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On Finding the Norm of the Error Functional of Interpolation Formulas in the Sobolev Space of Periodic functions

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Abstract. The main areas of application of various spaces of generalized functions lie in the theory of differential equations and in the theory of quadrature and cubature formulas. Therefore, there is a need to study the spaces of generalized functions, connected with various domains in R^n. Modern formulation of the problem of optimization of approximate integration formulas consists in minimizing the norm of the error functional formula on chosen normed spaces. Since in this paper we consider the space of functions defined on the n- dimensional torus T_n, i.e. on a manifold that is not a Euclidean space (we assume that the Euclidean structure is locally maintained), it is necessary to discuss the question of the invariance of the introduced norm under orthogonal transformations. The difficulties that arise are related to the fact that the space of periodic functions on R^n is not invariant under rotations. The space of derivatives of m order at point x forms m symmetric power of the introduced tangent space and is denoted by $S(T)^m$. It is required that the introduced norm at each point be invariant under orthogonal transformations of tangent space $T(x)$.

INTRODUCTION

In order to find an approximate representation of function f using elements of a specific finite-dimensional space, it is possible to use the value of this function on some finite set of points $x^{(\lambda)}, \lambda = 0, 1, \ldots, N$. The corresponding problem is called the interpolation problem and points $x^{(\lambda)}$ are called interpolation nodes.

Many researchers built theories of optimal interpolation formulas in [1, 2, 3, 4, 5, 6, 7, 8] and, optimal quadrature formulas (see, [9, 10, 11]).

Let mesh $\omega = \{0 = x_0 < x_2 < ... < x_N = 1\}$ be given on the segment $0 \leq x \leq 1$ and values $f(x^{(0)}), f(x^{(1)}), \ldots, f(x^{(N)})$ of the function $f(x)$ be specified at its nodes. It is required to construct an interpolation formula $P_f(x)$, coinciding with the function $f(x)$ in mesh nodes ω :

$$f(x^{(i)}) = P_f(x^{(i)}),\ i = 0, 1, 2, ..., N.$$

Consider an interpolation formula of the following form

$$f(x) \cong P_f(x) = \sum_{\lambda=1}^{N} C_\lambda(x) f(x^{(\lambda)}) \tag{1}$$

over Sobolev space $\tilde{W}_2^{(m)}(T_n)$. Here, $C_\lambda(x)$ and $x^{(\lambda)}$ are *the coefficients and nodes of the interpolation* formula (1), respectively, $f(x) \in \tilde{W}_2^{(m)}(T_n)$, and T_n is the n-dimensional torus.

Definition 1. Set $T_n = \{x = (x_1, x_2, ...x_n); x_k = \{t_k\}, t_k \in R\}$, where $\{t_k\} = t_k - [t_k]$, i.e. fractional part of t_k, is called *n-dimensional torus* T_n.

Definition 2. Space $\tilde{W}_2^{(m)}(T_n)$ is defined as the space of functions given on the n-dimensional torus T_n, having all generalized derivatives of m order summable with a square in the norm [12]

$$\left\| f(x) \,|\tilde{W}_2^{(m)}(T_n) \right\|^2 = \left(\int_{T_n} f(x)\,dx \right)^2 + \sum_{k \neq 0} |2\pi k|^{2m} \left| \widehat{f}_k \right|^2, \tag{2}$$

with the inner product

$$< f(x), \varphi(x) >_{\tilde{W}_2^{(m)}(T_n)} = \int_{T_n} f^{(m)}(x)\,\varphi^{(m)}(x)\,dx + \left(\int_{T_n} f(x)\,dx \right) \left(\int_{T_n} \varphi(x)\,dx \right), \tag{3}$$

where $\widehat{f}_k$ are the Fourier coefficients, i.e. $\widehat{f}_k = \int_{T_n} f(x)\,e^{2\pi i(k,x)}dx$.

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STATEMENT OF THE PROBLEM

The main purpose of this study is to find the norm of the error functional of the interpolation formula in the Sobolev space $\tilde{W}_2^{(m)}(T_n)$ for periodic functions.

One of the main tasks of the theory of interpolation is to find the maximum error of the interpolation formula $f(x) \cong P_f(x)$ in space $\tilde{W}_2^{(m)}(T_n)$. The value of this error at some point z is a functional over function f:

$$< \ell_N, f(x) > \equiv f(z) - P_f(z) = f(z) - \sum_{\lambda=1}^{N} C_\lambda(z) f(x^{(\lambda)}) = \\ = \int_0^1 (\delta(x-z) - \sum_{\lambda=1}^{N} C_\lambda(z)\delta(x-x^{(\lambda)})) f(x)dx, \tag{4}$$

where $\delta(x)$ is the Dirac delta function and

$$\ell_N(x) = \delta(x-z) - \sum_{\lambda=1}^{N} C_\lambda(z)\delta(x-x^{(\lambda)}), \tag{5}$$

is the error functional of the interpolation formula (1) is a linear and continuous functional from the space $\tilde{W}_2^{(m)}(T_n)$ to the conjugate space $\tilde{W}_2^{(m)*}(T_n)$, i.e. $\ell_N(x) \in \tilde{W}_2^{(m)*}(T_n)$.

