Image Reconstruction Algorithm using optimal interpolation formula in $W_2^{(1,0)}$ space.

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We apply interpolation to the convolution $\left(\mathcal{F}_D^{-1}A\right) * (\mathcal{R}_D f)$ where A is the low-pass filter. Denoting this interpolated function by $\mathcal{I}$, we then approximate the value $f(x_m, y_n)$ at each point in the image grid by

$$\begin{aligned} f(x_m, y_n) &\approx \frac{1}{2}\mathcal{B}_D\mathcal{I}(x_m, y_n) \\ &= \frac{1}{2N}\sum_{k=0}^{N-1}\mathcal{I}\left(x_m cos\left(\frac{k\pi}{N}\right) + y_n sin\left(\frac{k\pi}{N}\right), \frac{k\pi}{N}\right) \end{aligned}$$

As a test case, we will use the radially symmetric phantom shown on the left in Fig.1.

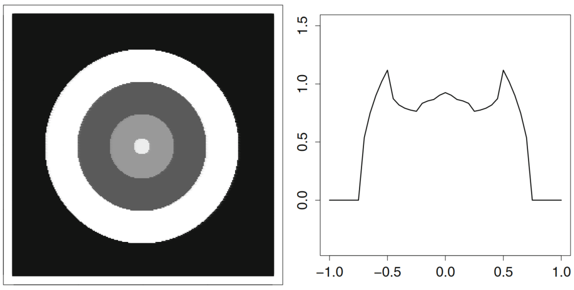

Fig.1. A radially symmetric attenuation function and its Radon transform.

- We use Python to compute the (discrete) Radon transform of this phantom;
- The next step is to choose a low-pass filter A and compute its discrete inverse Fourier transform, $\mathcal{F}_D^{-1}A$. When we interpret our low-pass filter as a $2L$-periodic discrete function that vanishes outside the interval $[-L, L]$, the effective sample spacing for $\mathcal{F}_D^{-1}A$ is given by $1/(2L)$. We then compute the discrete convolution $\left(\mathcal{F}_D^{-1}A\right)*(\mathcal{R}_D f)$. The two discrete functions must have the same sample spacing, so we want to have $1/(2L) = \tau$. In practice, the value of τ is determined by the scanner, and,therefore, we set $L = 1/(2\tau)$ for the low-pass filter.

  For our test case, we use the Shepp-Logan filter and compute a separate discrete convolution for each angle.

- Next, we select a method of interpolation. We will evaluate the discrete back projection only at a finite set of points $\{(x_m, y_n)\}$ that define the grid in which the final image is to be presented. We need interpolation in order to assign values to $\left(\mathcal{F}_D^{-1}A\right) * (\mathcal{R}_D f)$ at the points $\{(x_m cos(k\pi/N) + y_n sin(k\pi/N), k\pi/N)\}$.

We could execute an optimal interpolation formula

$$\varphi(x) \cong P_\varphi(x) = \sum_{\beta=0}^{N} C_\beta(x) \cdot \varphi(x_\beta),$$

with equal spaced nodes in $W_2^{(1,0)}$ space [1-3]. Coefficients of the optimal interpolation formula have the following form

$$\begin{aligned} \mathring{C}_\beta(z) = \ & \tfrac{1}{2(1-e^{2h})}\Big[\operatorname{sgn}(z-h\beta-h)\cdot(e^{h\beta+2h-z}-e^{z-h\beta}) \\ & +\operatorname{sgn}(z-h\beta+h)\cdot(e^{h\beta-z}-e^{z-h\beta+2h}) \\ & +(1+e^{2h})\cdot\operatorname{sgn}(z-h\beta)\cdot(e^{z-h\beta}-e^{h\beta-z})\Big], \quad \beta=0,1,...,N. \end{aligned}$$

In this case, for each angle in the scan, a separate spline is computed to fit the data given by the corresponding row of the matrix of filtered X-ray data. Then, for each point (x_m, y_n) in the image grid, we interpolate values at the desired points.

- Now that we have carried out the interpolation, we finish by applying the discrete back projection. For each point in the image grid, we compute the average, over all angles in the scan, of the corresponding interpolated values of the filtered X-ray data.

References

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3. **Hayotov A.R., Babaev S.S.** Calculation of the coefficients of optimal interpolation formulas in the space $W_2^{(2,1)}(0,1)$. *Uzbek Mathematical Journal*, 2014, no.3, pp.126-133. (in Russian)