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Global solvability of inverse coefficient problem for one fractional diffusion equation with initial non-local and integral overdetermination conditions

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Abstract
In this work, we consider an inverse problem of determining the coefficient at the lower term of a fractional diffusion equation. The direct problem is the initial-boundary problem for this equation with non-local initial and homogeneous Dirichlet conditions. To determine the unknown coefficient, an overdetermination condition of the integral form is specified with respect to the solution of the direct problem. Using Green's function for an ordinary fractional differential equation with a non-local boundary condition and the Fourier method, the inverse problem is reduced to an equivalent problem. Further, by using the fixed-point argument in suitable Sobolev spaces, the global theorems of existence and uniqueness for the solution of the inverse problem are obtained.

Keywords Nonlocal problems · The Caputo derivative · Subdiffusion equation · Inverse problem · Sobolev spaces · Mittag-Leffler functions

Mathematics Subject Classification Primary 34A12 · 35R11 · Secondary 33E12

1 Introduction

At present, in almost every area of modern technology and research, methods and tools of fractional calculus are used. Many fractional calculus applications have been successful because these new fractional-order models are often more accurate than integer-order models, i.e., the fractional-order model has more degrees of freedom

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than its classical analogue. Because of its ability to handle the dynamics of non-integer orders, fractional calculus is a powerful tool for explaining the memory and hereditary characteristics of various materials and processes. The theory of fractional calculus has been developed rapidly in many fields of science since the 19th century, including fractional geometry, fractional differential equations, and fractional dynamics due to its wide range of applications [1]–[3]. For example, in such areas as rheology, viscoelasticity, acoustics, optics, chemical, and statistical physics, robotics, control theory, electrical engineering, mechanical engineering, and bioengineering there have widely applied the fractional calculus [4]–[6].

Recently, it has increased extensive attention on inverse potential problems for time-fractional diffusion equations. Henceforth the inverse potential problems have been analyzed in many works [7]–[17] and references therein. Sun *et al* [7]–[9] investigated the uniqueness in determining the fractional order(s) and the potential simultaneously for the single-term and multi-term time-fractional diffusion equations, respectively, and gave a valid numerical method. Zhang and Zhou [10] considered an inverse potential term problem from the overdetermined final time data and gave an efficient regularized iterative algorithm based on mollification in the one-dimensional case. Wenjun Ma and Liangliang Sun [11] studied an inverse potential problem from an additional integral measured data over the domain. The well-posedness of the forward problem was investigated by using the well-known Rothe's method and recovered numerically the unknown coefficient by an iterative Tikhonov regularization method. Durdiev *et al* [12]–[15] analyzed inverse problems of determining unknown coefficients in the Cauchy problem for the fractional diffusion-wave equation. Local existence and uniqueness as a whole are proved and estimates of conditional stability are obtained. In addition, the inverse problems for the determination of potential terms from the generalized fractional derivative diffusion-wave equation is rapidly developing. For example, Durdiev and Turdiev [16], [17] studied inverse problems of determining the time-dependent coefficient in the fractional wave equation with Hilfer (generalized Riemann-Liouville) derivative. In these works, similar results to those mentioned above were obtained.

In the above literature, it is mainly the inverse potential problems of a single term for the time-fractional diffusion equations with initial conditions considered naturally. However, it does not involve cases including a fractional derivative, a nonlinear source and a nonlocal initial condition. This is because some traditional skills for the direct problem, such as well-posedness, the Banach fixed point method. We also point out recent papers [18]–[24] in which new classes of inverse problems of determining the time-dependent source and unknown order of the fractional derivative are investigated for various types of additional conditions. However, in all of the non-linear work mentioned above, results of local solvability have been obtained.

In this paper, we focus on an inverse time-dependent potential problem by the integral data over the domain. We first investigate the well-posedness of the direct problem by employing Fourier's method. Further, we use the Banach fixed point theorem, the Gauss-Seidel method and the coerciveness of the fractional derivative to obtain the existence and uniqueness of the inverse problem, and also illustrate an example. Furthermore, one of the main achievements of the paper is the result of the global solvability of the inverse problem with nonlocal initial conditions.

The rest of the paper is structured as follows: In the next section, we formulate the direct and inverse problems that will be investigated in this paper. Here we will also provide the necessary information from functional analysis and the theory of fractional calculus, which will be used throughout this thesis. At the end of this section, we will formulate the main results. Section 3 is devoted to constructing Green's function for the boundary problem of the linear ordinary fractional equation with nonlocal boundary conditions. Green's function will serve as a substitute for the eigenfunctions of the uniform elliptic operator in the construction of a solution to the direct problem. In Section 4, we will consider the solvability of the direct problem and construct a priori estimates that will be used in the investigation of an inverse problem, in addition, an equivalent formulation of the inverse problem is presented. Using the equivalent form and the contraction mapping principle, the result on the local in-time existence and uniqueness of solutions is established in Section 5. The main result of the paper is proved by the method of continuation of the solution in Section 6.

2 Formulation of problem and preliminaries

Let Ω be a bounded domain in $\mathbf{R}^n$ with sufficiently smooth boundary $\partial\Omega$. Consider the following fractional-diffusion equation in $Q_0^T := \{(x,t) : x \in \Omega \subset \mathbf{R}^n,\ 0 < t < T\}$:

$$\left(\partial_t^\alpha u\right)(x,t) = Au(x,t) + q(t)u(x,t) + f(x,t), \quad (x,t) \in Q_0^T, \tag{2.1}$$

with the Gerasimov–Caputo time fractional derivative ∂_t^α of order $0 < \alpha < 1$, defined by

$$\left(\partial_t^\alpha y\right)(t) = \left(I_{0+}^{1-\alpha} y'\right)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} y'(\tau)d\tau, \quad y \in W^{1,1}(0,T),$$

where I_{0+}^α is the Reimann-Liouville fractional integral of order α, that is

$$\left(I_{0+}^\alpha y\right)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds,$$

and $\Gamma(\cdot)$ is the Gamma function and the operator $-A$ is a symmetric uniformly elliptic operator defined on $D(-A) = H^2(\Omega) \cap H_0^1(\Omega)$ given by

$$Au(x,t) = \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\sum_{j=1}^n a_{ij}(x) \frac{\partial}{\partial x_j} u(x,t) \right), \quad (x,t) \in Q_0^T,$$

in which the coefficients satisfy

$$a_{ij} = a_{ji}, \quad 1 \le i, j \le n, \quad a_{ij} \in C^1(\bar{\Omega}),$$
$$\nu_1 \sum_{i=1}^{n} |\xi_i|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j$$
$$\le \nu_2 \sum_{i=1}^{n} |\xi_i|^2, \quad x \in \bar{\Omega}, \quad \xi \in \mathbf{R}^n, \ \nu_1, \nu_2 > 0.$$

We supplement the equation (2.1) with the nonlocal initial condition

$$u(x, 0) + \beta u(x, T) = \varphi(x), \quad x \in \Omega, \tag{2.2}$$

the boundary condition

$$u(x, t) = 0, \quad x \in \partial\Omega, \quad t \in (0, T), \tag{2.3}$$

and integral condition of the first kind

$$\int_{\Omega} \omega(x)u(x, t)dx = h(t), \quad 0 \le t \le T, \tag{2.4}$$

where $\beta \ge 0$ and $f(x, t), \varphi(x), \omega(x), h(t)$ are known functions. Besides, $h(t)$ is the measurement data representing the average temperature on a small part of Ω, because the weight function $\omega(x)$ is usually chosen to satisfy $\mathrm{supp}(\omega) \subset \Omega$ in applied sciences [26]. Now we pass to a rigorous statement of the main problem of our paper. If the function $q(t)$ is known, then the initial-boundary value problem (2.1)–(2.3) is called the *direct (forward) problem*.

We investigate the following inverse problem:

Inverse problem. Find $(u, q) \in C\left([0, T]; H^2(\Omega)\right) \times C[0, T]$ to satisfy (2.1)–(2.3) and the additional condition (2.4).

For the convenience of the reader, we present here the necessary definitions from functional analysis and fractional calculus theory.

For integers m, we denote $H^m(\Omega) = W^{m,2}(\Omega)$ (see [27]) and $H_0^m(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the norm of space $H^m(\Omega)$. For a given Banach space V on Ω, we use the notation $C^m([0, T]; V)$ to denote the following space:

$$C^m([0, T]; V) := \left\{u : \|\partial_t^j u(t)\|_V \text{ is continuous in } t \text{ on } [0, T] \text{ for all } 0 \le j \le m\right\}.$$

We endow $C^m([0, T]; V)$ with the following norm making it to be a Banach space:

$$\|u\|_{C^m([0,T];V)} = \sum_{j=0}^{m} \left(\max_{0 \le t \le T} \|\partial_t^j u(t)\|_V \right).$$

In addition, we define Banach space X_0^T by

$$X_0^T := C([0, T]; H^2(\Omega)).$$

Furthermore, we set

$$Y_0^T = X_0^T \times C[0,T]$$

endowed with the norm

$$\|(u,q)\|_{Y_0^T} := \|u\|_{X_0^T} + \|q\|_{C[0,T]}.$$

It is well-known that the operator $-A$ has only real and simple eigenvalues λ_k, and with suitable numbering, we have $0 < \lambda_1 < \lambda_2 < \cdots$, $\lim_{k\to\infty} \lambda_k = \infty$. By e_k, we denote the eigenfunction corresponding to λ_k, which satisfies $\|e_k\|^2_{L^2(\Omega)} = (e_k, e_k) = 1$, where $(\cdot,\cdot)$ denotes the inner product in the Hilbert space $L^2(\Omega)$ and λ_k, e_k satisfy $-Ae_k = \lambda_k e_k$, $e_k(x) = 0$, $x \in \partial\Omega$, $\{e_k\} \subset H^2(\Omega) \cap H_0^1(\Omega)$ is an orthonormal basis of $L^2(\Omega)$. Then for $\gamma \in \mathbf{R}$, we define a Hilbert space $D((-A)^\gamma)$ by

$$D((-A)^\gamma) := \left\{u \in L^2(\Omega) : \sum_{k=1}^\infty \lambda_k^{2\gamma} |(u,e_k)|^2 < \infty\right\},\ (-A)^\gamma u = \sum_{k=1}^\infty \lambda_k^\gamma (u,e_k)e_k$$

with the norm

$$\|u\|_{D((-A)^\gamma)} = \left(\sum_{k=1}^\infty \lambda_k^{2\gamma} |(u,e_k)|^2\right)^{1/2}$$