Nodes $x^{(\lambda)}$ and coefficients $C_\lambda(z)$ are the variable parameters of the interpolation formula (1). An optimal interpolation formula is the one whose error functional for a given number of nodes N has the minimum norm in space $W_2^m(R)$. If nodes $x^{(\lambda)}$ are lattice points, i.e. located at points of the form $x^{(\lambda)} = h\lambda$, then such an interpolation formula is called a lattice formula. Here h is a small parameter called the lattice spacing. Construction of lattice optimal interpolation formulas in the Sobolev space $\tilde{L}_2^{(m)}(H)$ of n variable periodic functions was considered in [1].

Optimal interpolation formulas in the space $L_2^{(m)}(R)$ were studied in [13].

In [14, 15], the problem of constructing optimal interpolation formulas of the form (1) in space $W_2^{(m,m-1)}(R)$ was considered, and a system of linear equations was obtained for the optimal coefficients.

The task of constructing optimal interpolation formulas of the form (1) with the error functional (5) in space $\tilde{W}_2^{(m)}(T_n)$ at fixed nodes $x^{(\lambda)}$ is to calculate the following quantity

$$\left\| \overset{0}{\ell}_N | \tilde{W}_2^{(m)*}(T_n) \right\| = \inf_{C_\lambda(z)} \left(\sup_{f, \|f\| \neq 0} \frac{|< \ell_N, f >|}{\left\| f | \tilde{W}_2^{(m)}(T_n) \right\|} \right). \tag{6}$$

This problem consists of two parts: first, we must calculate the norm $\|\ell_N\|$ of the error functional ℓ_N in the space $\tilde{W}_2^{(m)}(T_n)$, i.e. problem 1, and then minimize it by coefficients $C_\lambda(z)$ at fixed $x^{(\lambda)}$.

If there are such coefficients $C_\lambda(z) = \overset{0}{C}_\lambda(z)$, that equality (6) is achieved, then they are called optimal.

EXTREMAL FUNCTION AND THE NORM OF THE ERROR FUNCTIONAL

To find the norm of the error functional (5) in the space $\tilde{W}_2^{(m)}(T_n)$, we use the concept of the extremal function of the error functional ℓ_N introduced by S.L.Sobolev [12].

Definition 2. Function $\psi_\ell(x)$ is called an extremal function of functional ℓ_N if the following equality holds for it

$$< \ell_N, \psi_\ell > = \left\| \ell_N | \tilde{W}_2^{(m)^*}(T_n) \right\| \cdot \left\| \psi_\ell | \tilde{W}_2^{(m)}(T_n) \right\|. \tag{7}$$

In the space $\tilde{W}_2^{(m)}(T_n)$, using the Riesz theorem on the general form of a linear continuous functional on Hilbert spaces, the extremal function is expressed in terms of a given functional and the following equality holds

$$\left\| \ell_N | \tilde{W}_2^{(m)^*}(T_n) \right\| = \left\| \psi_\ell | \tilde{W}_2^{(m)}(T_n) \right\|. \tag{8}$$

Therefore, from (7) and (8) we conclude that

$$< \ell_N, \psi_\ell >= \left\| \ell_N | \tilde{W}_2^{(m)^*} (T_n) \right\|^2. \tag{9}$$

Let a function $f(x)$ belongs to some Banach space B, then $\ell_N(x)$ is a functional from conjugated space B^*. It is assumed that this space is compactly embedded in the space of continuous functions defined in the domain Ω:

$$B \to C(\Omega). \tag{10}$$

A functional $\ell_N(x)$ defined on B^* is linear and continuous, and due to condition (10) it is bounded, i.e. we have:

$$|< \ell_N, f >| \leq \|\ell_N | B^*\| \cdot \|f|B\|. \tag{11}$$

Estimate (11) shows that the quality of the interpolation formula is characterized by the norm of the error functional, determined by the following formula

$$\|\ell_N | B^*\| = \sup_{f,\ \|f|B\| \neq 0} \frac{|< \ell_N, f >|}{\|f|B\|}, \tag{12}$$

and is a function of unknown coefficients and nodes. Therefore, for computational practice, it is appropriate to be able to calculate the norm of the error functional (12) and estimate it. Finding the minimum norm of the error functional with respect to $C_\lambda(z)$ and $x^{(\lambda)}$ is the task of investigating the extrema of multivariable functions. The values of $C_\lambda(z)$ and $x^{(\lambda)}$, realizing this minimum, determine the optimal formula. Thus, the formula in which, for a given number of nodes N, the error functional has the minimum norm, we call the *optimal interpolation formula*.

