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On Finding the Coefficients of the Optimal Interpolation Formula in the Sobolev Space $\tilde{W}_2^{(m)}(T_1)$

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Abstract—A typical approximation problem is the interpolation problem. The classical method for solving it is to construct an interpolation polynomial. However, polynomials have a number of disadvantages, for instance, when used as a tool for approximating functions with singularities and functions with not very high smoothness. In practice, in order to approximate functions well, instead of constructing a high-degree interpolation polynomial, splines are used, which are very convenient to use. This paper examines the construction of interpolation splines using the Sobolev method, minimizing the norm in a certain Hilbert space. For the first time, Sobolev [12] posed the problem of finding the extremal function for the interpolation formula and calculating the norm of the error functional in the Sobolev space. In this work, the extremal function of the interpolation formula is found in explicit form in the Sobolev space $W_2^{(m)}\left(R^n\right)$; a function whose mth-order generalized derivatives are square integrable. Basically, the problem of constructing optimal interpolation formulas in the Sobolev space $\tilde{W}_2^{(m)}(T_1)$ for $m = 4$ is considered.

Keywords: generalized function, space, norm, error functional, interpolation formula, extremal function

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INTRODUCTION

A significant step in the theory of splines was the result of Halliday [1], which connects the Schoenberg cubic splines with the solution to the variational problem about the minimum of the squared norm for a function belonging to the space $L_2^{(2)}$. Next, the result of Halliday was generalized by de Boer [2].

At present, the theory of spline interpolation is developing rapidly. There are many works devoted to the theory of splines, for example, Alberg et al. [3], Arcangeli et al. [4], Berlinet and Thomas-Agnan [5], de Boer [2], Ignatov and Pevny [6], Korneichuk et al. [7], Laurent [8], and Stechkin and Subbotin [9].

Shadimetov and Mamatova [10] constructed composite lattice optimal cubature formulas in the Sobolev space using the variational method. In [11], they calculated a sharp upper bound for the error of the interpolation formula in the Sobolev space. Shadimetov and Mamatova proved existence and uniqueness of the optimal interpolation formula that yields the smallest error and presented an algorithm for finding the coefficients of the optimal interpolation formula. By implementing this algorithm, they found the optimal coefficients.

Let us assume that the values $f(x_0), f(x_1), \ldots, f(x_N)$ of the function $f(x)$ are given at $n+1$ randomly located points $\{x_i\}$ $\left(i = \overline{0, N}\right)$, which we call the interpolation nodes below.

It is required to construct an interpolation formula $P_f(x)$, that is,

$$f(x) \cong P_f(x) = \sum_{\lambda=0}^{N} C_\lambda(x) f(x_\lambda), \tag{1}$$

coinciding with the function $f(x)$ at the interpolation nodes:

$$f(x_i) = P_f(x_i),\ i = 0,1,\ldots,N; \tag{2}$$

here, the points $x_\lambda \in T_1$ and the parameters $C_\lambda(x)$ are called, respectively, the nodes and coefficients of the interpolation formula (1), and T_1 is a one-dimensional torus, that is, a circle of unit length.

An important problem in the interpolation theory is to find the maximum error of the interpolation formula $f(x) \cong P_f(x)$ over a given class of functions. The value of this function at some point z is a functional defined as

$$< \ell(x), f(x) >= \int_{-\infty}^{\infty} \ell(x) f(x) dx = f(z) - P_f(z) = f(z) - \sum_{\lambda=0}^{N} C_\lambda(z) f(x_\lambda), \tag{3}$$

where it is clear that $P_f(z) = \sum_{\lambda=0}^{N} C_\lambda(z) f(x_\lambda)$ is the interpolation formula and

$$\ell(x) = \delta(x-z) - \sum_{\lambda=0}^{N} C_\lambda(z)\delta(x - x_\lambda) \tag{4}$$

is the error functional of this interpolation formula, $C_\lambda(z)$ are the coefficients, x_λ are the nodes of the formula $P_f(z)$, $x_\lambda \in [0,1]$, $\delta(x)$ is the Dirac delta function, and $f(x) \in \tilde{W}_2^{(m)}(T_1)$.

Definition. The space $\tilde{W}_2^{(m)}(T_1)$ is defined as the space of functions defined by a one-dimensional T_1-circle with a unit length for which all mth-order generalized derivatives exist and are square summable [12].

