E-ISSN 2181-1466
9 772181 146004
ISSN 2181-6875
9 772181 687004

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Научный вестник Бухарского государственного университета
Scientific reports of Bukhara State University

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SCIENTIFIC REPORTS OF BUKHARA STATE UNIVERSITY
НАУЧНЫЙ ВЕСТНИК БУХАРСКОГО ГОСУДАРСТВЕННОГО УНИВЕРСИТЕТА

Ilmiy-nazariy jurnal
2023, № 1

Jurnal 2003-yildan boshlab **filologiya** fanlari bo‘yicha, 2015-yildan boshlab **fizika-matematika** fanlari bo‘yicha, 2018-yildan boshlab **siyosiy** fanlar bo‘yicha O‘zbekiston Respublikasi Vazirlar Mahkamasi huzuridagi Oliy attestatsiya komissiyasining dissertatsiya ishlari natijalari yuzasidan ilmiy maqolalar chop etilishi lozim bo‘lgan zaruriy nashrlar ro‘yxatiga kiritilgan.

Jurnal 2000-yilda tashkil etilgan.
Jurnal 1 yilda 6 marta chiqadi.

Jurnal O‘zbekiston matbuot va axborot agentligi Buxoro viloyat matbuot va axborot boshqarmasi tomonidan 2020-yil 24-avgust № 1103-sonli guvohnoma bilan ro‘yxatga olingan.

Muassis: Buxoro davlat universiteti
Tahririyat manzili: 200117, O‘zbekiston Respublikasi,Buxoro shahri Muhammad Iqbol ko‘chasi, 11-uy.
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ANIQ VA TABIIY FANLAR *** EXACT AND NATURAL SCIENCES *** ТОЧНЫЕ И ЕСТЕСТВЕННЫЕ НАУКИ

UDC: 519.682.5

LOWER BOUND OF THE ESSENTIAL SPECTRUM OF A FAMILY OF 3×3 OPERATOR MATRICES

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Abstract. *A family of 3×3 operator matrices $H(K)$, $K\in(-\pi,\pi]$ corresponding to the system of non-conserved and no more than three particles on a 1D lattice is considered. Considered family of operator matrices is defined as a linear, bounded and self-adjoint operator acting in the three-particle cut subspace of the Fock space. The number and location of the eigenvalues of the generalized Friedrichs model is investigated. We find a finite set $\Lambda\subset(-\pi,\pi]$ and give an estimate for the lower bound of the essential spectrum of operator matrix $H(K)$ for all $K\in\Lambda$.*

Keywords: *operator matrix; bosonic Fock space; annihilation and creation operators; generalized Friedrichs model.*

Аннотация. *Рассматривается семейство 3×3 операторных матриц $H(K)$, $K\in(-\pi,\pi]$, соответствующих системам с несохраняющимся и не более трёх частиц на одномерной решётке. Исследуемое семейство операторных матриц определено как линейный, ограниченный и самосопряжённый оператор в трёхчастичном обрезанном подпространстве фоковекого пространства. Исследовано число и местонахождение собственных значений обобщённой модели Фридрихса. Найдено конечное множество $\Lambda\subset(-\pi,\pi]$, и получена оценка для нижней грани существенного спектра операторной матрицы $H(K)$ при всех $K\in\Lambda$.*

Ключевые слова: *операторная матрица, бозонное пространство Фока, операторы уничтожения и рождения, обобщённая модель Фридрихса.*

Annotatsiya. *Bir o‘lchamli panjaradagi soni saqlanmaydigan va uchtadan oshmaydigan zarrachalar sistemasiga mos $H(K)$, $K\in(-\pi,\pi]$ – 3×3 operatorli matritsalar oilasi qaralgan. O‘rganilayotgan operatorli matritsalar oilasi Fok fazosining qirqilgan uch zarrachali qism fazosidagi chiziqli, chegaralangan va o‘z-o‘ziga qo‘shma operator sifatida aniqlangan. Umumlashgan Fridrixs modeli xos qiymatlari soni va joylashuv o‘rni tadqiq qilingan. Shunday $\Lambda\subset(-\pi,\pi]$ chekli to‘plam topilib, barcha $K\in\Lambda$ larda $H(K)$ operatorli matritsa muhim spektrining quyi chegarasi uchun baholash olingan.*