(see, e.g. [28]). We note that the norm $\|u\|_{D((-A)^\gamma)}$ is stronger than $\|u\|_{L^2(\Omega)}$ for $\gamma > 0$. Then, we have $D((-A)^\gamma) \subset H^{2\gamma}(\Omega)$ for $\gamma > 0$. In particular, $D((-A)^{\frac{1}{2}}) = H_0^1(\Omega)$. Since $D((-A)^\gamma) \subset L^2(\Omega)$, identifying the dual $(L^2(\Omega))'$ which itself, we have $D((-A)^\gamma) \subset L^2(\Omega) \subset (D((-A)^\gamma))'$. We set $D((-A)^{-\gamma}) = (D((-A)^\gamma))'$, which consists of bounded linear functionals on $D((-A)^\gamma)$. For $u \in D((-A)^\gamma)$ and $\varphi \in D((-A)^\gamma)$, the value obtained by operating u to φ is denoted by $\langle\cdot,\cdot\rangle_{-\gamma,\gamma}$. $D((-A)^{-\gamma})$ is a Hilbert space with the norm:

$$\|\varphi\|_{D((-A)^{-\gamma})} = \left(\sum_{k=1}^\infty \lambda_k^{-2\gamma} |\langle u,e_k\rangle_{-\gamma,\gamma}|^2\right)^{\frac{1}{2}}.$$

We further note that

$$\langle u,\varphi\rangle_{-\gamma,\gamma} = (u,\varphi) \quad \text{if } u \in L^2(\Omega) \text{ and } \varphi \in D((-A)^\gamma)$$

(see e.g., [25], Chapter V in [31]).

Let us remind the definition of the Mittag-Leffler function in [30]:

$$E_{\rho,\mu}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\rho k + \mu)}, \quad z \in \mathbf{C}$$

with $\mathrm{Re}(\rho) > 0$ and $\mu \in \mathbf{C}$. It is known that $E_{\rho,\mu}(z)$ is an entire function in $z \in \mathbf{C}$. If the parameter $\mu = 1$, then we have the classical (one-parameter) Mittag-Leffler function: $E_\rho(z) = E_{\rho,1}(z)$.

In what follows we need the asymptotic estimate of the Mittag-Leffler function with a sufficiently large negative argument. The well-known estimate has the form (see, e.g., [29], p.136)

$$|E_{\rho,\mu}(-t)| \leq \frac{C}{1+t}, \quad t > 0. \tag{2.5}$$

This estimate essentially follows from the following asymptotic estimate (see, e.g., [29], p.134)

$$E_{\rho,\mu}(-t) = \frac{t^{-1}}{\Gamma(\mu-\rho)} + O(t^{-2}). \tag{2.6}$$

For the Mittag-Leffler function with two parameters $E_{\rho,\rho}(-t)$ one can obtain a better estimate than (2.5). Indeed, using the asymptotic estimate (see, e.g., [29], p.134)

$$E_{\rho,\rho}(-t) = -\frac{t^{-2}}{\Gamma(-\rho)} + O(t^{-3}), \tag{2.7}$$

and the fact that $E_{\rho,\rho}(-t)$ is a real analytic, we can obtain the following inequality (see, [20])

$$|E_{\rho,\rho}(-t)| \leq \frac{C}{1+t^2}, \quad t > 0. \tag{2.8}$$

We will also use a coarser estimate with positive number λ and $0 < \varepsilon < 1$:

$$\left|t^{\rho-1}E_{\rho,\rho}(-\lambda t^\rho)\right| \leq \frac{Ct^{\rho-1}}{1+(\lambda t^\rho)^2} \leq C\lambda^{\varepsilon-1}t^{\varepsilon\rho-1}, \quad t > 0, \tag{2.9}$$

which is easy to verify (see, [20]).

Proposition 1 *Let* $0 < \rho < 1$. *Then*

$$E_\rho(x) > 0, \quad \frac{d}{dx}E_\rho(x) > 0, \quad x \in \mathbf{R}.$$

Moreover, the Mittag-Leffler function of negative argument $E_\rho(-x)$ *is monotonically decreasing function for all* $0 < \rho < 1$ *and*

$$0 < E_\rho(-x) < 1.$$

Proposition 2 (see [25]) *For* $\lambda > 0$, $\alpha > 0$ *and positive integer* $m \in \mathbf{N}$, *we have*

$$\frac{d^m}{dt^m} E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-m} E_{\alpha,\alpha-m+1}(-\lambda t^\alpha), \quad t > 0$$

and

$$\frac{d}{dt}\left(t E_{\alpha,2}(-\lambda t^\alpha)\right) = E_{\alpha,1}(-\lambda t^\alpha),\ \partial_t^\alpha(E_{\alpha,1}(-\lambda t^\alpha)) = -\lambda E_{\alpha,1}(-\lambda t^\alpha), \quad t \geq 0.$$

It is known (see [30], p. 96) that the following lemma holds.

Lemma 1 *If* $y(t)$ *belong to* $C^n[0, T]$ *or* $W^{n,1}(0, T)$, *then for* $c_j \in \mathbf{R}$ *we have*

$$\left(I_{0+}^\alpha \partial_t^\alpha y\right)(t) = y(t) - c_0 - c_1 t - c_2 t^2 - \ldots - c_{n-1} t^{n-1}, \quad n-1 < \alpha \leq n.$$

We now define a weak solution to (2.1)–(2.3), which is similar to the introduced in [25].

Definition 1 We call $u(x, t)$ a weak solution to (2.1)–(2.3) if (2.1) holds in $L^2(\Omega)$ and $u(\cdot, t) \in H_0^1(\Omega)$ for almost all $t \in (0, T)$ and $u \in C([0, T]; D((-A)^{-\gamma}))$,

$$\lim_{t\to 0} \|u(\cdot, t) + \beta u(\cdot, T-t) - \varphi\|_{D((-A)^{-\gamma})} = 0$$

with some $\gamma > 0$.

We make the following assumptions:

(K1) $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$, $f \in C([0, T]; D(-A)^\varepsilon)$ where $0 < \varepsilon < 1$;
(K2) $h(0) + \beta h(T) = (\omega, \varphi)$;
(K3) $\partial_t^\alpha h \in C[0, T]$ satisfies the following inequality:

$$|h(t)| \geq \frac{1}{h_0} > 0, \quad \text{for all} \quad t \in [0, T],$$

where h_0 is a positive constant;

(K4) $\gamma > \frac{n}{4} - \varepsilon - 1$ for $n > 3$;
(K5) $\omega(x) \in L^2(\Omega)$.

Remark 1 (K2) is the consistency condition for our problem (2.1)–(2.4), which guarantees that the inverse problem (2.1)–(2.4) is equivalent to (4.31) and (4.32) (see Lemma 7).

Our main results in this paper are the following existence and uniqueness theorems on the inverse problem solution.

Theorem 1 *Under hypotheses (K1)-(K5), there exists a solution* $(u, q) \in Y_0^T$ *of the inverse problem* (2.1)–(2.4) *for any* $T > 0$.

Theorem 2 *Let* $T > 0$. *Under hypotheses (K1)-(K5), if the inverse problem* (2.1)–(2.4) *has two solutions* $(u_i, q_i) \in Y_0^T$ $(i = 1, 2)$, *then* $(u_1, q_1) = (u_2, q_2)$ *for* $0 \leq t \leq T$.

3 Construction of the Green function

In this section, we will construct a formal solution to the ordinary fractional differential equation with nonlocal boundary conditions, which will be used in the proof of our main results.

To study problem (2.1)–(2.4), we consider the following fractional differential equation

$$\left(\partial_t^\alpha y\right)(t) = \gamma(t) y(t), \quad 0 < t < T, \tag{3.1}$$

with the boundary conditions

$$y(0) + \beta y(T) = 0, \tag{3.2}$$

where $\beta \geq 0$ are fixed numbers, $\gamma(t) \in C[0, T]$ is given function, and $y = y(t)$ is desired function.

Now, we give a new construction method of Green's function for the boundary value problem (3.1)–(3.2).

Lemma 2 *The fractional order boundary value problem* (3.1) *is equivalent to the integral equation*

$$y(t) = \int_0^T G_0(t, s)\gamma(s) y(s) ds, \tag{3.3}$$

where Green's function G_0 *is defined by*

$$G_0(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} -\frac{1}{1+\beta}\left(\beta(T-s)^{\alpha-1} - (1+\beta)(t-s)^{\alpha-1}\right), & s \in [0, t), \\ -\frac{\beta}{1+\beta}(T-s)^{\alpha-1}, & s \in (t, T]. \end{cases} \tag{3.4}$$

Proof Let $y \in C^1[0, T]$ or $W^{1,1}(0, T)$. Then, using Lemma 1, we have

$$y(t) = c_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \gamma(s) y(s) ds, \tag{3.5}$$

where c_0 is a real constant. Now, by employing the condition (3.2) we can obtain the coefficient c_0 as follow

$$c_0 = -\frac{\beta}{(1+\beta)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \gamma(s) y(s) ds.$$

Therefore, the solution of (3.1), (3.2) is

$$\begin{aligned} y(t) &= -\frac{\beta}{(1+\beta)\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\gamma(s)y(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\gamma(s)y(s)ds \\ &= \int_0^t \left(\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{\beta}{(1+\beta)\Gamma(\alpha)}(T-s)^{\alpha-1}\right)\gamma(s)y(s)ds \\ &\quad -\frac{\beta}{(1+\beta)\Gamma(\alpha)}\int_t^T (T-s)^{\alpha-1}\gamma(s)y(s)ds = \int_0^T G_0(t,s)\gamma(s)y(s)ds. \end{aligned}$$

□

Lemma 3 *Suppose that the function* $\gamma(t)$ *is continuous on the interval* $[0, T]$. *If* $\beta \geq 0$ *and*

$$\rho_0 := \frac{1+2\beta}{(1+\beta)\Gamma(1+\alpha)} T^{\alpha} \|\gamma\|_{C[0,T]} < 1, \tag{3.6}$$

then problem (3.1), (3.2) *has only a trivial solution.*

Proof From Lemma 2, we can conclude that the problem (3.1), (3.2) is equivalent to the integral equation (3.3).