The task of constructing the best interpolation formulas over Sobolev space $\tilde{W}_2^{(m)}(T_n)$ is to calculate the following quantity:

$$\left\| \ell_N(x) | \tilde{W}_2^{(m)*}(T_n) \right\| = \inf_{C_\lambda, x^{(\lambda)}} \sup_{\|f(x)\| \neq 0} \frac{|< \ell_N(x), f(x) >|}{\left\| f(x) | \tilde{W}_2^{(m)}(T_n) \right\|}, \tag{13}$$

where $\tilde{W}_2^{(m)*}(T_n)$ is the space conjugate to space $\tilde{W}_2^{(m)}(T_n)$. To estimate the error of the interpolation formula, it is necessary to solve the following problem.

Problem 1. Find the norm of the error functional (5) of the given interpolation formula.

First, we must calculate norm $\left\| \ell_N(x) | \tilde{W}_2^{(m)*}(T_n) \right\|$ of the error functional $\ell_N(x)$ in the space $\tilde{W}_2^{(m)}(T_n)$, and then, if necessary, to construct the best interpotation formula by varying $C_\lambda(z)$ and $x^{(\lambda)}$ $\lambda = \overline{1,N}$, it is necessary to solve the following problem.

Problem 2. Find such values of $C_\lambda(z)$ and $x^{(\lambda)}$, for which equality (13) is satisfied.

In this article, we solve Problem 1 for an interpolation formula of the form (1), i.e., calculate the norm $\left\| \ell_N(x) | \tilde{W}_2^{(m)*}(T_n) \right\|$ of the error functional $\ell_N(x)$ of the interpolation formula (1).

Since the space $\tilde{W}_2^{(m)}(T_n)$ with the inner product (3) becomes Hilbert space, then, based on the Riesz theorem on the general form of a continuous linear functional [12], there is unique function $\psi_\ell(x) \in \tilde{W}_2^{(m)}(T_n)$ for which

$$< \ell_N(x), f(x) >=< \psi_\ell(x), f(x) >_{\tilde{W}_2^{(m)}(T_n)}$$

and

$$\left\| \ell_N(x) | \tilde{W}_2^{(m)*}(T_n) \right\| = \left\| \psi_\ell(x) | \tilde{W}_2^{(m)}(T_n) \right\|.$$

Hence, in particular, for $f(x) = \psi_\ell(x)$ we have

$$< \ell_N(x), \psi_\ell(x) >=< \psi_\ell(x), \psi_\ell(x) >_{\tilde{W}_2^{(m)}(T_n)} = \left\| \psi_\ell(x) | \tilde{W}_2^{(m)}(T_n) \right\|^2$$

$$= \left\| \psi_\ell(x) | \tilde{W}_2^{(m)}(T_n) \right\| \cdot \left\| \ell_N(x) | \tilde{W}_2^{(m)*}(T_n) \right\| = \left\| \ell_N(x) | \tilde{W}_2^{(m)*}(T_n) \right\|^2. \tag{14}$$

The following theorem is valid.

Theorem 1. *The square of the norm of the error functional (5) of the general interpolation formula (1) over space* $\tilde{W}_2^{(m)}(T_n)$ *is*

$$\left\|\ell_N(x)|\tilde{W}_2^{(m)*}(T_n)\right\|^2 = \left|1-\sum_{\lambda=1}^{N} C_\lambda(z)\right|^2 + \frac{1}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\left|\cos 2\pi kz - \sum\limits_{\lambda=1}^{N} C_\lambda(z)\,e^{2\pi i(k,x^{(\lambda)})}\right|^2}{k^{2m}}, \tag{15}$$

where $C_\lambda(z)$ are the coefficients, $x^{(\lambda)}$ are the nodes of the interpolation formula (1).

Proof. It is known that the following equality is true for a function $f(x)\in\tilde{W}_2^{(m)}(T_n)$:

$$f(x)=\sum_k \widehat{f}_k e^{-2\pi i(k,x)},$$

where $\widehat{f}_k =< f(x), e^{2\pi i(k,x)} >= \int\limits_{T_n} f(x)\,e^{2\pi i(k,x)}dx$, i.e. Fourier coefficients.

Thus, we have

$$< \ell_N, f(x) >=< \ell_N(x), \sum_k \widehat{f}_k e^{-2\pi i(k,x)} >$$

$$=\sum_k \widehat{f}_k < \ell_N(x), e^{-2\pi i(k,x)} >= \sum_k \widehat{f}_k\widehat{\ell}_{-k} = \widehat{f}_0\widehat{\ell}_0 + \sum_{k\neq 0}\widehat{f}_k\widehat{\ell}_{-k}. \tag{16}$$

Here $\widehat{\ell}_0 = \int\limits_{T_n}\ell_N(x)\,dx$, $\widehat{\ell}_{-k} = \int\limits_{T_n}\ell_N(x)\,e^{-2\pi i(k,x)}dx$.

