RESEARCH ARTICLE | JUNE 08 2023

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AIP Conference Proceedings 2781, 020063 (2023)
https://doi.org/10.1063/5.0144842

Algorithm for Finding the Norm of the Error Functional of Hermite-Type Interpolation Formulas in the Sobolev Space of Periodic Functions

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Abstract. S.L. Sobolev [1] firstly posed the problem of finding an extremal function for an interpolation formula and calculating the norm of the error functional in the space W_2^m. In this paper, an extremal function of the interpolation formula was found in explicit form in the Sobolev space W_2^m, functions for which the generalized derivatives of order m are square integrable. In the present paper, we consider the problem of finding the norm of the error functional for interposional formulas of Hermite type in the space of S.L. Sobolev $\widetilde{W}_2^{(m)}(T_1)$.

INTRODUCTION

The problem of constructing interpolation formulas is one of the classical problems of computational mathematics and numerical analysis.

Theories of interpolation formulas have been constructed by many authors, for example [1, 2, 3, 4, 5]. Suppose that at $N+1$ arbitrarily located points $\{x_i\}$ $\left(i=\overline{0,N}\right)$, which we will call interpolation nodes throughout below, we have the value $f^{(\alpha)}(x_0), f^{(\alpha)}(x_1), ..., f^{(\alpha)}(x_N), \left(\alpha=\overline{0,m-1}\right)$ to the function $f(x)$.

If we know not only the values of the function $f(x)$ at points $x_0, x_1, ..., x_m$ of the segment $[0,1]$, but also the values of its derivatives of one order or another, then it is natural that with the correct use of all these data we can expect a more accurate result than in the case of using only the values of the function.

It is required to construct an interpolation formula of the Hermite type $P_f(x)$, i.e.,

$$f(x) \cong P_f(x) = \sum_{\alpha=0}^{m-1} \sum_{\lambda=0}^{N} (-1)^{\alpha} C_{\lambda}^{(\alpha)}(x) f^{(\alpha)}(x_{\lambda}), \tag{1}$$

where matching function $f(x)$ at the interpolation nodes is

$$f(x_i) = P_f(x_i),\ i = 0, 1, ...N.$$

An important problem in the theory of interpolation is to find the maximum error of the interpolation formula $f(x) \cong P_f(x)$ over a given class of functions. The value of this function at some point z is the functional defined as

$$< \ell_N^{(\alpha)}(x), f(x) >= \int_{-\infty}^{\infty} \ell_N^{(\alpha)}(x) f(x) dx = f(z) - P_f(z) = f(z) - \sum_{\alpha=0}^{m-1} \sum_{\lambda=0}^{N} (-1)^{\alpha} C_{\lambda}^{(\alpha)}(z) f^{(\alpha)}(x_{\lambda}), \tag{2}$$

where it is clear that $P_f(z) = \sum\limits_{\alpha=0}^{m-1} \sum\limits_{\lambda=0}^{N} (-1)^{\alpha} C_{\lambda}^{(\alpha)}(z) f^{(\alpha)}(x_{\lambda})$ is an interpolation formula of the Hermite type and

$$\ell_N^{(\alpha)}(x) = \delta(x-z) - \sum_{\alpha=1}^{m-1} \sum_{\lambda=0}^{N} C_{\lambda}^{(\alpha)}(z) \delta^{(\alpha)}(x - x_{\lambda}) \tag{3}$$

is the error functional of this interpolation formula, $C_{\lambda}^{(\alpha)}(z)$ are coefficients, and x_{λ} are nodes of the formula $P_f(z), x_{\lambda} \in [0,1]$, $\delta(x)$ is the Dirac's delta function and $f(x) \in \widetilde{W}_2^{(m)}(T_1)$.

Definition 1 *The space $\widetilde{W}_2^{(m)}(T_1)$ is defined as the space of functions given by the one-dimensional torus T_1, a circle of length equal to one, having all generalized derivatives of order m summable with a square [6].*

Modern Problems of Applied Mathematics and Information Technology (MPAMIT 2021)
AIP Conf. Proc. 2781, 020063-1–020063-6; https://doi.org/10.1063/5.0144842
Published by AIP Publishing. 978-0-7354-4535-2/$30.00

Space $\widetilde{W}_2^{(m)}(T_1)$ becomes Hilbert space with the inner product

$$< f(x), \phi(x)|\widetilde{W}_2^{(m)}\ (T_1) >= \int_{T_1} f^{(m)}(x)\,\phi^{(m)}(x)\,dx + \left(\int_{T_1} f(x)\,dx\right)\left(\int_{T_1} \phi(x)\,dx\right).$$