Let us introduce the notation

$$P(y(t)) = \int_0^T G_0(t,s)\gamma(s)y(s)ds. \tag{3.7}$$

Then the equation (3.3) can be rewritten as

$$y(t) = P(y(t)). \tag{3.8}$$

The operator P is continuous in the space $C[0, T]$. Indeed, for $y(t) \in C[0, T]$, we have

$$|P(y(t))| \leq \int_0^T |G(t,s)\gamma(s)y(s)|\, ds \leq \rho_0 \|y\|_{C[0,T]},$$

that is, operator P is bounded. Hence, it is continuous.

Now we prove that P is a contraction operator in the space $C[0, T]$. It is easy to see that the inequality

$$\|P(y_1(t)) - P(y_2(t))\|_{C[0,T]} \leq \rho_0 \|y_1(t) - y_2(t)\|_{C[0,T]} \tag{3.9}$$

holds for any functions $y_1(t), y_2(t) \in C[0, T]$.

In view of (3.6) and (3.9) it is clear that the operator P is contractive in $C[0, T]$. Therefore, the operator P has a unique fixed point $y(t)$ in the space $C[0, T]$ which is a solution of equation (3.8). Thus, the integral equation (3.3) has a unique solution in $C[0, T]$. Consequently, problem (3.1), (3.2) also has a unique solution in the indicated space. Since $y(t) = 0$ is a solution to problem (3.1) and (3.2), it follows that this problem has a unique trivial solution. □

Lemma 4 *Assume $\beta \geq 0$. Then, the solution to the boundary-value problem*

$$\left(\partial_t^\alpha y_1\right)(t) + \lambda y_1(t) = f(t), \quad t > 0,$$
$$y_1(0) + \beta y_1(T) = a$$

with $0 < \alpha < 1$ and $\lambda, a \in \mathbf{R}$ has the form

$$y_1(t) = \frac{a}{\rho(T)} E_{\alpha,1}(-\lambda t^\alpha) + \int_0^T G(t,s) f(s) ds,$$

where

$$G(t,s) = -\frac{1}{\rho(T)} \begin{cases} \beta(T-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(T-s)^\alpha) E_{\alpha,1}(-\lambda t^\alpha) \\ \quad -\rho(T)(t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha), & s \in [0,t) \\ \beta(T-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(T-s)^\alpha) E_{\alpha,1}(-\lambda t^\alpha), & s \in (t,T]. \end{cases}$$

The proof of this Lemma 4 is analogical to the proof of Lemma 2. □

4 Direct Problem

This section will study the direct problem (2.1)–(2.3). We first consider the following initial boundary problem

$$\begin{cases} \partial_t^\alpha u(x,t) - Au(x,t) = F(x,t), & (x,t) \in Q_0^T, \\ u(x,0) + \beta u(x,T) = \varphi(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega, \ 0 < t < T. \end{cases} \tag{4.1}$$

Let $u_k(t) = (u(\cdot,t), e_k)$, $k \geq 1$. Assume that problem (4.1) has a unique solution u which is given by

$$u(x,t) = \sum_{k=1}^\infty u_k(t) e_k(x), \tag{4.2}$$

where u_k, $(k = 1, 2, ...)$ are solutions of the nonlocal problems:

$$\left(\partial_t^\alpha u_k\right)(t) + \lambda_k u_k(t) = F_k(t), \quad 0 < t < T, \tag{4.3}$$
$$u_k(0) + \beta u_k(T) = \varphi_k, \tag{4.4}$$

where $F_k(t) = (F(\cdot,t), e_k)$, $\varphi_k = (\varphi, e_k)$. Using Lemma 4, we have

$$u_k(t) = \frac{\varphi_k}{\rho_k(T)} E_{\alpha,1}(-\lambda_k t^\alpha) + \int_0^T G_k(t,s) F_k(s) ds,$$

where $\rho_k(T)$ is $\rho(T)$ with λ_k instead of λ.

Thus, we have

$$u(x,t) = \sum_{k=1}^{\infty} \frac{(\varphi, e_k)}{\rho_k(T)} E_{\alpha,1}(-\lambda_k t^\alpha) e_k(x) + \sum_{k=1}^{\infty} \Big(\int_0^T G_k(t,s)(F, e_k)(s) ds \Big) e_k(x). \tag{4.5}$$

Based on the method of [25], we will prove the well-posedness of the problem (4.1). We first split (4.1) into the following two initial and boundary value problems:

$$\begin{cases} \partial_t^\alpha v(x,t) - Av(x,t) = 0, & (x,t) \in Q_0^T, \\ v(x,0) + \beta v(x,T) = \varphi(x), & x \in \Omega, \\ v(x,t) = 0, & x \in \partial\Omega,\ 0 < t < T, \end{cases} \tag{4.6}$$

and

$$\begin{cases} \partial_t^\alpha w(x,t) - Aw(x,t) = F(x,t), & (x,t) \in Q_0^T, \\ w(x,0) + \beta w(x,T) = 0, & x \in \Omega, \\ w(x,t) = 0, & x \in \partial\Omega,\ 0 < t < T. \end{cases} \tag{4.7}$$

Lemma 5 *(i) Let $\varphi \in L^2(\Omega)$. Then there exists a unique weak solution $v \in C([0,T]; L^2(\Omega)) \cap C((0,T]; H^2(\Omega) \cap H_0^1(\Omega))$ to (4.6) with $\partial_t^\alpha v \in C((0,T]; L^2(\Omega))$. Moreover, there exists a constant $c_1 > 0$ satisfying*

$$\begin{cases} \|v\|_{C([0,T];L^2(\Omega))} \le \|\varphi\|_{L^2(\Omega)}, \\ \|v(\cdot,t)\|_{H^2(\Omega)} + \|\partial_t^\alpha v(\cdot,t)\|_{L^2(\Omega)} \le c_1 t^{-\alpha} \|\varphi\|_{L^2(\Omega)}, \end{cases} \tag{4.8}$$

and we have

$$v(x,t) = \sum_{k=1}^{\infty} \frac{(\varphi, e_k)}{\rho_k(T)} E_{\alpha,1}(-\lambda_k t^\alpha) e_k(x) \tag{4.9}$$

in $C([0,T]; L^2(\Omega)) \cap C((0,T]; H^2(\Omega) \cap H_0^1(\Omega))$.

(ii) We assume that $\varphi \in H_0^1(\Omega)$. Then the unique weak solution v further belongs to $L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega))$, $\partial_t^\alpha v \in L^2(Q_0^T)$ and there exists a constant $c_2 > 0$ satisfying the following inequality:

$$\|v\|_{L^2(0,T;H^2(\Omega))} + \|\partial_t^\alpha v\|_{L^2(Q_0^T)} \le c_2 \|\varphi\|_{H^1(\Omega)} \tag{4.10}$$

and we have (4.9) in the corresponding space on the right-hand side of (4.10).

(iii) Let $\varphi \in H^2(0,1) \cap H_0^1(0,1)$. *Then the unique weak solution* v *belong to* $C([0,T]; H^2(\Omega) \cap H_0^1(\Omega))$, $\partial_t^\alpha v \in C([0,T]; L^2(\Omega)) \cap C((0,T]; H_0^1(\Omega))$ *and the following inequality holds:*

$$\|v\|_{C([0,T];H^2(\Omega))} + \|\partial_t^\alpha v\|_{C([0,T];L^2(\Omega))} \le c_3 \|\varphi\|_{H^2(\Omega)} \tag{4.11}$$

and we have (4.9) *in the corresponding space on the right-hand side of* (4.11).

Proof **(i)** We will show that $v(x,t)$ certainly gives the weak solution to (4.6). It is easy to see that from the definition of β and Proposition 1, we have

$$\rho_k(T) = 1 + \beta E_{\alpha,1}(-\lambda_k T^\alpha) \ge 1, \quad \forall k \in \mathbf{N}.$$

Taking into account this relation and Proposition 1, we obtain

$$\|v(\cdot,t)\|^2_{L^2(\Omega)} = \sum_{k=1}^{\infty} \left| \frac{(\varphi, e_k)}{\rho_k(T)} E_{\alpha,1}(-\lambda_k t^\alpha) \right|^2 \le \|\varphi\|^2_{L^2(\Omega)} \tag{4.12}$$

and this is the first inequality of (4.8).

Moreover, by (2.5), we have

$$\|Av(\cdot,t)\|^2_{L^2(\Omega)} = \sum_{k=1}^{\infty} \left| \lambda_k \frac{(\varphi, e_k)}{\rho_k(T)} E_{\alpha,1}(-\lambda_k t^\alpha) \right|^2 \le C^2 t^{-2\alpha} \|\varphi\|^2_{L^2(\Omega)}, \tag{4.13}$$

for $t > 0$. In (4.12), since $\sum\limits_{k=1}^{\infty} \frac{(\varphi,e_k)}{\rho_k(T)} E_{\alpha,1}(-\lambda_k t^\alpha) e_k$ is convergent in $L^2(\Omega)$ uniformly in $t \in [0,T]$, we see that $u \in C([0,T]; L^2(\Omega))$. Moreover in (4.13), since $\sum\limits_{k=1}^{\infty} \lambda_k \frac{(\varphi,e_k)}{\rho_k(T)} E_{\alpha,1}(-\lambda_k t^\alpha) e_k$ is convergent in $L^2(\Omega)$ uniformly in $t \in [\varepsilon, T]$ with any given $\varepsilon > 0$, we see that $Av \in C((0,T]; L^2(\Omega))$, that is $v \in C((0,T]; H^2(\Omega) \cap H_0^1(\Omega))$. Therefore we obtain that $v \in C([0,T]; L^2(\Omega)) \cap C((0,T]; H^2(\Omega) \cap H_0^1(\Omega))$. By (4.6) we see that $\partial_t^\alpha v \in C((0,T]; L^2(\Omega))$.