The space $\tilde{W}_2^{(m)}(T_1)$ becomes Hilbert if we introduce the scalar product

$$< f(x), \varphi(x) >= \int_{T_1} f^{(m)}(x)\varphi^{(m)}(x)\,dx + \left(\int_{T_1} f(x)\,dx\right)\left(\int_{T_1} \varphi(x)\,dx\right).$$

The norm is determined by the formula

$$\left\| f \,|\, \tilde{W}_2^{(m)}(T_1)\right\|^2 = \left(\int_{T_1} f(x)\,dx\right)^2 + \sum_{k\neq 0} |2\pi k|^{2m} \left|\hat{f}_k\right|^2.$$

1. PROBLEM STATEMENT

The error functional $\ell(x)$ of an interpolation formula $P_f(z)$ is a linear continuous functional in the space $\tilde{W}_2^{(m)}(T_1)$.

We estimate error (3) of the interpolation formula $P_f(z)$ using the maximum error of this formula on the unit ball of the Hilbert space $\tilde{W}_2^{(m)}(T_1)$, that is, using the norm of functional (4):

$$\left\| \ell \,|\, \tilde{W}_2^{(m)*}(T_1)\right\| = \sup_{\left\| f | \tilde{W}_2^{(m)}(T_1)\right\|=1} < \ell, f >,$$

where $\tilde{W}_2^{(m)*}(T_1)$ is the space conjugate to the space $\tilde{W}_2^{(m)}(T_1)$.

Therefore, in order to estimate error (3) of the interpolation formula $P_f(z)$, it is enough to solve the following problem.

Problem 1. Calculate the norm of the error functional $\ell(x)$ of the interpolation formula $P_f(z)$ under consideration. It is clear that the norm of the error functional $\ell(x)$ depends on the coefficients $C_\lambda(z)$ and nodes x_λ. If

$$\left\| \overset{o}{\ell} \,|\, \tilde{W}_2^{(m)*}(T_1)\right\| = \inf_{C_\lambda(x), x_\lambda} \left\| \ell \,|\, \tilde{W}_2^{(m)*}(T_1)\right\|, \tag{5}$$

then the functional $\overset{o}{\ell}(x)$ is called the optimal error functional, and the corresponding interpolation formula is called the optimal interpolation formula.

Thus, there arises

Problem 2. Find the values of the coefficients $C_\lambda(z)$ and nodes x_λ of the interpolation formula $P_f(z)$, which satisfy equality (5).

The coefficients $C_\lambda(z)$ and nodes x_λ satisfying equality (5) are called the optimal coefficients and optimal nodes of the interpolation formula $P_f(z)$.

Sobolev solved [12] the problem of interpolation of functions of n variables in the space $L_2^{(m)}(\Omega)$. In the space $L_2^{(m)}(R)$ problems 1 and 2 were investigated in [13]. In [14], the problem of constructing optimal interpolation formulas of the form (1) with the interpolation condition (2) (with fixed nodes x_λ) was considered in the space $W_2^{(m,m-1)}(0,1)$ and a system of linear equations was obtained for the optimal coefficients. An algorithm for calculating the coefficients of optimal interpolation formulas in the space $W_2^{(m,m-1)}(0,1)$ was given in work [6].

In work [15] we considered the problem of constructing optimal interpolation formulas in the Sobolev space $\tilde{W}_2^{(m)}(T_1)$ at $m = 3$. In the current paper, we consider the solution to this problem in the space $\tilde{W}_2^{(m)}(T_1)$ in the case when $m = 4$.

2. AN ALGORITHM FOR CONSTRUCTING AN OPTIMAL INTERPOLATION FORMULA IN THE PERIODIC SPACE $\tilde{W}_2^{(m)}(T_1)$

Theorem 1 ([15]). *The squared norm of the error functional of the interpolation formula* (1) *over the space* $\tilde{W}_2^{(m)}(T_1)$ *is*

$$\left\|\ell|\tilde{W}_2^{(m)*}(T_1)\right\|^2 = \left|1-\sum_{\lambda=1}^{N}C_\lambda(z)\right|^2 + \frac{1}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\left|\cos 2\pi kz - \sum_{\lambda=1}^{N}C_\lambda(z)e^{2\pi ikx^{(\lambda)}}\right|^2}{k^{2m}},$$

where $C_\lambda(z)$ *are the coefficients and* $x^{(\lambda)}$ *are the nodes of the interpolation formula of type* (1).

Theorem 2 ([15]). *The function*

$$\psi_\ell(x) = 1-\sum_{\lambda=1}^{N}C_\lambda(z) + \frac{1}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\hat{\ell}_k e^{-2\pi ikx}}{k^{(2m)}} \tag{6}$$

is the extremal function for the interpolation formula (1), *and* $\psi_\ell(x)\in\tilde{W}_2^{(m)}(T_1)$.