Kalit so’zlar: *operatorli matritsa; bozonli Fok fazosi; yo‘qotish va paydo qilish operatorlari; umumlashgan Fridrixs modeli*

Introduction. Block operator matrices are matrices the entries of which are linear operators between Banach or Hilbert spaces [1, 2]. Every bounded linear operator can be written as a block operator matrix if the space in which it acts is decomposed in two or more more components. Operator matrices arise in various areas of mathematics and its applications: in system theory as Hamiltonians, in the discretization of partial differential equations as large partitioned matrices due to sparsity patterns, in saddle point problems in non-linear analysis, in evolution problems as linearizations of second order Cauchy problems and as linear operators describing coupled systems of partial differential equations. Such systems occur widely in mathematical physics, e.g. in fluid mechanics, magnetohydrodynamics, theory of solid-state physics, quantum field theory, statistical physics and quantum mechanics.

Notice that one special class of block operator matrices are Hamiltonians associated with systems of non-conserved number of quasi-particles on a lattice. Their number can be unbounded as in the case of spin-boson models or bounded as in the case of "truncated" spin-boson models. It is remarkable that lattice spin-boson model with at most two photons can be represented as a 6×6 operator matrix which is unitarily equivalent to a 2×2 block diagonal operator with the 3×3 operator matrices on the diagonal [3,4].

In the present paper we consider a family of 3×3 operator matrices $H(K)$, $K\in \mathrm{T}:=(-\pi;\pi]$. These operator matrices are associated with the lattice systems describing two identical bosons and one particle, another nature in interactions, without conservation of the number of particles. They act in the direct sum of zero-, one- and two-particle subspaces of the bosonic Fock space. We discuss the case where the dispersion function $\varepsilon(\cdot)$ has the form $\varepsilon(x)=1-\cos(nx)$ with $n>1$. We denote by Λ the set of points T where the function $\varepsilon(\cdot)$ takes its (global) minimum. Under some smoothness assumptions on the parameters of $H(K)$, we estimate the lower bound of the essential spectrum of $H(K)$.

Family of 3×3 operator matrices and main results

Let T be the one-dimensional torus, $\mathrm{H}_0 := \mathbb{C}$ be the field of complex numbers, $\mathrm{H}_1 := L_2(\mathrm{T})$ be the Hilbert space of square integrable (complex) functions defined on T and $\mathrm{H}_2 := L_2^s(\mathrm{T}^2)$ be the Hilbert space of square integrable (complex) symmetric functions defined on T^2. The Hilbert space $\mathrm{H} := \mathrm{H}_0 \oplus \mathrm{H}_1 \oplus \mathrm{H}_2$ is called three-particle cut subspace of a bosonic Fock space $\mathrm{F}_s(L_2(\mathrm{T}))$ over $L_2(\mathrm{T})$, respectively.

In the present paper we consider a family of 3×3 operator matrices $H(K)$, $K\in\mathrm{T}$ acting in the Hilbert space H as

$$H(K):=\begin{pmatrix} H_{00}(K) & H_{01} & 0 \\ H_{01}^* & H_{11}(K) & H_{12} \\ 0 & H_{12}^* & H_{22}(K) \end{pmatrix}$$

with the entries

$$H_{00}(K)f_0 = w_0(K)f_0,\quad H_{01}f_1=\int_{\mathrm{T}} v(t)f_1(t)dt,$$

$$(H_{11}(K)f_1)(x)=w_1(K;x)f_1(x),\quad (H_{12}f_2)(x)=\int_{\mathrm{T}} v(t)f_2(x,t)dt,$$

$$(H_{22}(K)f_2)(x,y)=w_2(K;x,y)f_2(x,y),\quad f_i\in\mathrm{H}_i,\quad i=0,1,2$$

where H_{ij}^* $(i<j)$ denotes the adjoint operator to H_{ij}.