We have to prove

$$\lim_{t \to 0} \|v(\cdot,t) + \beta v(\cdot, T-t) - \varphi\|_{L^2(\Omega)} = 0. \tag{4.14}$$

In fact,

$$\|v(\cdot,t) + \beta v(\cdot,T-t) - \varphi\|^2_{L_2(\Omega)}$$
$$= \sum_{k=1}^{\infty} \left| \frac{(\varphi, e_k)}{\rho_k(T)} \underbrace{\Big(E_{\alpha,1}(-\lambda_k t^\alpha) + \beta E_{\alpha,1}(-\lambda_k (T-t)^\alpha) - \rho_k(T) \Big)}_{S_{1k}(t)} \right|^2$$

and $\lim_{t\to 0} S_{1k}(t) = 0$ for each $k \in \mathbf{N}$ and

$$\sum_{k=1}^{\infty}(\varphi, e_k)^2 S_{1k}^2(t) \le 3\sum_{k=1}^{\infty}(\varphi, e_k)^2\Bigg[\left(\frac{C}{1+\lambda_k t^\alpha}\right)^2 + \left(\frac{C\beta}{1+\lambda_k (T-t)^\alpha}\right)^2 + \left(1+\frac{C\beta}{1+\lambda_k T^\alpha}\right)^2\Bigg] < \infty$$

for $0 \le t \le T$ and $\beta \ge 0$. Then, the Lebesgue theorem yields (4.14).

Next, we prove the uniqueness of the weak solution to (4.6) within the class given in Definition 1. Under the condition $\varphi = 0$, we have to prove that problem (4.6) has only a trivial solution. Since $e_k(x)$ is the eigenfunctions to the following eigenvalue problem:

$$(Ae_k)(x) = -\lambda_k e_k(x), \quad x \in \Omega, \quad e_k(x) = 0, \quad x \in \partial\Omega$$

in terms of the regularity of v, taking the duality pairing $\langle\cdot,\cdot\rangle_{-\gamma,\gamma}$ of (4.6) with e_k and setting $v_k(t) = \langle v(\cdot,t), e_k\rangle_{-\gamma,\gamma}$, we obtain

$$\partial_t^\alpha v_k(t) = -\lambda_k v_k(t), \quad \text{almost all} \quad t \in (0,T).$$

Since $v(\cdot,t) \in L^2(\Omega)$ for almost all $t \in (0,T)$ and $v_k(t) \equiv \langle v(\cdot,t), e_k\rangle_{-\gamma,\gamma} = (v(\cdot,t), e_k)$, where $\langle v(\cdot,t), e_k\rangle_{-\gamma,\gamma}$ denotes the duality pairing between $D((-A)^{-\gamma})$ and $D((-A)^{\gamma})$, it follows from $\lim_{t\to 0}\|v(\cdot,t)+\beta v(\cdot,T-t)\|_{D((-A)^{-\gamma})} = 0$ that $v_k(0)+\beta v_k(T) = 0$. Due to the existence and uniqueness of the boundary-value problem for the ordinary fractional differential equation (see, Lemma 3), we obtain that $v_k(t) = 0$, $k = 1, 2, ...$ Since $\{e_k\}_{k\in\mathbf{N}}$ is a complete orthonormal system in $L^2(\Omega)$, we have $v = 0$ in Q_0^T.

Moreover, by (2.5), we have

$$\|v(\cdot,t)\|_{H^2(\Omega)}^2 \le c_1'\|Av(\cdot,t)\|_{L^2(\Omega)}^2 \le c_1'\sum_{k=1}^{\infty}\lambda_k^2\left|\frac{(\varphi,e_k)}{\rho_k(T)}E_{\alpha,1}(-\lambda_k t^\alpha)\right|^2 \le c_1' C^2 t^{-2\alpha}\|\varphi\|_{L^2(\Omega)}^2. \tag{4.15}$$

From (4.12) and (4.15), we get the estimates (4.8).

(ii) Let $\varphi \in H_0^1(\Omega)$. By (2.5), we have

$$\|v(\cdot,t)\|^2_{H^2(\Omega)} \le c_1' \|Av(\cdot,t)\|^2_{L^2(\Omega)} \le c_1' \sum_{k=1}^{\infty} \lambda_k^2 \left| \frac{(\varphi, e_k)}{\rho_k(T)} E_{\alpha,1}(-\lambda_k t^\alpha) \right|^2$$
$$\le c_1' \sum_{k=1}^{\infty} \left| \lambda_k^{\frac{1}{2}} (\varphi, e_k)(\lambda_k t^\alpha)^{\frac{1}{2}} E_{\alpha,1}(-\lambda_k t^\alpha) \right|^2 t^{-\alpha}$$
$$\le c_1' C^2 \sum_{k=1}^{\infty} \left| \left((-A)^{\frac{1}{2}} \varphi, e_k\right) \frac{(\lambda_k t^\alpha)^{\frac{1}{2}}}{1+\lambda_k t^\alpha} \right|^2 t^{-\alpha} \le \frac{c_1' C^2}{4} \|\varphi\|^2_{H^1(\Omega)} t^{-\alpha}. \tag{4.16}$$

By $0 < \alpha < 1$, we see $\|v\|_{L^2(0,T;H^2(\Omega))} \le c_2' \|\varphi\|_{H^1(\Omega)}$. Therefore we have $u \in L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega))$.

Since $\partial_t^\alpha (E_{\alpha,1}(-\lambda_k t^\alpha)) = -\lambda_k E_{\alpha,1}(-\lambda_k t^\alpha)$ (e.g., [30], p. 98), we have

$$\int_0^T \|\partial_t^\alpha v(\cdot,t)\|^2_{L^2(\Omega)} dt = \int_0^T \sum_{k=1}^{\infty} \frac{|(\varphi, e_k)|^2}{\rho_k^2(T)} \lambda_k^2 |E_{\alpha,1}(-\lambda_k t^\alpha)|^2 dt$$
$$\le C^2 \int_0^T \sum_{k=1}^{\infty} (\varphi, e_k)^2 \lambda_k \left(\frac{(\lambda_k t^\alpha)^{\frac{1}{2}}}{1+\lambda_k t^\alpha} \right)^2 t^{-\alpha} dt \le c_2'' \|\varphi\|^2_{H^1(\Omega)},$$

where we have used the following

$$\max_{y \ge 0} \frac{y^\theta}{1+y} = \frac{\left(\frac{\theta}{1-\theta}\right)^\theta}{1+\frac{\theta}{1-\theta}}, \quad 0 < \theta < 1. \tag{4.17}$$

By (4.6) we have $\partial_t^\alpha v = Av$, which yields $\partial_t^\alpha v \in L^2(Q_0^T)$. Thus the proof of Lemma 5 (ii) is complete.

(iii) Let $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$. Then by Proposition 1, we have

$$\|v(\cdot,t)\|^2_{H^2(\Omega)} \le c_3' \|Av(\cdot,t)\|^2_{L^2(\Omega)}$$
$$\le c_3' \sum_{k=1}^{\infty} \lambda_k^2 (\varphi, e_k)^2 E_{\alpha,1}(-\lambda_k t^\alpha)^2 \le c_3' \|\varphi\|^2_{H^2(\Omega)}, \quad t \ge 0.$$

By (4.6) obtain

$$\|\partial_t^\alpha v(\cdot,t)\|^2_{L^2(\Omega)} \le c_3'' \|\varphi\|^2_{H^2(\Omega)}, \quad t > 0.$$

Combining the last two estimates, we get (3.5). This completes the proof of Lemma 5. □

Using the above Lemma 5, we obtain the following result.

Corollary 1 *Let $\varphi \in L^2(\Omega)$ and $F = 0$. Then for the unique weak solution $v \in C([0,\infty); L^2(\Omega)) \cap C((0,\infty); H^2(\Omega) \cap H_0^1(\Omega))$ to (4.6), there exists a positive constant c_4 satisfying*

$$\|v(\cdot,t)\|_{L^2(\Omega)} \le \frac{c_4}{1+\lambda_1 t^\alpha}\|\varphi\|_{L^2(\Omega)}, \quad t \ge 0. \tag{4.18}$$

Moreover, there exists a positive constant c_5 such that

$$\begin{cases} v \in C^\infty\left((0,\infty); L^2(\Omega)\right), \\ \|\partial_t^m v(\cdot,t)\|_{L^2(\Omega)} \le \frac{c_5}{t^m}\|\varphi\|_{L^2(\Omega)}, \quad t>0,\ m \in \mathbf{N}. \end{cases} \tag{4.19}$$

Proof By (2.5), we have

$$\begin{aligned} \|v(\cdot,t)\|^2_{L^2(\Omega)} &= \sum_{k=1}^{\infty} \frac{(\varphi,e_k)^2}{\rho_k^2(T)} E_{\alpha,1}(-\lambda_k t^\alpha)^2 \le \sum_{k=1}^{\infty} (\varphi,e_k)^2 \left(\frac{C}{1+\lambda_k t^\alpha}\right)^2 \\ &\le \left(\frac{C}{1+\lambda_1 t^\alpha}\right)^2 \|\varphi\|^2_{L^2(\Omega)}, \quad t \ge 0. \end{aligned}$$