The main result of this work is the following theorem.

Theorem 3. *In the periodic Sobolev space* $\tilde{W}_2^{(m)}(T_1)$ *there exists a unique optimal interpolation formula of the form* (1) *with the error functional* (4) *the coefficients of which for* $m = 4$ *have the following form*:

$$C_{[\beta]}(z) = \frac{1+\frac{1}{(2\pi)^8}\frac{1}{N^8}\sum_{k\neq 0}\frac{\cos 2\pi k(z-h\beta)}{k^8}}{N\left(1+\frac{1}{(2\pi)^8}\frac{1}{N^8}\sum_{k\neq 0}\frac{1}{k^8}\right)},$$

where $\beta = \overline{1,N}$, $N = 2,3,\ldots$.

Proof. Because by the conditions of Theorem 2 $\psi_\ell(x)$ is an extremal function for the interpolation formula (1), that is, $\psi_\ell(x) \in \tilde{W}_2^{(m)}(T_1)$, it is known [12] that by the Babushka theorem the optimality condition of the interpolation formula can be written as

$$< \delta(x - x^{(\beta)}), \psi_\ell(x) >= \psi_\ell(x^{(\beta)}) = 0, \tag{7}$$

where $\psi_\ell(x)$ is the extremal function of the interpolation formula (1) in the space $\tilde{W}_2^{(m)}(T_1)$. It is known [8] that $\hat{\ell}_k = \cos 2\pi kz - \sum_{\lambda=1}^{N} C_\lambda(z) e^{2\pi i k x^{(\lambda)}}$ in equality (6); hence, the extremal function at $m = 4$ can be represented as

$$\psi_\ell(x) = 1 - \sum_{\lambda=1}^{N} C_\lambda(z) + \frac{1}{(2\pi)^8} \sum_{k\neq 0} \frac{\left(\cos 2\pi kz - \sum_{\lambda=1}^{N} C_\lambda(z) e^{2\pi i k x^{(\lambda)}}\right) e^{-2\pi i k x}}{k^8}, \tag{8}$$

where $x^{(\lambda)}$ and $C_\lambda(z)$ are the nodes and coefficients of the interpolation formula (1). Because we consider interpolation formulas with uniformly distributed nodes, we have $x^{(\beta)} = h\beta$ and $x^{(\lambda)} = h\lambda$ $(\beta = 1, \ldots, N, \lambda = 1, \ldots, N)$. Then, taking into account (7), from (8) we obtain

$$\psi_\ell(h\beta) = 0, \ \beta = \overline{1, N},$$

that is, for $\psi_\ell(h\beta)$ we have the following system of equations:

$$1 - \sum_{\lambda=1}^{N} C_\lambda(z) + \frac{1}{(2\pi)^8} \sum_{k\neq 0} \frac{\left(\cos 2\pi kz - \sum_{\lambda=1}^{N} C_\lambda(z) e^{2\pi i k x^{(\lambda)}}\right) e^{-2\pi i k x^{(\beta)}}}{k^8} = 0, \ \text{where } \beta = \overline{1, N}. \tag{9}$$

Transforming (9), we have

$$1 - \sum_{\lambda=1}^{N} C_\lambda(z) + \frac{1}{(2\pi)^8} \sum_{k\neq 0} \frac{(\cos 2\pi kz)\, e^{-2\pi i k h\beta}}{k^8} - \frac{1}{(2\pi)^8} \sum_{k\neq 0} \frac{\left(\sum_{\lambda=1}^{N} C_\lambda(z) e^{2\pi i k (h\beta - h\lambda)}\right)}{k^8} = 0,$$

or

$$\sum_{\beta=1}^{N} C_\beta(z) + \frac{1}{(2\pi)^8} \sum_{k\neq 0} \frac{\left(\sum_{\beta=1}^{N} C_\beta(z) e^{2\pi i k (h\beta - h\lambda)}\right)}{k^8} = 1 + \frac{1}{(2\pi)^8} \sum_{k\neq 0} \frac{\cos 2\pi k (z - h\beta)}{k^8}. \tag{10}$$

After some transformations, from (10) we obtain

$$\sum_{\beta=1}^{N} C_\beta(z) \left[1 + \frac{1}{(2\pi)^8} \sum_{k\neq 0} \frac{\cos 2\pi k (h\beta - h\lambda)}{k^8}\right] = 1 + \frac{1}{(2\pi)^8} \sum_{k\neq 0} \frac{\cos 2\pi k (z - h\beta)}{k^8}. \tag{11}$$