Here $w_0(\cdot)$ is a real-valued bounded function on T, the function $v(\cdot)$ is a real-valued analytic on T, the functions $w_1(\cdot;\cdot)$ and $w_2(\cdot;\cdot,\cdot)$ are defined by the equalities

$$w_1(K;x):=l_1\varepsilon(x)+l_2\varepsilon(K-x)+1,\quad w_2(K;x,y):=l_1\varepsilon(x)+l_1\varepsilon(y)+l_2\varepsilon(K-x-y),$$

respectively, with $l_1,l_2>0$ and

$$\varepsilon(x):=1-\cos(nx),\quad n\in N.$$

Under these assumptions the operator $H(K)$ is bounded and self-adjoint.

We remark that the operators H_{01} and H_{12}, resp. H_{01}^* and H_{12}^* are called annihilation resp. creation operators, respectively. It is clear that

$$H_{01}^*: H_0\to H_1,\quad (H_{01}^*f_0)(x)=v(x)f_0,\quad f_0\in H_0;$$

$$H_{12}^*: H_1\to H_2,\ \ (H_{12}^*f_1)(x,y)=\frac{1}{2}(v(x)f_1(y)+v(y)f_1(x)),\quad f_1\in H_1.$$

To study the essential spectrum of the operator $H(K)$ we introduce a family of bounded self-adjoint operators (generalized Friedrichs models) $h(k)$, $k\in\mathrm{T}$, which acts in $\mathrm{H}_0\oplus\mathrm{H}_1$ as

$$h(k):=\begin{pmatrix} h_{00}(k) & h_{01} \\ h_{01}^* & h_{11}(k) \end{pmatrix},$$

where

$$h_{00}(k)f_0=(l_2\varepsilon(k)+1)f_0,\quad h_{01}f_1=\frac{1}{\sqrt{2}}\int_{\mathrm{T}} v(t)f_1(t)dt,$$

$$(h_{11}(k)f_1)(y)=E_k(y)f_1(y),\quad E_k(y):=l_1\varepsilon(y)+l_2\varepsilon(k-y).$$

Let the operator $h_0(k)$, $k\in\mathrm{T}$ acts in $\mathrm{H}_0\oplus\mathrm{H}_1$ as

$$h_0(k):=\begin{pmatrix} 0 & 0 \\ 0 & h_{11}(k) \end{pmatrix}.$$

The perturbation $h(k)-h_0(k)$ of the operator $h_0(k)$ is a self-adjoint operator of rank 2, and thus, according to the Weyl theorem, the essential spectrum of the operator $h(k)$ coincides with the essential spectrum of $h_0(k)$. It is evident that $\sigma_{\text{ess}}(h_0(k))=[E_{\min}(k);E_{\max}(k)]$, where the numbers $E_{\min}(k)$ and $E_{\max}(k)$ are defined by

$$E_{\min}(k):=\min_{y\in\mathrm{T}} E_k(y) \quad \text{and} \quad E_{\max}(k):=\max_{y\in\mathrm{T}} E_k(y).$$

This yields $\sigma_{\text{ess}}(h(k))=[E_{\min}(k);E_{\max}(k)]$.

For any $k\in\mathrm{T}$ we define an analytic function $\Delta(k;\cdot)$ (the Fredholm determinant associated with the operator $h(k)$ in $\mathbb{C}\setminus[E_{\min}(k);E_{\max}(k)]$ by

$$\Delta(k;z):=l_2\varepsilon(k)+1-z-\frac{1}{2}\int_{\mathrm{T}}\frac{v^2(t)dt}{E_k(t)-z}.$$

Note that for the discrete spectrum of $h(k)$ the equality

$$\sigma_{\text{disc}}(h(k))=\{z\in\mathbb{C}\setminus[E_{\min}(k);E_{\max}(k)]:\ \Delta(k;z)=0\}$$

holds (see Lemma 1).

The following theorem [5, 6] describes the location of the essential spectrum of the operator $H(K)$ by the spectrum of the family $h(k)$ of generalized Friedrichs models.