Further, by Proposition 2, we have

$$\partial_t^m v(\cdot,t) = -\sum_{k=1}^{\infty} \frac{(\varphi,e_k)}{\rho_k(T)} \lambda_k t^{\alpha-m} E_{\alpha,\alpha-m+1}(-\lambda_k t^\alpha) e_k$$

for $m \in \mathbf{N}$ and $t > 0$, so that

$$\|\partial_t^m v(\cdot,t)\|^2_{L^2(\Omega)} \le C^2 \sum_{k=1}^{\infty} (\varphi,e_k)^2 \left(\frac{\lambda_k t^\alpha}{1+\lambda_k t^\alpha}\right)^2 t^{-2m} \le \frac{C^2}{t^{2m}}\|\varphi\|^2_{L^2(\Omega)}.$$

□

Now, we study the problem of (4.7) and we have:

Lemma 6 *Let $\varphi = 0$ and $F \in C([0,T]; D((-A)^\varepsilon))$, where $0 < \varepsilon < 1$. Then there exists a unique weak solution $w \in C([0,T]; H^2(\Omega) \cap H_0^1(\Omega))$ to (4.7) such that $\partial_t^\alpha w \in C([0,T]; L^2(\Omega))$. In particular, for any γ satisfying (K4), we have $w \in C([0,T]; D((-A)^{-\gamma}))$,*

$$\lim_{t\to 0} \|w(\cdot,t) + \beta w(\cdot,T-t)\|_{D((-A)^{-\gamma})} = 0 \tag{4.20}$$

and if $n = 1, 2, 3$, then

$$\lim_{t\to 0} \|w(\cdot,t) + \beta w(\cdot,T-t)\|_{L^2(\Omega)} = 0. \tag{4.21}$$

Moreover, there exists a constant $c_6 > 0$ *such that*

$$\|w\|_{C([0,T];H^2(\Omega))} + \|\partial_t^\alpha w\|_{C([0,T];L^2(\Omega))} \le c_6 \|F\|_{C([0,T];D((-A)^\varepsilon))} \tag{4.22}$$

for all $t \in [0,T]$ *and we have*

$$w(x,t) = \sum_{k=1}^{\infty} e_k(x) \int_0^T G_k(t,s)(F(\cdot,s), e_k)ds \tag{4.23}$$

in the corresponding space on the right-hand side of (4.22).

Proof Here we will show only regularity and estimate (4.22). We rewrite the standard form of (4.23) as follows:

$$\begin{aligned} w(x,t) &= \sum_{k=1}^{\infty} e_k(x) \int_0^t (F(\cdot,s), e_k)(t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (t-s)^\alpha) ds \\ &- \sum_{k=1}^{\infty} \frac{\beta E_{\alpha,1}(-\lambda_k t^\alpha)}{\rho_k(T)} e_k(x) \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (T-s)^\alpha)(F(\cdot,s), e_k) ds \\ &=: I_1 + I_2. \end{aligned}$$

By (2.5) and (2.9) for any $0 < \varepsilon < 1$, we have

$$\begin{aligned} \|I_1(\cdot,t)\|^2_{H^2(\Omega)} &\le \sum_{k=1}^{\infty} \lambda_k^2 \left| \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (t-s)^\alpha) \left(F(\cdot,s), e_k\right) ds \right|^2 \\ &\le \sum_{k=1}^{\infty} \max_{0\le s\le t} \left|\left((-A)^\varepsilon F(\cdot,s), e_k\right)\right|^2 \left| \int_0^t s^{\varepsilon\alpha-1} ds \right|^2 \\ &\le c_{10} \|F\|^2_{C([0,t];D((-A)^\varepsilon))} t^{2\varepsilon\alpha}, \quad t \in [0,T]. \end{aligned} \tag{4.24}$$

Similarly, we have

$$\begin{aligned} &\|I_2(\cdot,t)\|^2_{H^2(\Omega)} \\ &\le \sum_{k=1}^{\infty} \lambda_k^2 \left(\frac{\beta C}{1+\lambda_k t^\alpha} \right)^2 \left| \int_0^T \left((T-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (T-s)^\alpha) \left(F(\cdot,s), e_k\right) ds \right|^2 \\ &\le \left(\frac{\beta C}{1+\lambda_1 t^\alpha} \right)^2 \sum_{k=1}^{\infty} \max_{0\le s\le T} \left|\left((-A)^\varepsilon [F](\cdot,s), e_k\right)\right|^2 \left| \int_0^T s^{\alpha\varepsilon-1} ds \right|^2 \\ &\le c_{11} T^{2\varepsilon\alpha} \|F\|^2_{C([0,T];D((-A)^\varepsilon))}. \end{aligned} \tag{4.25}$$

On the other hand, using Proposition 2 and the formula (3.1.34) in [30], we have

$$
\begin{aligned}
&\partial_t^\alpha w(x,t) = \sum_{k=1}^{\infty} (F(\cdot,t), e_k)\, e_k(x) \\
&\quad - \sum_{k=1}^{\infty} \lambda_k \int_0^t (F(\cdot,s), e_k)(t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (t-s)^\alpha) ds e_k(x) + \\
&\quad + \beta \sum_{k=1}^{\infty} \frac{\lambda_k E_{\alpha,1}(-\lambda_k t^\alpha)}{\rho_k(T)} \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (T-s)^\alpha)(F(\cdot,s), e_k) ds e_k(x) \\
&=: I_3 + I_4 + I_5.
\end{aligned}
$$

Taking account this $D((-A)^\varepsilon) \subset L^2(\Omega)$, we have

$$
\begin{aligned}
\|I_3(\cdot,t)\|^2_{L^2(\Omega)} &= \sum_{k=1}^{\infty} (F(\cdot,t), e_k)^2 \\
&= \|F\|^2_{L^2(\Omega)} \le c_9 \|F\|^2_{D((-A)^\varepsilon)}, \quad t \in [0,T].
\end{aligned} \tag{4.26}
$$

Furthermore, by (2.9) we have

$$
\begin{aligned}
\|I_4(\cdot,t)\|^2_{L^2(0,1)} &\le \sum_{k=1}^{\infty} \left| \lambda_k \int_0^t (F(\cdot,s), e_k)(t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k (t-s)^\alpha) ds \right|^2 \\
&\le c'_{10} \sum_{k=1}^{\infty} \max_{0\le s\le t} |((-A)^\varepsilon F(\cdot,s), e_k)|^2 \left| \int_0^t s^{\varepsilon\alpha-1} ds \right|^2 \\
&\le c_{10} \|F\|^2_{C([0,t];D((-A)^\varepsilon))} t^{2\varepsilon\alpha}, \quad t \in [0,T].
\end{aligned} \tag{4.27}
$$

Argue similarly to the previous one, we get

$$
\|I_5(\cdot,t)\|^2_{L^2(0,1)} \le c_{11} T^{2\varepsilon\alpha} \|F\|^2_{C([0,T];D((-A)^\varepsilon))}. \tag{4.28}
$$

Summing up (4.24)–(4.28) yields (4.22).

Finally, we have to prove (4.20). In fact, by Propositions 1 and 2, we have

$$
\begin{aligned}
&\int_0^t \left| s^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k s^\alpha) \right| ds = \int_0^t s^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k s^\alpha) ds \\
&= -\frac{1}{\lambda_k} \int_0^t \frac{d}{ds} E_{\alpha,1}(-\lambda_k s^\alpha) ds = \frac{1}{\lambda_k} \left(1 - E_{\alpha,1}(-\lambda_k t^\alpha)\right), \quad t > 0.
\end{aligned} \tag{4.29}
$$

Using the (4.29), we have

$$
\begin{aligned}
&\|w(\cdot,t)+\beta w(\cdot,T-t)\|^2_{D((-A)^{-\gamma})}\\
&=\sum_{k=1}^{\infty}\lambda_k^{-2\gamma}\Big|\int_0^t (F(\cdot,s),e_k)(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(t-s)^{\alpha})ds\\
&\quad-\frac{\beta E_{\alpha,1}(-\lambda_k t^{\alpha})}{\rho_k(T)}\int_0^T (F(\cdot,s),e_k)(T-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(T-s)^{\alpha})ds\\
&\quad+\beta\int_0^{T-t}(F(\cdot,s),e_k)(T-t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(T-t-s)^{\alpha})ds\\
&\quad-\frac{\beta^2 E_{\alpha,1}(-\lambda_k(T-t)^{\alpha})}{\rho_k(T)}\int_0^T (F(\cdot,s),e_k)(T-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(T-s)^{\alpha})ds\Big|^2\\
&\le\sum_{k=1}^{\infty}\lambda_k^{-2(\gamma+\varepsilon+1)}\max_{0\le t\le T}|(-A)^{\varepsilon}F(\cdot,t),e_k|^2\Big[\underbrace{1-E_{\alpha,1}(-\lambda_k t^{\alpha})}\\
&\quad\underbrace{-\frac{\beta}{\rho_k(T)}E_{\alpha,1}(-\lambda_k t^{\alpha})(1-E_{\alpha,1}(-\lambda_k T^{\alpha}))}\\
&\quad\underbrace{+\beta(1-E_{\alpha,1}(-\lambda_k T^{\alpha}))-\frac{\beta^2}{\rho_k(T)}E_{\alpha,1}(-\lambda_k(T-t)^{\alpha})\left(1-E_{\alpha,1}(-\lambda_k T^{\alpha})\right)}_{S_{2k}(t)}\Big]^2\\
&\le c_{12}\|F\|^2_{C([0,T];D((-A)^{\varepsilon}))}\sum_{k=1}^{\infty}\lambda_k^{-2(\gamma+\varepsilon+1)}S_{2k}^2(t).
\end{aligned}
$$

Since $\lambda_k\ge c'_{12}k^{\frac{2}{n}},\ k\in\mathbf{N}$ (see [32], p. 407), we have

$$
\frac{1}{\lambda_k^{2(\gamma+\varepsilon+1)}}\le\frac{c''_{12}}{k^{\frac{4(\gamma+\varepsilon+1)}{n}}}.
$$

By (K4), we have $\frac{4(\gamma+\varepsilon+1)}{n}>1$, and $\sum\limits_{k=1}^{\infty}\lambda^{-2(\gamma+\varepsilon+1)}S_{2k}^2(t)<\infty$. Since $\lim\limits_{t\to 0}S_{2k}(t)=0$ for each $n\in\mathbf{N}$, the Lebesgue theorem implies $\lim\limits_{t\to 0}\|w(\cdot,t)+\beta w(\cdot,T-t)\|_{D((-A)^{-\gamma})}=0$. The uniqueness of the weak solution is verified similarly to Lemma 5. This completes the proof of Lemma 6. □

As a result combining Lemmas 5, 6, and Corollary 1, we obtain the following result:

Corollary 2 *Let $\varphi\in H^2(\Omega)\cap H_0^1(\Omega)$ and $F\in C([0,T];D((-A)^{\varepsilon}))$, where $0<\varepsilon<1$. Then there exists a unique weak solution $u\in X_0^T$ to (4.1), such that*

$$
\|u\|_{X_0^T}\le c_{13}\Big(\|\varphi\|_{H^2(\Omega)}+T^{\alpha\varepsilon}\|F\|_{C([0,T];D((-A)^{\varepsilon}))}\Big). \tag{4.30}
$$

At the end of this section, we give a lemma to show an equivalent form of our inverse problem.