The corresponding norm is determined by the formula

$$\left\| f(x)|\widetilde{W}_2^{(m)}(T_1)\ \right\|^2 = \left(\int_{T_1} f(x)\,dx\right)^2 + \sum_{k\neq 0} |2\pi k|^{2m} \left|\hat{f}_k\right|^2 \tag{4}$$

STATEMENT OF THE PROBLEM

It is known that [6]

$$\left\|\ell_N^{(\alpha)}(x)|\widetilde{W}_2^{(m)*}(T_1)\right\| = \sup_{\|\varphi\|\neq 0} \frac{\left|< \ell_N^{(\alpha)}, \varphi >\right|}{\left\|\varphi|\widetilde{W}_2^{(m)}(T_1)\ \right\|}. \tag{5}$$

The error (2) of the interpolation formula $P_f(z)$is estimated using the maximum error of this formula on the unit ball of the Hilbert space $\widetilde{W}_2^{(m)}(T_1)$ i.e., using the norm of functional (3)

$$\left\|\ell_N^{(\alpha)}(x)|\widetilde{W}_2^{(m)*}(T_1)\right\| = \sup_{\left\|f(x)|\widetilde{W}_2^{(m)}(T_1)\right\|=1} \left|< \ell_N^{(\alpha)}, f >\right|, \tag{6}$$

where $\widetilde{W}_2^{(m)*}(T_1)$ is the conjugate space of space $\widetilde{W}_2^{(m)}(T_1)$ space.

Hence, in order to estimate the error (2) of the interpolation formula $P_f(z)$, it is sufficient to solve the following problem.

Problem 1. Calculate the norm of the error functional $\ell^{(\alpha)}(x)$ of the considered interpolation formula $\ell^{(\alpha)}(x)$.

It is clear that the norm of the error functional $\ell(x)$ depends on the coefficients and nodes $C_\lambda^{(\alpha)}(z)$.

$$\left\|\overset{o}{\ell}{}_N^{(\alpha)}(x)|\widetilde{W}_2^{(m)*}(T_1)\right\| = \inf_{C_\lambda^{(\alpha)}(x), x_\lambda} \left\|\ell_N^{(\alpha)}(x)|\widetilde{W}_2^{(m)*}(T_1)\right\|, \tag{7}$$

then functional $\overset{o}{\ell}{}_N^{(\alpha)}(x)$ is called the optimal error functional, and the corresponding interpolation formula is called the optimal interpolation formula.

Thus, the following problem arises.

Problem 2. Find the values of the coefficients $C_\lambda^{(\alpha)}(z)$ and nodes x_λ of the interpolation formula $P_f(z)$ that satisfy equality (5).

Coefficients $C_\lambda^{(\alpha)}(z)$ and nodes x_λ satisfying equality (7) are called optimal coefficients and optimal nodes of the interpolation formulas $P_f(z)$.

In the space $L_2^{(m)}(R)$, problems 1 and 2 were studied in [7]. In [8], the problem of constructing optimal interpolation formulas of the form (2) with the interpolation condition (1) (for fixed nodes x_λ) in space $W_2^{(m,m-1)}(0,1)$ was considered, and a system of linear equations was obtained for the optimal coefficients. An algorithm for calculating the coefficients of optimal interpolation formulas in space $W_2^{(m,m-1)}(0,1)$ is given in [9, 10, 11, 12, 13].

NORM AND EXTREMAL FUNCTION OF THE ERROR FUNCTIONAL OF AN INTERPOLATION FORMULA OF HERMITE TYPE IN THE SPACE $\widetilde{W}_2^{(m)}(T_1)$

It is known that the problem of estimating the error of an interpolation formula on functions of some space B is equivalent to calculating the value of the norm of the error functional in the space B conjugate to B^* or, which is the same, finding an extremal function for a given interpolation formula. To solve this problem, as B we take the space $\widetilde{W}_2^{(m)}(T_1)$.

In this paper, we will deal with the solution of Problem 1 for an interpolation formula of the Hermite type of the form (1), i.e. by calculating the norm $\left\|\ell_N^{(\alpha)}(x)|W_2^{(m)*}(T_1)\right\|$ of the error functional $\ell_N^{(\alpha)}(x)$ of the interpolation formula (1).