Theorem 1. *For the essential spectrum of $H(K)$ the equality*

$$\sigma_{\text{ess}}(H(K))=\bigcup_{x\in\mathrm{T}}\{\sigma_{\text{disc}}(h(K-x))+l_1\varepsilon(x)\}\cup[m_K;M_K] \tag{1}$$

holds, where the numbers m_K and M_K are defined by

$$m_K:=\min_{x,y\in\mathrm{T}} w_2(K;x,y) \quad \text{and} \quad M_K:=\max_{x,y\in\mathrm{T}} w_2(K;x,y).$$

Let Λ be a subset of T given by

$$\Lambda:=\left\{0;\pm\frac{2}{n}\pi;\pm\frac{4}{n}\pi;\ldots;\pm\frac{n'}{n}\pi\right\}\cup\Pi_n,$$

where

$$n':=\begin{cases} n-2, & \text{if } n \text{ is even}\\ n-1, & \text{if } n \text{ is odd}\end{cases} \qquad \text{and} \qquad \Pi_n:=\begin{cases}\{\pi\}, & \text{if } n \text{ is even}\\ \varnothing, & \text{if } n \text{ is odd}\end{cases}$$

Direct calculation shows that the cardinality of Λ is equal to n. It is easy to check that for any $K\in\Lambda$ the function $w_2(K;\cdot,\cdot)$ has non-degenerate zero minimum at the points of $\Lambda\times\Lambda$, that is, $m_K=0$ for $K\in\Lambda$.

Since $0\in\Lambda$ the definition of the functions $w_1(\cdot;\cdot)$ and $w_2(\cdot;\cdot,\cdot)$ imply the identity $h(0)\equiv h(k)$ for all $k\in\Lambda$.

For $\delta>0$ and $a\in\mathrm{T}$ we set

$$U_\delta(a):=\{x\in\mathrm{T}:|x-a|<\delta\}.$$

We remark that, if $v(x')=0$ for some $x'\in\mathrm{T}$, then from analyticity of $v(\cdot)$ on T it follows that there exist positive numbers C_1,C_2 and δ such that the inequalities

$$C_1|x-x'|^{\beta(x')}\le|v(x)|\le C_2|x-x'|^{\beta(x')}, \quad x\in U_\delta(x') \tag{2}$$

hold for some $\beta(x')\in N$.

Since for any $K\in\Lambda$ the function $w_2(K;\cdot,\cdot)$ has non-degenerate zero minimum at the points $(x',y')\in\Lambda\times\Lambda$, we obtain the following expansion

$$w_2(K;x,y)=\frac{n^2}{2}\Big[(l_1+l_2)|x-x'|^2+2l_2(x-x',y-y')+(l_1+l_2)|y-y'|^2\Big]$$
$$+O(|x-x'|^4)+O(|y-y'|^4)$$

as $|x-x'|,|y-y'|\to 0$ for $K,x',y'\in\Lambda$. Therefore, for any $K\in\Lambda$ there exist $C_1,C_2>0$ and $\delta>0$ such that estimates hold

$$C_1(|x-x'|^2+|y-y'|^2)\le w_2(K;x,y)\le C_2(|x-x'|^2+|y-y'|^2),(x,y)\in U_\delta(x')\times U_\delta(y'); \quad (3)$$

$$w_2(K;x,y)\ge C_1,(x,y)\not\in\bigcup_{x'\in\Lambda}U_\delta(x')\times\bigcup_{y'\in\Lambda}U_\delta(y'). \quad (4)$$

Hence, if $v(x')=0$ for all $x'\in\Lambda$, then using the inequalities (2)–(4) one can easily seen that the integral

$$\int_{\mathrm{T}}\frac{v^2(t)dt}{w_2(K;x,t)}$$

is positive and finite for any $K\in\Lambda$ and $x\in\mathrm{T}$.

For $x'\in\Lambda$ the Lebesgue dominated convergence theorem yields

$$\Delta(-x';-l_1\varepsilon(x'))=\lim_{x\to x'}\Delta(-x;-l_1\varepsilon(x)),$$

and hence, if $v(x')=0$ for all $x'\in\Lambda$, then the function $\Delta(-x;-l_1\varepsilon(x))$ is a continuous with respect to x on T.