Lemma 7 *Let (K1)-(K5) be held. Then our inverse problem* (2.1)–(2.4) *is equivalent to following problem:*

$$\begin{cases} \left(\partial_t^\alpha u\right)(x,t) - Au(x,t) = q(t)u(x,t) + f(x,t), & (x,t) \in Q_0^T, \\ u(x,0) + \beta u(x,T) = \varphi(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega,\ 0 < t < T, \end{cases} \tag{4.31}$$

and

$$h(t)q(t) = \partial_t^\alpha h(t) - (f(\cdot,t),\omega) - (Au(\cdot,t),\omega), \quad t \in [0,T]. \tag{4.32}$$

Proof Obviously, the solution $(u(x,t), q(t)) \in X_0^T \times C[0,T]$ of our inverse problem (2.1)–(2.4) is also a solution to the problem (4.31), (4.32) in $X_0^T \times C[0,T]$. Because the problem (4.31) is the same as (2.1)–(2.3). Therefore, we should show only (4.32). Multiplying both sides of Eq. (2.1) by a function $\omega(x)$ and integrating from Ω with respect to x gives

$$\begin{aligned} &\left(\partial_t^\alpha \int_\Omega \omega(x)u(x,t)dx\right)(t) - \int_\Omega \omega(x)Au(x,t)dx \\ &\quad = q(t)\int_\Omega \omega(x)u(x,t)dx + \int_\Omega \omega(x)f(x,t)dx, \end{aligned} \tag{4.33}$$

for all $0 < t < T$. Taking into account the condition (K3), and fractional differentiating (2.4) α^{th} order, we have

$$\partial_t^\alpha \left(\int_\Omega \omega(x)u(x,t)dx\right)(t) = \left(\partial_t^\alpha h\right)(t), \quad 0 < t < T. \tag{4.34}$$

From (4.33), taking into account (2.4) and (4.34) we arrive at (4.32).

Now, suppose that $(u(x,t), q(t)) \in X_0^T \times C[0,T]$ is a solution to the problem (4.31), (4.32). In order to prove that (u,q) is also a solution of (2.1)–(2.4), it suffices to prove that (u,q) satisfies (2.4). By the equation (4.31), we have:

$$\begin{aligned} &\int_\Omega \omega(x)\partial_t^\alpha u(x,t)dx - \int_\Omega \omega(x)Au(x,t)dx \\ &\quad = q(t)\int_\Omega \omega(x)u(x,t)dx + \int_\Omega \omega(x)f(x,t)dx, \end{aligned} \tag{4.35}$$

for $0 < t \leq T$.

Together with (4.32) and (K2), we obtain that $y(t) = \int_{\Omega} \omega(x)u(x,t)dx - h(t)$ satisfies

$$\begin{cases} \partial_t^{\alpha} y(t) = q(t)y(t), & 0 < t < T, \\ y(0) + \beta y(T) = 0. \end{cases} \tag{4.36}$$

Therefore, we have

$$y(t) = \int_0^T G_0(t,s)q(s)y(s)ds$$

for all $t \in [0, T]$.

Lemma 3 enables us to conclude that the problem (4.36) has only a trivial solution satisfying (3.6). Then, $\int_{\Omega} \omega(x)u(x,t)dx - h(t) = 0,\ 0 \le t \le T$, i.e., the condition (2.4) is satisfied. This completes the proof of this lemma. □

5 Existence of the solution to the inverse problem

We are now in a position to prove the existence of a solution to our inverse problem, i.e. Theorem 1, which proceeds by a fixed point argument. First, we define the function set

$$\begin{aligned} B_{r,T} = \Big\{ &(\bar{u}, \bar{q}) \in Y_0^T :\ \bar{u}(x,0) + \beta \bar{u}(x,T) = \varphi(x),\ x \in \Omega,, \\ &\bar{u}(x,t) = 0,\ x \in \partial\Omega,\ t \in (0,T),\ \|\bar{u}\|_{X_0^T} + \|\bar{q}\|_{C[0,T]} \le r \Big\}. \end{aligned}$$

Here r is a large constant depending on the initial data φ, measurement data h, and the number β. Throughout, we use M to denote a constant that depends on Ω, α, β, the initial data φ, the known functions f, ω and measurement data h, but independent of r and T.

For given $(\bar{u}, \bar{q}) \in B_{r,T}$, we consider

$$\begin{cases} \partial_t^{\alpha} u(x,t) - Au = F(x,t), & (x,t) \in Q_0^T, \\ u(x,0) + \beta u(x,T) = \varphi(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega,\ t \in (0,T), \end{cases} \tag{5.1}$$

and

$$h(t)q(t) = \left(\partial_t^{\alpha} h\right)(t) - \int_{\Omega} \omega(x) f(x,t)dx + \sum_{k=1}^{\infty} \lambda_k (\omega, e_k)(u(\cdot,t), e_k), \tag{5.2}$$

where

$$F(x,t) = \bar{q}(t)\bar{u}(x,t) + f(x,t).$$

According to Corollary 2, the unique solution $u \in X_0^T$ of the problem (5.1), given by (4.5) satisfies (4.30).

Furthermore

$$\|\bar{q}(t)\bar{u}(\cdot,t)\|^2_{C([0,T];D((-A)^{\varepsilon}))} = \max_{0\le t\le T}\left|\sum_{k=1}^{\infty}\lambda_k^{2\varepsilon}(\bar{q}(t)\bar{u}(\cdot,t),e_k)^2\right|$$

$$\le \|\bar{q}\|^2_{C[0,T]}\|\bar{u}\|^2_{C([0,T];D((-A)^{\varepsilon}))}. \tag{5.3}$$

Using this result together with $f \in C([0,T];D((-A)^{\varepsilon}))$, we have

$$\bar{q}(t)\bar{u}(x,t) + f(x,t) \in C([0,T];D((-A)^{\varepsilon})). \tag{5.4}$$

Therefore, Corollary 2 ensures that there exists a unique solution $u \in X_0^T$ to (5.1). Then (5.2) defines the function in terms of u. Furthermore, by (5.2) we have

$$\begin{aligned}
&\|q\|_{C[0,T]} \le h_0\Big[\|\partial_t^{\alpha}h\|_{C[0,T]} + \max_{0\le t\le T}\Big|\int_{\Omega}\omega(x)f(x,t)dx\Big| \\
&+\max_{0\le t\le T}\Big|\sum_{k=1}^{\infty}\lambda_k(\omega,e_k)(u(\cdot,t),e_k)\Big|\Big] \le h_0\Big[\|\partial_t^{\alpha}h\|_{C[0,T]} \\
&+\Big|\int_{\Omega}\omega(x)\|f(x,\cdot)\|_{C[0,T]}dx\Big| + \Big(\sum_{k=1}^{\infty}|(\omega,e_k)|^2\Big)^{1/2}\max_{0\le t\le T}\Big(\sum_{k=1}^{\infty}\lambda_k^2|(u,e_k)|^2\Big)^{1/2}\Big] \\
&\le h_0\Big[\|\partial_t^{\alpha}h\|_{C[0,T]} + \Big(\int_{\Omega}|\omega(x)|^2dx\Big)^{1/2}\Big(\int_{\Omega}\|f(x,\cdot)\|^2_{C[0,T]}dx\Big)^{1/2} \\
&+\|\omega\|_{L^2(\Omega)}\|u\|_{X_0^T}\Big] \\
&\le h_0\Big[\|\partial_t^{\alpha}h\|_{C[0,T]} + \|\omega\|_{L^2(\Omega)}\|f\|_{C([0,T];L^2(\Omega))} + \|\omega\|_{L^2(\Omega)}\|u\|_{X_0^T}\Big] \\
&\le h_0\left[\|\partial_t^{\alpha}h\|_{C[0,T]} + \|\omega\|_{L^2(\Omega)}\|f\|_{C([0,T];D((-A)^{\varepsilon}))} + \|\omega\|_{L^2(\Omega)}\|u\|_{X_0^T}\right].
\end{aligned} \tag{5.5}$$

Based on (4.30), (5.3) and (5.4) this implies that $q \in C[0,T]$. Thus, the mapping

$$S : B_{r,T} \to Y_0^T, \qquad (\bar{u},\bar{q}) \mapsto (u,q) \tag{5.6}$$

given by (5.1) and (5.2), is well defined.

The next lemma shows that S is a contraction map on $B_{r,T}$.