Since the space $W_2^{(m)}(T_1)$ is a Hilbert space, then, based on the Riesz theorem on the general form of a linear continuous functional [6], there is a unique function $\psi_\ell(x) \in W_2^{(m)}(T_1)$ for which

$$< \ell_N^{(\alpha)}(x), f(x) >=< \psi_\ell(x), f(x) >_{W_2^{(m)}(T_1)} \tag{8}$$

and

$$\left\|\ell_N^{(\alpha)}(x)|W_2^{(m)*}(T_1)\right\| = \left\|\psi_\ell(x)|W_2^{(m)}(x)\right\|.$$

In particular, from (8) for $f(x) = \psi_\ell(x)$ we have

$$\begin{aligned} < \ell_N^{(\alpha)}(x), \psi_\ell(x) >&=< \psi_\ell(x), \psi_\ell(x) >_{W_2^{(m)}(T_1)} = \left\|\psi_\ell(x)|W_2^{(m)}(T_1)\right\|^2 \\ &= \left\|\psi_\ell(x)|W_2^{(m)}(T_1)\right\| \cdot \left\|\ell_N^{(\alpha)}(x)|W_2^{(m)*}(T_1)\right\| = \left\|\ell_N^{(\alpha)}(x)|W_2^{(m)*}(T_1)\right\|^2. \end{aligned} \tag{9}$$

The main result of this work is the following theorem.

Theorem 1 *The squared norm of the error functional of the Hermite-type interpolation formula* (1) *in the Sobolev space $W_2^{(m)}(T_1)$ is*

$$\left\|\ell_N^{(\alpha)}(x)|W_2^{(m)*}(T_1)\right\|^2 = \left|1 - \sum_{\alpha=0}^{m-1}\sum_{\lambda=1}^{N} C_\lambda^{(\alpha)}(z)\right|^2 + \frac{1}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\left|\cos 2\pi kz - \sum\limits_{\alpha=0}^{m-1}\sum\limits_{\lambda=1}^{N} C_\lambda^{(\alpha)}(z)(2\pi i)^\alpha k^\alpha e^{-2\pi i k x_\lambda}\right|^2}{k^{2m}} \tag{10}$$

where $C_\lambda^{(\alpha)}$ are coefficients, $x^{(\lambda)}$ are nodes of the interpolation formula (1).

Proof. It is known that the following equality holds for function $f(x) \in \widetilde{W}_2^{(m)}(T_1)$:

$$f(x) = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{-2\pi i k x} = \sum_k \hat{f}_k e^{-2\pi i k x},$$

where

$$\hat{f}_k =< f(x), e^{-2\pi i k x} >= \int_{T_1} f(x) e^{-2\pi i k x} dx,$$

i.e., Fourier coefficients. Thus, we have

$$\begin{aligned} < \ell_N^{(\alpha)}, f(x) >=< \ell_N^{(\alpha)}(x), \sum_{k=-\infty}^{\infty} \hat{f}_k e^{-2\pi i k x} >= \sum_{k=-\infty}^{\infty} \hat{f}_k < \ell_N^{(\alpha)}(x), e^{-2\pi i k x} > &= \sum_{k=-\infty}^{\infty} \hat{f}_k \hat{\ell}_{-k}^{(\alpha)} \\ &= \hat{f}_0 \hat{\ell}_0^{(\alpha)} + \sum_{k\neq 0} \hat{f}_k \hat{\ell}_{-k}^{(\alpha)}. \end{aligned} \tag{11}$$

Here,

$$\hat{\ell}_0^{(\alpha)} = \int_{T_1} \ell_N^{(\alpha)}(x) dx, \ \hat{\ell}_{-k}^{(\alpha)} = \int_{T_1} \ell_N^{(\alpha)}(x) e^{-2\pi i k x} dx.$$

Now we sequentially calculate the value of the Fourier coefficients $\hat{\ell}_0^{(\alpha)}$ and $\hat{\ell}_{-k}^{(\alpha)}$.

$$\hat{\ell}_0^{(\alpha)} = \int_{T_1} \ell_N^{(\alpha)}(x)\,dx = \int_{T_1} [\delta(x-z)]dx - \int_{T_1} \left[\sum_{\alpha=1}^{m-1}\sum_{\lambda=1}^{N} C_\lambda^{(\alpha)}(z)\delta^{(\alpha)}(x-x_\lambda)\right]dx = 1 - \sum_{\alpha=1}^{m-1}\sum_{\lambda=1}^{N} C_\lambda^{(z)}(z), \tag{12}$$

$$\hat{\ell}_{k\,-}^{(\alpha)} =< \ell_k^{(\alpha)}(x), e^{2\pi ikx} >=< \delta(x-z) - \sum_{\alpha=1}^{m-1}\sum_{\lambda=1}^{N} C_\lambda^{(\alpha)}(z)\delta^{(\alpha)}(x-x_\lambda), e^{2\pi ikx} >=< \delta(x-z), e^{2\pi ikx} >$$