Set

$$\tau_{\mathrm{ess}}(K):=\min\{\lambda:\lambda\in\sigma_{\mathrm{ess}}(H(K))\}.$$

Then $\tau_{\mathrm{ess}}(K)\in\sigma_{\mathrm{ess}}(H(K))$ and it is called the lower bound of the essential spectrum of $H(K)$.

The main result of the present paper is the following theorem.

Theorem 2. *Let $K\in\Lambda$. For the lower bound $\tau_{\mathrm{ess}}(K)$ the following assertions hold.*

(i) If $v(x')\ne 0$ for some $x'\in\Lambda$, then $\tau_{\mathrm{ess}}(K)<0$;

(ii) If $v(x')=0$ for all $x'\in\Lambda$ and $\min_{x\in\mathrm{T}}\Delta(-x;-l_1\varepsilon(x))<0$ then $\tau_{\mathrm{ess}}(K)<0$;

(iii) If $v(x')=0$ for all $x'\in\Lambda$ and $\min_{x\in\mathrm{T}}\Delta(-x;-l_1\varepsilon(x))\ge 0$, then $\tau_{\mathrm{ess}}(K)=0$.

This theorem is play a key role in the study of the number of eigenvalues of $H(K)$, $K\in\Lambda$.

Lower bound of the essential spectrum of $H(K)$

In this section first we study the discrete spectrum of the generalized Friedrichs model $h(k)$ and then we prove Theorem 2.

The following lemma is a simple consequence of the Birman-Schwinger principle and the Fredholm theorem.

Lemma 1. *For any fixed $k\in\mathrm{T}$ the operator $h(k)$ has an eigenvalue $z(k)\in\mathbb{R}\setminus[E_{\min}(k);E_{\max}(k)]$ if and only if $\Delta(k;z(k))=0$.*

In the next two lemmas we describe the number and location of the eigenvalues of $h(k)$.

Lemma 2. *If $v(y')\ne 0$ for some $y'\in\Lambda$, then for all $x'\in\Lambda$ the operator $h(x')$ has a unique negative eigenvalue.*

Proof. Since the function $w_2(0;\cdot,\cdot)$ has the non-degenerate zero global minimum at the points $(0,0)\in\mathrm{T}^2$, by the implicit function theorem there exist $\delta>0$ and an analytic function $y_0(\cdot)$ on $U_\delta(0)$ such that for any $x\in U_\delta(0)$ the point $y_0(x)$ is the unique non-degenerate minimum of the function $w_2(x,\cdot)$ and $y_0(0)=0$. Therefore, we have $w_2(x,y_0(x))=m_x$ for any $x\in U_\delta(0)$.

Let $w_2(\cdot,\cdot)$ be the function defined on $U_\delta(x)\times\mathrm{T}$ as $w_2(x,y):=w_2(x,y+y_0(x))$. Then for any $x\in U_\delta(x)$ the function $w_2(x,\cdot)$ has a unique non-degenerate zero minimum at the point $x\in\mathrm{T}$. Now using the equality

$$\int_{\mathrm{T}}\frac{v_1^2(t)dt}{w_2(x,t)-m_x}=\int_{\mathrm{T}}\frac{v_1^2(t+y_0(x))dt}{w_2(x,t)},\quad x\in U_\delta(0),$$

the continuity of the function $v_1(\cdot)$, the conditions $v_1(0)\neq 0$ and $y_0(0)=0$ it is easy to see that $\lim\limits_{z\to m_x-0}\Delta(x;z)=-\infty$ for all $x\in U_\delta(0)$.

Since for any $x\in \mathrm{T}$ the function $\Delta(x;\cdot)$ is continuous and monotonically decreasing on $(-\infty, m_x)$ the equality

$$\lim_{z\to-\infty}\Delta(x;z)=\infty \tag{5}$$

implies that for any $x\in U_\delta(0)$ the function $\Delta(x;\cdot)$ has a unique zero $z=z(x)$, lying in $(-\infty, m_x)$. By Lemma 1 the number $z(x)$ is the eigenvalue of $h(x)$.