Lemma 8 *Let (K1)–(K5) be held. Then there exists a sufficiently small τ and a suitable large r such that S is a contraction map on $B_{r,T}$ for all $T \in (0,\tau]$, where τ and r are two positive constants depending on α, β and the known functions φ, f and the measurement data h.*

Proof First, we prove $S(B_{r,T}) \subset B_{r,T}$ for sufficiently small T and suitable large r. To simplify the calculations, we restrict $T \in (0, 1]$. By (4.30), we obtain

$$\|u\|_{X_0^T} \leq M + T^{\alpha\varepsilon} M + T^{\alpha\varepsilon} r^2. \tag{5.7}$$

On the other hand, by (K3), (5.3), and (5.6), we have

$$\|q\|_{C[0,T]} \leq h_0 \Big\{ \|\partial_t^\alpha h\|_{C[0,T]} + \|\omega\|_{L^2(\Omega)} \|f\|_{C([0,T];D((-A)^\varepsilon))} + \|\omega\|_{L^2(\Omega)} \|u\|_{X_0^T} \Big\} \leq M + M^2 + M\|u\|_{X_0^T}. \tag{5.8}$$

Hence, by (5.7) and (5.8), we have

$$\|(u, q)\|_{Y_0^T} \leq M\zeta_1(T) \left(r^2 + 1\right) + M, \tag{5.9}$$

where

$$\zeta_1(T) = T^{\alpha\varepsilon}$$

and therefore satisfies $\lim\limits_{T\to+0} \zeta_1(T) = 0$. Now we take r, such that $r = 2M$ with the constant M in (5.9). Then there exists a sufficiently small $\tau_1 > 0$, such that

$$\|(\bar{u}, \bar{q})\|_{Y_0^T} \leq r, \tag{5.10}$$

for all $T \in (0, \tau_1]$, that is, S maps $B_{r,T}$ into itself for each fixed $T \in (0, \min\{1, \tau_1\}]$.

Next, we estimate the increment of operator S. To this end, we deduce the differences $(u - U, q - Q)$ from (5.1), (5.2) to yield

$$\begin{cases} \partial_t^\alpha (u - U)(x, t) - A(u - U)(x, t) = \bar{q}(\bar{u} - \bar{U}) + \bar{U}(\bar{q} - \bar{Q}), & (x, t) \in Q_0^T, \\ (u - U)(x, 0) + \beta(u - U)(x, T) = 0, & x \in \Omega, \\ (u - U)(x, t)0, & x \in \partial\Omega, \ t \in (0, T), \end{cases} \tag{5.11}$$

and

$$h(t)\,(q - Q)\,(t) = \sum_{k=1}^{\infty} \lambda_k(\omega, e_k)\big(u(\cdot, t) - U(\cdot, t), e_k\big). \tag{5.12}$$

Using Corollary 2, we get

$$\|u - U\|_{X_0^T} \leq c_{13} r T^{\alpha\varepsilon} (\|\bar{u} - \bar{U}\|_{X_0^T} + \|\bar{q} - \bar{Q}\|_{C[0,T]}), \tag{5.13}$$

Further, by (K3), (5.12), and (5.13), we have

$$\|q-Q\|_{C[0,T]} \le h_0 \left(\sum_{k=1}^{\infty}(\omega, e_k)^2\right)^{1/2} \max_{0\le t\le T}\left(\sum_{k=1}^{\infty}\lambda_k^2\big(u(\cdot,t)-U(\cdot,t),e_k\big)^2\right)^{1/2}$$
$$\le h_0\|\omega\|_{L^2(\Omega)}\|u-U\|_{C([0,T];H^2(\Omega))} \le h_0\|\omega\|_{L^2(\Omega)}\|u-U\|_{X_0^T}$$

$$\le c_{13}T^{\alpha\varepsilon} r h_0\|\omega\|_{L^2(\Omega)}(\|\bar{u}-\bar{U}\|_{X_0^T}+\|\bar{q}-\bar{Q}\|_{C[0,T]}). \tag{5.14}$$

Therefore, by (5.13) and (5.14), we obtain

$$\|(u-U, q-Q)\|_{Y_0^T} \le MrT^{\alpha\varepsilon}\|(\bar{u}-\bar{U}, \bar{q}-\bar{Q})\|_{Y_0^T}. \tag{5.15}$$

Hence we can choose sufficiently small $\tau_2 > 0$ such that

$$MrT^{\alpha\varepsilon} \le \frac{1}{2} \tag{5.16}$$

for all $T \in (0, \tau_2]$ to obtain

$$\|S(\bar{u},\bar{q})-S(\bar{U},\bar{Q})\|_{Y_0^T} \le \frac{1}{2}\|(\bar{u}-\bar{U}, \bar{q}-\bar{Q})\|_{Y_0^T}. \tag{5.17}$$

Estimates (5.10) and (5.17) show that S is a contraction map on $B_{r,T}$ for all $T \in (0, \tau]$, if we choose $\tau \le \min\{1, \tau_1, \tau_2\}$. □

6 Proof of the main results

In this section, we give proof of the global in time uniqueness of solutions to our inverse problem, i.e., Theorem 1.

Proof of Theorem 1 For S is a contraction map on $B_{r,T}$ for all $T \in (0, \tau]$, the Banach fixed point theorem concludes that there exists a unique solution $(u, q) \in X_0^T \times C[0, T]$ of the inverse problem (4.31) and (4.32).

Next we show that we could extend the solution (u, q) in $(0, \tau]$ to a larger interval $[\tau, 2\tau]$. To do this, we consider

$$\begin{cases} \partial_t^{\alpha} v(x,t) - Av(x,t) = p(t)v(x,t) + f(x,t), & (x,t) \in Q_\tau^T, \\ v(x,\tau) + \beta v(x,T) = u(x,\tau) + \beta u(x,T), & x \in \Omega \\ v(x,t) = 0, \quad x \in \partial\Omega, \ t \in (\tau, T), \end{cases} \tag{6.1}$$

and

$$h(t)p(t) = \left(\partial_t^\alpha h\right)(t) - \int_\Omega \omega(x) f(x,t)dx$$
$$+ \sum_{k=1}^{\infty} \lambda_k(\omega, e_k)(v(\cdot,t), e_k), \quad t \in [\tau, T]. \tag{6.2}$$

Obviously, if we prove that there exists a solution $(v, p) \in Y_\tau^T$ with some $T \geq 2\tau$, then $(\tilde{u}, \tilde{q})$ defined by

$$(\tilde{u}, \tilde{q}) = \begin{cases} (u, q), & t \in [0, \tau], \\ (v, p), & t \in [\tau, 2\tau] \end{cases} \tag{6.3}$$

is a solution of the inverse problem (4.31) and (4.32) on the larger interval $[0, 2\tau]$.

We repeat a similar fixed pointed argument to prove the existence of (v, p). Define an operator

$$K : \tilde{B}_{\tilde{r},T} \to Y_\tau^T, \qquad (\bar{v}, \bar{p}) \to (v, p) \tag{6.4}$$

with $(\bar{v}, \bar{p}) \in \tilde{B}_{\tilde{r},T}$, where

$$\tilde{B}_{\tilde{r},T} = \big\{(\bar{v}, \bar{p}) \in Y_\tau^T : \ \bar{v}(x,\tau) + \beta\bar{v}(x,T) = u(x,\tau) + \beta u(x,T), \ x \in \Omega$$
$$\bar{v}(x,t) = 0, \ x \in \partial\Omega, \ t \in (\tau, T), \ \|\bar{v}\|_{X_\tau^T} + \|\bar{p}\|_{C[\tau,T]} \leq \tilde{r}\big\}.$$

Here v is the solution to the initial and boundary value problem

$$\begin{cases} \partial_t^\alpha v(x,t) - Av(x,t) = \bar{p}(t)\bar{v}(x,t) + f(x,t), & (x,t) \in Q_\tau^T, \\ v(x,\tau) + \beta v(x,T) = u(x,\tau) + \beta u(x,T), & x \in \Omega \\ v(x,t) = 0, \quad x \in \partial\Omega, \ t \in (\tau, T). \end{cases} \tag{6.5}$$

Furthermore, p is the solution of (6.2) in terms of v. Additionally, we have $F(x,t) = \bar{p}(t)\bar{v}(x,t) + f(x,t) \in C([0,T]; D((-A)^\varepsilon))$, $u(\cdot,\tau), u(\cdot,T) \in H^2(\Omega)$. In fact, the first property comes from (5.4), according to Corollary 2 the functions $u(\cdot,\tau), u(\cdot,T)$ the same as (5.4) at $t \in \{\tau, T\}$, therefore we can conclude that $u(\cdot,\tau) \in H^2(\Omega)$.

Besides, by (4.30) we have

$$\|v\|_{X_\tau^T} \leq c_{13}\Big[\|u(\cdot,\tau)\|_{H^2(\Omega)} + \beta\|u(\cdot,T)\|_{H^2(\Omega)} + (T-\tau)^{\alpha\varepsilon}(\tilde{r}^2 + M)\Big] \tag{6.6}$$

and from (6.2) via (K1), (K3), (K5) we get

$$\|p\|_{C[\tau,T]} \le h_0 \left\{ \|\partial_t^\alpha h\|_{C[\tau,T]} + \|\omega\|_{L^2(\Omega)} \Big[\|f\|_{C([\tau,T];D((-A)^\varepsilon))} + c_{13}(\|u(\cdot,\tau)\|_{H^2(\Omega)} + \beta\|u(\cdot,T)\|_{H^2(\Omega)} + (T-\tau)^{\alpha\varepsilon}(\tilde{r}^2 + M)) \Big] \right\}. \tag{6.7}$$