Multiplying both sides of (11) by the number

$$a = \frac{1}{N\left(D_4(0) + 2D_4(1) + 2\sum_{\gamma=2}^{\infty} D_4[h\gamma]\right)\left(1 + \frac{1}{(2\pi)^8 N^8} \sum_{k\neq 0} \frac{1}{k^8}\right)}, \tag{12}$$

we have

$$\sum_{\beta=1}^{N} C_\beta(z) a \left[1 + \frac{1}{(2\pi)^8} \sum_{k\neq 0} \frac{\cos 2\pi k (h\beta - h\lambda)}{k^8}\right] = a\left(1 + \frac{1}{(2\pi)^8} \sum_{k\neq 0} \frac{\cos 2\pi k (z - h\beta)}{k^8}\right). \tag{13}$$

Let us designate

$$a\left[1+\frac{1}{(2\pi)^8}\sum_{k\neq 0}\frac{\cos 2\pi k\,(h\beta-h\lambda)}{k^8}\right]=\nu_4\,(h\beta-h\lambda)\quad\left(\lambda=\overline{1,N}\right),$$

$$a\left(1+\frac{1}{(2\pi)^8}\sum_{k\neq 0}\frac{\cos 2\pi k\,(z-h\beta)}{k^8}\right)=f_4\,(h\beta). \tag{14}$$

From (13) we obtain the equation

$$\sum_{\beta=1}^{N}C_\beta\,(z)\,\nu_4\,(h\beta-h\lambda)=f_4\,(h\beta)\,\left(\lambda=\overline{1,N}\right). \tag{15}$$

Redesignating $\nu_4\,(h\beta)=\nu_4\,[\beta]$ and $f_4\,(h\beta)=f_4\,[\beta]$, we can write system (15) as a convolution of functions of a discrete argument:

$$C_\beta\,(z)*\nu_4\,[\beta]=f_4\,[\beta],\quad \beta=\overline{0,N}, \tag{16}$$

$$C_\beta\,(z)=0,\quad h\beta\notin T_1.$$

Next, we need

Theorem 4 ([2]). *A discrete analogue $D_m\,[\beta]$ of the differential operator $\left(1-\frac{d^2}{(2\pi)^2\,dx^2}\right)^m$ at $m=4$ satisfying the equality*

$$D_4\,[\beta]*\nu_4\,[\beta]=\delta[\beta], \tag{17}$$

where

$$\delta[\beta]=\begin{cases}1, & \beta=0;\\ 0, & \beta\neq 0,\end{cases}$$

is determined by the formula

$$D_4\,[\beta]=B\begin{cases}c+\sum\limits_{i=1}^{3}\frac{A_i}{\lambda_i}, & \beta=0;\\ 1+\sum\limits_{i=1}^{3}A_i, & |\beta|=1;\\ \sum\limits_{i=1}^{3}A_i\lambda_i^{|\beta|-1}, & |\beta|\geq 2,\end{cases}$$

where

$$B=\frac{2^2\cdot 3}{\pi^2\cdot a_1},\quad a_1=-15a+120hb-12\cdot 16\pi h^2a+32\pi^2h^3b,$$

$$c=\frac{a_2}{a_1}+8b,\quad a_2=90ab-240h\left(1+2b^2\right)+12\cdot 16\pi^2a\cdot 2\,(1+b)+32\pi^2h^3\cdot 8\left(b^2-1\right),$$

$$A_i=\frac{\lambda_i^8+b_1'\lambda_i^7+b_2'\lambda_i^6+b_3'\lambda_i^5+b_4'\lambda_i^4+b_5'\lambda_i^3+b_6'\lambda_i^2+b_7'+1}{\lambda_i^2-1}\quad\left(i=\overline{1,3}\right),$$

$$b_1'=-8b,\quad b_2'=4\left(b^2+1\right),\quad b_3'=8b\left(1-4b^2\right),\quad b_4'=2\left(24b^2+8b^4+3\right);$$

here,

$$b_1' = b_7',\ \ b_2' = b_6',\ \ b_3' = b_5',$$

λ_i *are the roots of the polynomial* $P_6(\lambda) = \left(\lambda^6 + b_1\lambda^5 + b_2\lambda^4 + b_3\lambda^3 + b_4\lambda^2 + b_5\lambda + 1\right)\ \left(i = \overline{1,3}\right)$,

