbers and $L_2(\mathbb{T}^3)$ be the Hilbert space of square integrable (complex) functions defined on the three-dimensional torus $\mathbb{T}^3$. Denote by $\mathcal{H}$ the direct sum of spaces $\mathcal{H}_0 := \mathbb{C}$ and $\mathcal{H}_1 := L_2(\mathbb{T}^3)$, that is, $\mathcal{H} := \mathcal{H}_0 \oplus \mathcal{H}_1$.

In the present note we consider a family of generalized Friedrichs models $h(k)$, $k \in \mathbb{T}^3$, which acts in $\mathcal{H}$ as

$$h(k) := \begin{pmatrix} h_{00}(k) & h_{01} \\ h_{01}^* & h_{11}(k) \end{pmatrix},$$

where

$$h_{00}(k)f_0 = (l_2\varepsilon(k)+1)f_0, \quad h_{01}f_1 = \int_{\mathbb{T}^3} v(t)f_1(t)dt,$$

$$(h_{11}(k)f_1)(q) = E_k(q)f_1(q), \quad E_k(q) := l_1\varepsilon(q) + l_2\varepsilon(k-q)$$

with $l_1, l_2 > 0$ and the dispersion function $\varepsilon(\cdot)$ is defined by

$$\varepsilon(q) := \sum_{i=1}^{3}(1-\cos(nq^{(i)})), \quad q = (q^{(1)}, q^{(2)}, q^{(3)}) \in \mathbb{T}^3, \quad n \in \mathbb{N}.$$

Here $f_i \in \mathcal{H}_i$, $i = 0, 1$; the function $v(\cdot)$ is either even or odd function on each variable and there exist all second order continuous partial derivatives of $v(\cdot)$ on $\mathbb{T}^3$. Then the family of operators $h(k)$, $k \in \mathbb{T}^3$ is bounded and self-adjoint in $\mathcal{H}$.

We remark that the operators h_{01} resp. h_{01}^* are called annihilation resp. creation operators, respectively.

Let Λ be a subset of $\mathbb{T}^3$ given by

$$\Lambda := \left\{ (p^{(1)}, p^{(2)}, p^{(3)}) : p^{(i)} \in \left\{ 0, \pm\frac{2}{n}\pi; \pm\frac{4}{n}\pi; \dots; \pm\frac{n'}{n}\pi \right\} \cup \Pi_n, \ i = 1, 2, 3 \right\},$$

where

$$n' := \begin{cases} n-2, & \text{if } n \text{ is even} \\ n-1, & \text{if } n \text{ is odd} \end{cases} \quad \text{and} \quad \Pi_n := \begin{cases} \{\pi\}, & \text{if } n \text{ is even} \\ \emptyset, & \text{if } n \text{ is odd} \end{cases}$$

Direct calculation shows that the cardinality of Λ is equal to n^3 and for any fixed $k \in \Lambda$ the function $E_k(\cdot)$ has the non-degenerate zero minimum at the points of Λ.

Using the Weyl theorem, for the essential spectrum of $h(k)$ we obtain the equality $\sigma_{\text{ess}}(h(k)) = [E_{\min}(k); E_{\max}(k)]$, where the numbers $E_{\min}(k)$ and $E_{\max}(k)$ by the rule

$$E_{\min}(k) := \min_{q\in\mathbb{T}^3} E_k(q) \quad \text{and} \quad E_{\max}(k) := \max_{q\in\mathbb{T}^3} E_k(q).$$

For any $k \in \mathbb{T}^3$ we define an analytic function $\Delta(k\,;\cdot)$ (the Fredholm determinant associated with the operator $h(k)$) in $\mathbb{C} \setminus [E_{\min}(k); E_{\max}(k)]$ by

$$\Delta(k\,;z) := l_2\varepsilon(k) + 1 - z - \int_{\mathbb{T}^3} \frac{v^2(t)dt}{E_k(t)-z}.$$

A simple consequence of the Birman-Schwinger principle and the Fredholm theorem imply that

$$\sigma_{\rm disc}(h(k)) = \{z \in \mathbb{C} \setminus [E_{\min}(k); E_{\max}(k)] : \, \Delta(k\,; z) = 0\}.$$

Since for any $k \in \Lambda$ the function $E_k(\cdot)$ has non-degenerate zero minimum at the points of Λ and the function $v(\cdot)$ is a continuous on $\mathbb{T}^3$, for any $k \in \mathbb{T}^3$ the integral

$$\int_{\mathbb{T}^3} \frac{v^2(t)dt}{E_k(t)}$$

is positive and finite. The Lebesgue dominated convergence theorem and the equality $\Delta(\mathbf{0}\,;0) = \Delta(k\,;0)$ for $k \in \Lambda$ yield

$$\Delta(\mathbf{0}\,;0) = \lim_{k\to k'} \Delta(k\,;0), \quad k' \in \Lambda,$$

where $\mathbf{0} := (0,0,0) \in \mathbb{T}^3$.

Since $\mathbf{0} \in \Lambda$ the definition of $h(k)$ imply the identity $h(\mathbf{0}) \equiv h(k)$ for all $k \in \Lambda$.

Moreover, $\sigma_{\rm ess}(h(\mathbf{0})) = [0; 6(l_1 + l_2)]$.

Let us denote by $C(\mathbb{T}^3)$ and $L_1(\mathbb{T}^3)$ the Banach spaces of continuous and integrable functions on $\mathbb{T}^3$, respectively.