Lemma 3. *Let* $v_1(x')=0.$ *for any* $x'\in\Lambda$.

(i) *If* $\min\limits_{x\in T}\Delta(x;m)\geq 0$, *then for any* $x\in\mathrm{T}$ *the operator* $h(x)$ *has no eigenvalues, lying on the left of* m.

(ii) *If* $\min\limits_{x\in T}\Delta(x;m)<0$, *then there exists a non-empty set* $G\subset\mathrm{T}$ *such that for any* $x\in G$ *the operator* $h(x)$ *has a unique eigenvalue* $z(x)$, *lying on the left of* m.

Proof. First we recall that if $v_1(0)=0$, then the function $\Delta(\cdot;m)$ is a continuous on T. Let $\min\limits_{x\in\mathrm{T}}\Delta(x;m)\geq 0$. Since for any $x\in\mathrm{T}$ the function $\Delta(x;\cdot)$ is monotonically decreasing on $(-\infty,m)$ we have $\Delta(x;z)>\Delta(x;m)\geq\min\limits_{x\in\mathrm{T}}\Delta(x;m)\geq 0$, that is, $\Delta(x;z)>0$ for all $x\in\mathrm{T}$ and $z<m$. Therefore, by Lemma 1 for any $x\in\mathrm{T}$ the operator $h(x)$ has no eigenvalues in $(-\infty,m)$.

Now we suppose that $\min\limits_{x\in\mathrm{T}}\Delta(x;m)<0$ and introduce the following subset of T:

$$G:=\{x\in\mathrm{T}:\Delta(x;m)<0\}.$$

Since $\Delta(\cdot;m)$ is a continuous on the compact set T, there exists a point $x^0\in\mathrm{T}$ such that $\min\limits_{x\in\mathrm{T}}\Delta(x;m)=\Delta(x^0;m)$, that is, $x^0\in G$. So, the set G is a non-empty. Note that if $\max\limits_{x\in\mathrm{T}}\Delta(x;m)<0$, then $\Delta(x;m)<0$ for all $x\in\mathrm{T}$ and hence $G=\mathrm{T}$.

Since for any $x\in\mathrm{T}$ the function $\Delta(x;\cdot)$ is a continuous and monotonically decreasing on $(-\infty,m]$ by the equality (5) for any $x\in G$ there exists a unique point $z(x)\in(-\infty,m)$ such that $\Delta(x;z(x))=0$. By Lemma 1 for any $x\in G$ the point $z(x)$ is the unique eigenvalue of $h(x)$.

By the construction of G the inequality $\Delta(x;m)\geq 0$ holds for all $x\in\mathrm{T}\setminus G$. In this case one can see that for any $x\in\mathrm{T}\setminus G$ the operator $h(x)$ has no eigenvalues in $(-\infty,m)$.

Proof of Theorem 2. Let $v_1(0)\neq 0$. Then by Lemma 2 there exists $\delta>0$ such that for any $x\in U_\delta(0)$ the operator $h(x)$ has a unique eigenvalue $z(x)$, lying on the left of m_x. In particular, $z(0)<m_0$. Since $m=\min\limits_{x\in\mathrm{T}}m_x=m_0$ it follows that $\min\sigma\leq z(0)<m$, that is, $E_{\min}<m$.

Let $v_1(0)=0$. Then two cases are possible: $\min\limits_{x\in\mathrm{T}}\Delta(x;m)\geq 0$ or $\min\limits_{x\in\mathrm{T}}\Delta(x;m)<0$. In the case $\min\limits_{x\in\mathrm{T}}\Delta(x;m)\geq 0$, by the part (i) of Lemma 3 for any $x\in\mathrm{T}$ the operator $h(x)$ has no eigenvalues in $(-\infty,m)$, that is, $\min\sigma\geq m$. By Theorem 1 it means that $E_{\min}=m$.

For the case $\min\limits_{x\in\mathrm{T}}\Delta(x;m)<0$, using the part (ii) of Lemma 3 we obtain $\min\sigma\leq z(x')<m$ for all $x'\in G$, that is, $E_{\min}<m$.

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