$$b_1 = \frac{a_2}{a_1},\ \ b_2 = \frac{a_3}{a_1},\ \ b_3\frac{a_4}{a_1};$$

here,

$$a_3 = -45a\left(1+4b^2\right) + 120hb\left(11+4b^2\right) + 12\cdot 16\pi h^2 a(8b-11) + 32\pi^2 h^3 b\left(4b^2 - 13\right),$$

$$a_4 = 60ab(3+2b) - 120h4\left(1+4b^2\right) + 12\cdot 16\pi h^2 a4\left(1+2b^2+3b\right) + 32\pi^2 h^3 16\left(2-b^2\right),$$

$$\lambda_1 = \frac{1}{2}\left\{\frac{1}{2}\left[b_1 - \left(A+B+\frac{b_1}{3}\right) + \left(\left(b_1 - \left(A+B+\frac{b_1}{3}\right)\right)^2 - 4\left(b_2 - \left(A+B+\frac{b_1}{3}\right)\right.\right.\right.\right.$$

$$\left.\left.\left.\times\left(b_1 - \left(A+B+\frac{b_1}{3}\right)\right) - 3\right)\right)^{1/2}\right] + \frac{1}{4}\left[b_1 - \left(A+B+\frac{b_1}{3}\right) + \left(\left(b_1 - \left(A+B+\frac{b_1}{3}\right)\right)^2\right.\right.$$

$$\left.\left.\left. - 4\left(b_2 - \left(A+B+\frac{b_1}{3}\right)\right)\left(b_1 - \left(A+B+\frac{b_1}{3}\right)\right) - 3\right)\right)^{1/2}\right]^2 - 4\right)^{1/2}\right\},$$

$$\lambda_2 = \frac{1}{2}\left\{\frac{1}{2}\left[b_1 - \left(A+B+\frac{b_1}{3}\right) + \left(\left(b_1 - \left(A+B+\frac{b_1}{3}\right)\right)^2 - 4\left(b_2 - \left(A+B+\frac{b_1}{3}\right)\right.\right.\right.\right.$$

$$\left.\left.\left.\times\left(b_1 - \left(A+B+\frac{b_1}{3}\right)\right) - 3\right)\right)^{1/2}\right] - \left(\frac{1}{4}\left[b_1 - \left(A+B+\frac{b_1}{3}\right) + \left(\left(b_1 - \left(A+B+\frac{b_1}{3}\right)\right)^2\right.\right.\right.$$

$$\left.\left.\left. - 4\left(b_2 - \left(A+B+\frac{b_1}{3}\right)\right)\left(b_1 - \left(A+B+\frac{b_1}{3}\right)\right) - 3\right)\right)^{1/2}\right]^2 - 4\right)^{1/2}\right\},$$

$$A = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{p^3}{9} + \frac{q^2}{4}}},\ \ B = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{p^3}{9} + \frac{q^2}{4}}};$$

here,

$$p = -\frac{b_1^2}{3} + b_2 - 3,\ \ q = 2\cdot\frac{b_1^3}{9} - \frac{b_1(b_2-3)}{3} + 2b_1 - b_3$$

and h is a small parameter.

Applying the operator $D_4[\beta]$ to both sides of Eq. (16), we get

$$C_\beta(z)\cdot D_4[\beta] * \nu_4[\beta] = D_4[\beta] * f_4[\beta],\ \ \beta = \overline{0,N}. \tag{18}$$

According to (16), (17), and Theorem 3, from (18) we have

$$C_\beta(z) = D_4[\beta] * f_4[\beta],\ \ [\beta] = [0,1]. \tag{19}$$

Substituting (14) into (19), we get

$$
\begin{aligned}
C_\beta(z) &= D_4[\beta] * a \cdot \left(1 + \frac{1}{(2\pi)^8}\sum_{k\neq 0}\frac{\cos 2\pi k(z-h\beta)}{k^8}\right) \\
&= a \cdot \left(D_4[\beta] * 1 + \frac{1}{(2\pi)^8}\sum_{k\neq 0}\frac{e^{2\pi i k z}}{k^8} D_4[\beta] * e^{-2\pi i k h\beta}\right).
\end{aligned}
\tag{20}
$$