$$- < \sum_{\alpha=1}^{m-1}\sum_{\lambda=1}^{N} C_\lambda^{(z)}(z)\delta^{(\alpha)}(x-x_\lambda), e^{2\pi ikx} >= e^{-2\pi ikz} - \sum_{\alpha=1}^{m-1}\sum_{\lambda=1}^{N} C_\lambda^{(\alpha)}(z)(2\pi ik)^\alpha e^{2\pi ikx_\lambda}$$

$$= cos2\pi kz - \sum_{\alpha=1}^{m-1}\sum_{\lambda=1}^{N} C_\lambda^{(\alpha)}(z)(2\pi ik)^\alpha e^{2\pi ikx_\lambda},$$

i.e.,

$$\hat{\ell}_k^{(\alpha)} = cos2\pi kz - \sum_{\alpha=1}^{m-1}\sum_{\lambda=1}^{N} C_\lambda^{(\alpha)}(z)(2\pi ik)^\alpha e^{2\pi ikx_\lambda}. \tag{13}$$

Keeping in mind (12), applying the Cauchy-Schwarz inequality to the right-hand side of (11) and taking into account (2), we obtain the estimate

$$\left|< \ell_N^{(\alpha)}, f(x) >\right| = \left|\hat{f}_0\,\hat{\ell}_0^{(\alpha)} + \sum_{k\neq 0}\hat{f}_k\,\hat{\ell}_k^{(\alpha)}\right| \leq \left|\hat{f}_0\,\hat{\ell}_0^{(\alpha)}\right| + \left|\sum_{k\neq 0}\hat{f}_k\,\hat{\ell}_k^{(\alpha)}(2\pi ik)^m\cdot\frac{1}{(2\pi ik)^m}\right|$$

$$\leq \left|\hat{f}_0\,\hat{\ell}_0^{(\alpha)}\right| + \sum_{k\neq 0}\left|\hat{f}_k\right|\left|\hat{\ell}_k^{(\alpha)}\right|(2\pi ik)^m\frac{1}{|(2\pi ik)^m|} \leq \left\{\left|\hat{f}_0\right|^2 + \sum_{k\neq 0}\left|\hat{f}_k\right|^2|2\pi k|^{2m}\right\}^{\frac{1}{2}}\cdot\left\{\left|\hat{\ell}_0^{(\alpha)}\right|^2 + \sum_{k\neq 0}\frac{\left|\hat{\ell}_k^{(\alpha)}\right|^2}{|2\pi k|^{2m}}\right\}^{\frac{1}{2}}$$

$$= \left\|f(x)|\widetilde{W}_2^{(m)}(T_1)\right\|\cdot\left\{\left|1 - \sum_{\alpha=0}^{m-1}\sum_{\lambda=1}^{N} C_\lambda^{(\alpha)}(z)\right|^2 + \frac{1}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\left|\hat{\ell}_k^{(\alpha)}\right|^2}{k^{2m}}\right\}^{\frac{1}{2}}. \tag{14}$$

Taking into account (2), (13) and (14), we obtain

$$\left\|\ell_N^{(\alpha)}(x)|\widetilde{W}_2^{(m)^*}(T_1)\right\|^2 \leq \left|1 - \sum_{\alpha=0}^{m-1}\sum_{\lambda=1}^{N} C_\lambda^{(\alpha)}(z)\right|^2 + \frac{1}{(2\pi)^{2m}}\sum_{k\neq o}\frac{\left|\cos 2\pi kz - \sum_{\alpha=0}^{m-1}\sum_{\lambda=1}^{N} C_\lambda^{(\alpha)}(z)(2\pi i)^\alpha k^\alpha e^{2\pi ikx_\lambda}\right|^2}{k^{2m}}. \tag{15}$$

There is a function $\psi_\ell(x) \in \widetilde{W}_2^{(m)}(T_1)$ such that the equality is attained in inequality (15).

Indeed, consider the following function $\psi_\ell(x)$

$$\psi_\ell(x) = 1 - \sum_{\alpha=0}^{m-1}\sum_{\lambda=1}^{N} C_\lambda^{(\alpha)}(z) + \frac{1}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\hat{\ell}_k^{(\alpha)}e^{2\pi kix}}{k^{2m}}.$$