We set $T - \tau \le 1$. Then using (6.8) and (6.9), we have

$$\|K(\tilde{v},\tilde{p})\|_{Y_\tau^T} \le h_0\big(\|\partial_t^\alpha h\|_{C[\tau,T]} + \|\omega\|_{L^2(\Omega)}\|f\|_{C([\tau,T];D((-A)^\varepsilon))}\big) + +c_{13}(h_0\|\omega\|_{L^2(\Omega)} + 1)\Big[\|u(\cdot,\tau)\|_{H^2(\Omega)} +\beta\|u(\cdot,T)\|_{H^2(\Omega)} + (M + \tilde{r}^2)(T-\tau)^{\alpha\varepsilon}\Big] \tag{6.8}$$

and by a similar calculation to (5.15), we have

$$\|K(\bar{v}_1,\bar{p}_1) - K(\bar{v}_2,\bar{p}_2)\|_{Y_\tau^T} \le M(T-\tau)^{\alpha\varepsilon}\|(\bar{v}_1-\bar{v}_2,\bar{p}_1-\bar{p}_2)\|_{Y_\tau^T}, \tag{6.9}$$

where M is the same as the ones in Lemma 8. We choose $\tilde{r}$ such that $\tilde{r} \ge r$ and

$$h_0\big(\|\partial_{0t}^\alpha h\|_{C[\tau,T]} + \|\omega\|_{L^2(\Omega)}\|f\|_{C([\tau,T];D((-A)^\varepsilon))}\big) + +c_{13}(h_0\|\omega\|_{L^2(\Omega)} + 1)\big(\|u(\cdot,\tau)\|_{H^2(\Omega)} + \|u(\cdot,T)\|_{H^2(\Omega)}\big) \le \frac{\tilde{r}}{2}.$$

It is easy to see that if we choose $\tilde{r}$ larger, then we could get larger $T-\tau$ to satisfy

$$M(T-\tau)^{\alpha\varepsilon}\Big(\|u(\cdot,\tau)\|_{H^2(\Omega)} + \|u(\cdot,T)\|_{H^2(\Omega)}\Big) \le \frac{\tilde{r}}{2}. \tag{6.10}$$

Furthermore noticing that (6.12) and (5.16) we have the same structure, we can choose $T - \tau = \tau$ to satisfy (6.12), which yields $\|K(\tilde{u},\tilde{q})\|_{Y_\tau^T} \le \tilde{r}$, i.e. $K(\tilde{B}_{\tilde{r},T}) \subset \tilde{B}_{\tilde{r},T}$. Additionally,

$$\|K(\tilde{u}_1,\tilde{q}_1) - K(\tilde{u}_2,\tilde{q}_2)\|_{Y_\tau^T} \le \frac{1}{2}\|(\tilde{u}_1-\tilde{u}_2,\tilde{q}_1-\tilde{q}_2)\|_{Y_\tau^T} \tag{6.11}$$

for $T = 2\tau$, because (6.11) is the same as (5.17), if we replace T in (5.17) by $T - \tau$. Hence we prove that K is a contraction operator on $\tilde{B}_{\tilde{r},T}$ for $T = 2\tau$.

Repeating the extension process limited times, we could obtain a solution $(u,q) \in X_0^T \times C[0,T]$ of the inverse problem (4.31) and (4.32) for any T. Lemma 7 shows that the inverse problem (4.31) and (4.32) is equivalent to the inverse problem (2.1)–(2.4). Consequently, the inverse problem (2.1)–(2.2) also admits a unique solution (u,q) in the space $X_0^T \times C[0,T]$ for any T. □

Proof of Theorem 2 By Lemma 7, we know that our inverse problem is equivalent to the inverse problem (4.31) and (4.32). Hence in the following proof, we turn to prove the uniqueness of the solution to (4.31) and (4.32).

We set $(\hat{u}, \hat{q}) = (u_1 - u_2, q_1 - q_2)$ and

$$\sigma = \inf \left\{ t \in (0, T] : \|(\hat{u}, \hat{q})\|_{Y_0^t} > 0 \right\}. \tag{6.12}$$

It suffices to prove that $\sigma = T$.

If (6.12) is not true, then we have $\sigma < T$. Choose l such that $0 < l < T - \sigma$. Next, we consider

$$\begin{cases} \partial_t^\alpha \hat{u}(x,t) - A\hat{u}(x,t) = \hat{F}(x,t), \quad (x,t) \in Q_\sigma^{\sigma+l}, \\ \hat{u}(x,\sigma) + \beta\hat{u}(x,\sigma+l) = 0, \quad x \in \Omega, \\ \hat{u}(x,t) = 0, \quad x \in \partial\Omega, \ t \in (\sigma, \sigma+l), \end{cases} \tag{6.13}$$

and

$$h(t)\hat{q}(t) = \sum_{k=1}^{\infty} \lambda_k(\omega, e_k)(\hat{u}(\cdot,t), e_k), \quad t \in [\sigma, \sigma+l], \tag{6.14}$$

where

$$\hat{F}(x,t) = q_1(t)\hat{u}(x,t) + \hat{q}(t)u_2(x,t).$$

By (4.23), we can write the solution $\hat{u}$ as

$$\hat{u}(x,t) = \sum_{k=1}^{\infty} \int_0^T G_k(t,s)(\hat{F}(\cdot,s), e_k)ds, \quad t \in [\sigma, \sigma+l]. \tag{6.15}$$

Then similar to the proof of Corollary 2, we have

$$\|\hat{u}\|_{X_\sigma^{\sigma+\varepsilon}} \le c_{13} l^{\alpha\varepsilon} \|\hat{F}\|_{C([\sigma,\sigma+l]; D((-A)^\varepsilon)} \tag{6.16}$$

and

$$\max_{\sigma \le t \le \sigma+l} \left| \sum_{k=1}^{\infty} \lambda_k(\omega, e_k)(\hat{u}(\cdot,t), e_k) \right| \le \|\omega\|_{L^2(\Omega)} \|\hat{u}\|_{C([\sigma,\sigma+l]; H^2(\Omega))}$$

$$\le \|\omega\|_{H^2(\Omega)} \|\hat{u}\|_{X_\sigma^{\sigma+l}} \le M l^{\alpha\varepsilon} \|\hat{F}\|_{C([\sigma,\sigma+l]; D((-A)^\varepsilon))} \tag{6.17}$$

with the same M in Lemma 6 and estimates (5.3). By (5.3), and noting that $(u_1, q_1) = (u_2, q_2)$ on $[0, \sigma]$, we have

$$\|\hat{F}\|_{C([\sigma,\sigma+l]; D((-A)^\varepsilon))} \le M(\|\hat{u}\|_{X_\sigma^{\sigma+l}} + \|\hat{q}\|_{C[\sigma,\sigma+l]}), \tag{6.18}$$

which yields, together with (6.16), that

$$\|\hat{u}\|_{X_\sigma^{\sigma+l}} \leq M l^{\alpha\varepsilon}(\|\hat{u}\|_{X_\sigma^{\sigma+l}} + \|\hat{q}\|_{C[\sigma,\sigma+l]}). \tag{6.19}$$

On the other hand, by (6.14), (6.17), and (6.18) we have the following estimate for $\hat{q}$

$$\|\hat{q}\|_{C[\sigma,\sigma+l]} \leq M h_0 l^{\alpha\varepsilon}(\|\hat{u}\|_{X_\sigma^{\sigma+l}} + \|\hat{q}\|_{C[\sigma,\sigma+l]}). \tag{6.20}$$

Therefore, by (6.19) and (6.20) we have

$$\|(\hat{u},\hat{q})\|_{Y_\sigma^{\sigma+l}} \leq M l^{\alpha\varepsilon}\|(\hat{u},\hat{q})\|_{Y_\sigma^{\sigma+l}}$$

implying

$$\|(\hat{u},\hat{q})\|_{Y_\sigma^{\sigma+l_0}} = 0$$

for some sufficiently small positive constant l_0. This means that $(u_1 - q_1, u_2 - q_2)$ vanishes in some neighborhood of σ. But this is not compatible with the definition of σ. We conclude that $\sigma = T$.

At the end of this section, consider the following example.

Example 1 Let $\beta = 1$. Consider the case $n = 1$ and $Q_0^T = (0,1) \times (0, T = \pi)$, and $A := -\partial_x^2$. We consider the following inverse problem

$$\begin{cases} \partial_t^\alpha u - \partial_x^2 u = q(t)u + 2\sin(\pi x)\Big[t^{1-\alpha}E_{2,2-\alpha}(-t^2) \\ \qquad\qquad +(2+\sin t)\left(\pi^2 - (1+t)e^{-t}\right)\Big], \\ u(x,0) + u(x,\pi) = 8\sin(\pi x), \quad u(0,t) = u(1,t) = 0, \quad t \in (0,\pi), \\ \int\limits_0^1 \sin(\pi x)u(x,t)dx = 2 + \sin t, \qquad t \in [0,\pi]. \end{cases} \tag{6.21}$$

Note, that the all given data satisfy conditions (K1)-(K5). Furthermore, according to Theorem 1, using the given data, there exists a unique solution to the above inverse problem, and from Lemma 7, it has the form

$$u(x,t) = 2(2+\sin t)\sin(\pi x), \quad q(t) = (1+t)e^{-t}. \tag{6.22}$$

Now, let $T = 2\pi$ and using (6.21) and (6.22), we consider the following problem

$$\begin{cases} \partial_t^\alpha v - \partial_x^2 v = p(t)v + 2\sin(\pi x)\Big[t^{1-\alpha}E_{2,2-\alpha}(-t^2) \\ \qquad\qquad +(2+\sin t)\left(\pi^2 - (1+t)e^{-t}\right)\Big], \\ v(x,\pi) + v(x,2\pi) = 8\sin(\pi x), \quad v(0,t) = v(1,t) = 0, \quad t \in (\pi,2\pi), \\ \int\limits_0^1 \sin(\pi x)v(x,t)dx = 2 + \sin t, \qquad t \in [\pi,2\pi]. \end{cases} \tag{6.23}$$

We can easily see that, as above, $v(x,t) = 2(2+\sin t)\sin(\pi x),\ p(t) = (1+t)e^{-t}$ is a solution of the problem (6.23). Therefore, we can continue the same process for any $T = \pi n$, where $n \in \mathbf{N}$.

7 Conclusions

This paper investigated the multi-dimensional fractional-diffusion equation with Robin-type initial and Dirichlet boundary conditions. We derived Green's function and corresponding integral operator and then examined the fixed point theorem for the operator. Theorems of global existence and uniqueness of the solution to the inverse problem are proved.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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