Because

$$
D_4[\beta] * 1 = \sum_{\beta=-\infty}^{\infty} D_4(h\beta) = D_4(0) + 2D_4(1) + 2\sum_{\beta=2}^{\infty} D_4(h\beta) \tag{21}
$$

and

$$
D_4[\beta] * e^{2\pi i k h\beta} = \sum_{\gamma=-\infty}^{\infty} D_4[\gamma] e^{2\pi i k h(\gamma-\beta)}
$$

$$
= e^{-2\pi i k h\beta}\left[D_4(0) + 2D_4(1)\cos 2\pi k h + 2\sum_{\gamma=2}^{\infty} D_4(h\gamma) e^{2\pi i k h\gamma}\right], \tag{22}
$$

in view of (21) and (22), from (20) we obtain

$$
\begin{aligned}
C_\beta(z) = a \cdot \Bigg(& D_4[0] + 2D_4[1] + 2\sum_{\gamma=2}^{\infty} D_4[h\gamma] + \frac{1}{(2\pi)^8} \\
& \times \sum_{k\neq 0}\frac{\cos 2\pi k(z-h\beta)}{k^8}\left(D_4[0] + 2D_4[1]\cos 2\pi k h + 2\sum_{\gamma=2}^{\infty} D_4[h\gamma] e^{2\pi k h\gamma}\right)\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
C_\beta(z) = a \cdot \Bigg(& D_4[0] + 2D_4[1] + 2\sum_{\gamma=2}^{\infty} D_4[h\gamma] + \frac{1}{(2\pi)^8} \\
& \times \sum_{k\neq 0}\frac{\cos 2\pi k(z-h\beta)}{k^8}\left([D_4[0] + 2D_4[1]\cos 2\pi k h] + 2\sum_{\gamma=2}^{\infty} D_4[h\gamma]\cos 2\pi k h\gamma\right)\Bigg).
\end{aligned}
\tag{23}
$$

Because, when $kh \in Z$, where Z is the set of integers, $\cos 2\pi k h = 1$ and

$$
2\sum_{\gamma=2}^{\infty} D_4[h\gamma]\cos 2\pi k h\gamma = 2\sum_{\gamma=2}^{\infty} D_4[h\gamma],
$$

after some transformations, from (23) we have

$$
\begin{aligned}
C_\beta(z) &= a \cdot \left(D_4[0] + 2D_4[1] + 2\sum_{\gamma=2}^{\infty} D_4[h\gamma]\frac{1}{(2\pi)^8 N^8}\sum_{k\neq 0}\frac{\cos 2\pi k(z-h\beta)}{k^8}\right) \\
&\times \left(\left[D_4[0] + 2D_4[1] + 2\sum_{\gamma=2}^{\infty} D_4[h\gamma]\right]\right) = a \cdot \left(D_4[0] + 2D_4[1] + 2\sum_{\gamma=2}^{\infty} D_4[h\gamma]\right) \\
&\times \left(1 + \frac{1}{(2\pi)^8 N^8}\sum_{k\neq 0}\frac{\cos 2\pi k(z-h\beta)}{k^8}\right).
\end{aligned}
\tag{24}
$$

Due to (12), for the optimal coefficients of the interpolation formula (1), from (24) we obtain

$$
C_\beta(z) = \frac{\left(1 + \frac{1}{(2\pi)^8 N^8}\sum_{k\neq 0}\frac{\cos 2\pi k(z-h\beta)}{k^8}\right)}{N\left(1 + \frac{1}{(2\pi)^8}\frac{1}{N^8}\sum_{k\neq 0}\frac{1}{k^8}\right)},
$$

where $h = \frac{1}{N}$ and $\beta = \overline{1, N},\ N = 2, 3, \ldots$. Thus, we have determined the coefficients of the optimal interpolation formula (1).

□

CONCLUSIONS

Interpolation has important theoretical and practical significance in approximating functions with given tabular values. In mathematics and its applications, many practical problems are solved using interpolation. A distinction was made between classical and variational interpolation methods. This article considered the problem of constructing an interpolation formula based on the variational method. Here, we constructed an optimal interpolation formula in Hilbert space. In the variational approach, splines were understood as elements of a Hilbert or Banach space that minimize certain functionals. This work was devoted specifically to variational methods. In this work, using the method of Sobolev, we solved a minimization problem of interpolation.

The result of this work is a new optimal interpolation formula in the Sobolev space $\tilde{W}_2^{(4)}(T_1)$; here, using a discrete analog $D_4[\beta]$ of the differential operator $\left(1 - \frac{d^2}{(2\pi)^2 dx^2}\right)^4$, we obtained the coefficients of the optimal interpolation formula in the Sobolev space $\tilde{W}_2^{(4)}(T_1)$.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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