We calculate the value of the functional $\ell_N^{(\alpha)}(x)$ for the function $\psi_\ell(x)$

$$< \ell_N^{(\alpha)}(x), \psi_\ell(x) >=< \ell_N^{(\alpha)}(x), 1 - \sum_{\alpha=0}^{m-1}\sum_{\lambda=1}^{N} C_\lambda^{(\alpha)}(z) > + < \ell_N^{(\alpha)}(x), \sum_{k\neq 0}\frac{\hat{\ell}_k^{(\alpha)}e^{2\pi ikx}}{(2\pi)^{2m}k^{2m}} >= \left|1 - \sum_{\alpha=0}^{m-1}\sum_{\lambda=1}^{N} C_\lambda^{(\alpha)}(z)\right|^2$$

$$+ \frac{1}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\hat{\ell}_k^{(\alpha)}\hat{\ell}_k^{(\alpha)}}{k^{2m}} = \left|1 - \sum_{\alpha=0}^{m-1}\sum_{\lambda=1}^{N} C_\lambda^{(\alpha)}(z)\right|^2 + \frac{1}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\left|\hat{\ell}_k^{(\alpha)}\right|^2}{k^{2m}}. \tag{16}$$

Thus, from (16) we have

$$<\ell_N^{(\alpha)}(x),\psi_\ell(x)>=\left|1-\sum_{\alpha=0}^{m-1}\sum_{\lambda=1}^{N}C_\lambda^{(\alpha)}(z)\right|^2+\frac{1}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\left|\hat{\ell}_k^{(\alpha)}\right|^2}{k^{2m}}. \quad (17)$$

Taking into account equalities (8), (9) from (16) we obtain that

$$\left\|\psi_\ell(x)|\widetilde{W}_2^{(m)}(T_1)\right\|^2=\left|1-\sum_{\alpha=0}^{m-1}\sum_{\lambda=1}^{N}C_\lambda^{(\alpha)}(z)\right|^2+\frac{1}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\left|\hat{\ell}_k^{(\alpha)}\right|^2}{k^{2m}}. \quad (18)$$

or

$$\left\|\ell_N^{(\alpha)}(x)|\widetilde{W}_2^{(m)}(T_1)\right\|^2=\left|1-\sum_{\alpha=0}^{m-1}\sum_{\lambda=1}^{N}C_\lambda^{(\alpha)}(z)\right|^2+\frac{1}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\left|\hat{\ell}_k^{(\alpha)}\right|^2}{k^{2m}}. \quad (19)$$

Substituting the value $\hat{\ell}_k^{(\alpha)}$ from (13) into equality (19), we have

$$\left\|\ell_N^{(\alpha)}(x)|\widetilde{W}_2^{(m)}(T_1)\right\|^2=\left|1-\sum_{\alpha=0}^{m-1}\sum_{\lambda=1}^{N}C_\lambda^{(\alpha)}(z)\right|^2+\frac{1}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\left|cos2\pi kz-\sum\limits_{\alpha=1}^{m-1}\sum\limits_{\lambda=1}^{N}C_\lambda^{(\alpha)}(z)(2\pi ik)^\alpha e^{2\pi ikx^{(\lambda)}}\right|^2}{k^{2m}}. \quad (20)$$

The theorem is proved.

Based on this theorem, the error functional of the interpolation formula (1) for functions of class $\widetilde{W}_2^{(m)}(T_1)$has the estimate

$$\left|<\ell_N^{(\alpha)},f(x)>\right|\leq\left\{\left|\hat{f}_0\right|^2+\sum_{k\neq 0}\left|\hat{f}_k\right|^2|2\pi ik|^{2m}\right\}^{\frac{1}{2}}\left\{\left|1-\sum_{\alpha=0}^{m-1}\sum_{\lambda=1}^{N}C_\lambda^{(\alpha)}(z)\right|^2\right.$$

$$\left.+\frac{1}{(2\pi)^{2m}}\times\sum_{k\neq 0}\frac{\left|\cos 2\pi kz-\sum\limits_{\alpha=0}^{m-1}\sum\limits_{\lambda=1}^{N}C_\lambda^\alpha(2\pi i)^\alpha k^\alpha e^{2\pi ix_\lambda}\right|^2}{k^{2m}}\right\}^{\frac{1}{2}}, \quad (21)$$

which was required to be proved.

In order to find an approximate representation of a function using elements of a certain finite-dimensional space, it is possible to use the values of this function at a certain finite set of points x_λ, $\lambda=\overline{0,N}$.

CONCLUSION

There is polynomial and spline interpolation. Our research is conducted by polynomial interpolation. In this paper, we consider with equally spaced nodes, interpolation formulas. An optimal interpolation formula is constructed with finding the optimal coefficients.

ACKNOWLEDGMENTS

We would like to thank our colleagues at Bukhara State University and the Institute of Mathematics named V.I. Romanovskiy for making convenient research facilities. We much appreciate the reviewers for their thoughtful comments and efforts toward improving our manuscript.

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