Definition 1. *The operator $h(\mathbf{0})$ is said to have a zero energy resonance, if the number 1 is an eigenvalue of the integral operator given by*

$$(G\psi)(q) = \frac{v(q)}{l_1 + l_2} \int_{\mathbb{T}^3} \frac{v(t)\psi(t)}{\varepsilon(t)} dt, \quad \psi \in C(\mathbb{T}^3)$$

and at least one (up to a normalization constant) of the associated eigenfunctions ψ satisfies the condition $\psi(p') \neq 0$ for some $p' \in \Lambda$.

We notice that in Definition 1 the requirement of the existence of the eigenvalue 1 of G corresponds to the existence of a solution of $h(\mathbf{0})f = 0$ and the condition $\psi(p') \neq 0$ for some $p' \in \Lambda$ implies that the solution f of this equation does not belong to $\mathcal{H}$.

The following result establishes in which cases the bottom of the essential spectrum is a threshold energy resonance or eigenvalue.

Theorem 1. *The following statements are hold.*

(i) *The operator $h(\mathbf{0})$ has a zero eigenvalue if and only if $\Delta(\mathbf{0}\,;0) = 0$ and $v(q') = 0$ for all $q' \in \Lambda$;*

(ii) *The operator $h(\mathbf{0})$ has a zero energy resonance if and only if $\Delta(\mathbf{0}\,;0) = 0$ and $v(q') \neq 0$ for some $q' \in \Lambda$.*

Let I be an identity operator on $\mathcal{H}$.

Theorem 2. *If the operator $h(\mathbf{0})$ has either a zero energy resonance or a zero eigenvalue, then for any $k \in \Lambda$ and $p \in \mathbb{T}^3$ the operator $h(k-p) + l_1\varepsilon(p)I$ is non-negative.*

Set

$$\Lambda_0 := \{q' \in \Lambda : v(q') \neq 0\}.$$

Now we formulate a result (zero energy expansion for the Fredholm determinant, leading to behaviors of the zero energy resonance).

Theorem 3. *Let the operator $h(\mathbf{0})$ have a zero energy resonance and $k, p' \in \Lambda$. Then the following decomposition*

$$\Delta(k-p\,; z - l_1\varepsilon(p)) = \frac{4\pi^2}{n^2(l_1+l_2)^{3/2}} \Big(\sum_{q'\in\Lambda_0} v^2(q')\Big) \sqrt{\frac{l_1^2 + 2l_1 l_2}{l_1 + l_2}|p-p'|^2 - \frac{2z}{n^2}}$$
$$+ O(|p-p'|^2) + O(|z|)$$

holds for $|p - p'| \to 0$ *and* $z \to -0$.

We remark that Theorems 1, 2 and 3 are play key role in the spectral analysis of the family of 3×3 operator matrices, associated with the lattice systems describing two identical bosons and one particle, another nature in interactions, without conservation of the number of particles.

Reference:

1. *S.N.Lakaev.* Some spectral properties of the generalized Friedrichs model, (Russian), Trudy Sem. Petrovsk. 11 (1986), pp. 210–238, Translation in J. Soviet Math. 45:6 (1989), pp. 1540–1565.

On invariant sets of a quadratic non-stochastic operator

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Non-linear dynamical systems arise in many problems of biology, physics and other sciences. In particular, quadratic dynamical systems describe the behavior of populations of different species with population models [1, 2, 3]. Let $E = \{1, 2, \ldots, m\}$. A distribution on the set E is a probability measure $x = (x_1, \ldots, x_m)$, i.e., an element of the simplex:

$$S^{m-1} = \{x \in R : x_i \geq 0, \sum_{i=1}^{m} x_i = 1\}.$$

In general, a quadratic operator V, $V : x \in R^m \to x' = V(x) \in R^m$ is defined by:

$$V : x_k' = \sum_{i,j=1}^{m} P_{ij,k} x_i x_j, \quad k = 1, \ldots, m \tag{1}$$

In this talk we are interested to a non-stochastic quadratic mapping of simplex to itself, i.e. $V : S^{m-1} \to S^{m-1}$.

Definition. [3] A quadratic operator (1), preserving a simplex, is called non-stochastic (QnSO) if at least one of its coefficients $P_{ij,k}$, $i \neq j$ is negative.

Consider the following example of QnSO on the two-dimensional simplex S^2.

$$\begin{cases} x' = \frac{1}{2}(z-y)^2 + \frac{3}{2}x(y+z) \\ y' = \frac{1}{2}(x-z)^2 + \frac{3}{2}y(x+z) \\ z' = \frac{1}{2}(y-x)^2 + \frac{3}{2}z(x+y). \end{cases} \tag{2}$$

Fixed points. The fixed points are solutions to the system (2)

$$\begin{cases} x = \frac{1}{2}(z-y)^2 + \frac{3}{2}x(y+z) \\ y = \frac{1}{2}(x-z)^2 + \frac{3}{2}y(x+z) \\ z = \frac{1}{2}(y-x)^2 + \frac{3}{2}z(x+y). \end{cases}$$

By full analysis this system one obtains the following family of fixed points:

$$a_1 = (0, \frac{1}{2}, \frac{1}{2}), \quad a_2 = (\frac{1}{2}, 0, \frac{1}{2}), \quad a_3 = (\frac{1}{2}, \frac{1}{2}, 0), \quad a_4 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}).$$

Thus a_1, a_2 and a_3 are saddle, but a_4 is an attracting fixed point.