Applying the Cauchy-Schwarz inequality to the right side of (16) and taking into account (2), we obtain the following estimate

$$|< \ell_N, f(x) >| = \left|\widehat{f}_0\widehat{\ell}_0 + \sum_{k\neq 0}\widehat{f}_k\widehat{\ell}_{-k}\right| \leq |\widehat{f}_0\widehat{\ell}_0|$$

$$+\left|\sum_{k\neq 0}\widehat{f}_k\widehat{\ell}_{-k}|2\pi k|^m\cdot\frac{1}{|2\pi k|^m}\right| \leq |\widehat{f}_0\widehat{\ell}_0| + \sum_{k\neq 0}\left|\widehat{f}_k\right|\left|\widehat{\ell}_{-k}\right||2\pi k|^m\frac{1}{|2\pi k|^m}$$

$$\leq \left\{\left|\widehat{f}_0\right|^2 + \sum_{k\neq 0}\left|\widehat{f}_k\right|^2|2\pi k|^{2m}\right\}^{1/2}\cdot\left\{\left|\widehat{\ell}_0\right|^2 + \sum_{k\neq 0}\frac{\left|\widehat{\ell}_{-k}\right|^2}{|2\pi k|^{2m}}\right\}^{1/2}$$

$$= \left\|f(x)|\tilde{W}_2^{(m)}(T_n)\right\|\cdot\left\{\left|\widehat{\ell}_0\right|^2 + \frac{1}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\left|\widehat{\ell}_k\right|^2}{|k|^{2m}}\right\}^{1/2}. \tag{17}$$

With (2), (12), and (17), we obtain

$$\left\|\ell_N(x)|\tilde{W}_2^{(m)*}(T_n)\right\| \leq \left\{\left|\widehat{\ell}_0\right|^2 + \frac{1}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\left|\widehat{\ell}_k\right|^2}{|k|^{2m}}\right\}^{1/2}, \tag{18}$$

We calculate $\widehat{\ell}_k$:

$$\widehat{\ell}_k = \int\limits_{T_n} \ell_N(x)\, e^{2\pi i(k,x)} dx = <\ell_N(x), e^{2\pi i(k,x)}>$$

$$= <\delta(x-z) - \sum_{\lambda=1}^{N} C_\lambda(z)\,\delta(x-x^{(\lambda)}), e^{2\pi i(k,x)}> = <\delta(x-z) - \sum_{\lambda=1}^{N} C_\lambda(z)\,\delta(x-x^{(\lambda)}), e^{2\pi i(k,x)}>$$

$$= <\delta(x-z), e^{2\pi i(k,x)}> - \sum_{\lambda=1}^{N} C_\lambda(z) <\delta(x-x^{(\lambda)}), e^{2\pi i(k,x)}> = \cos 2\pi(k,z) - \sum_{\lambda=1}^{N} C_\lambda(z)\, e^{2\pi i(k,x^{(\lambda)})},$$

i.e.

$$\widehat{\ell}_k = \cos 2\pi(k,z) - \sum_{\lambda=1}^{N} C_\lambda(z)\, e^{2\pi i(k,x^{(\lambda)})}, \tag{19}$$

Hence, for $k=0$ we have

$$\widehat{\ell}_0 = 1 - \sum_{\lambda=1}^{N} C_\lambda(z) \tag{20}$$

Thus, with (19) and (20), from (18) we obtain

$$\left\|\ell_N(x)\,|\tilde{W}_2^{(m)*}(T_n)\right\|^2 \le \left|1 - \sum_{\lambda=1}^{N} C_\lambda(z)\right|^2 + \frac{1}{(2\pi)^{2m}} \sum_{k\neq 0} \frac{\left|\cos 2\pi(k,z) - \sum\limits_{\lambda=1}^{N} C_\lambda(z)\, e^{2\pi i(k,x^{(\lambda)})}\right|^2}{k^{2m}}. \tag{21}$$

There is such function from $\tilde{W}_2^{(m)}(T_n)$ that inequality (21) achieves equality.
Indeed, consider the following function $u(x)$:

$$u(x) = 1 - \sum_{\lambda=1}^{N} C_\lambda(z) + \frac{1}{(2\pi)^{2m}} \sum_{k\neq 0} \frac{\widehat{\ell}_{-k} e^{-2\pi i(k,x)}}{|k|^{2m}}.$$

Calculating the value of functional $\ell_N(x)$ on function $u(x)$, we obtain

$$<\ell(x), u(x)> = <\ell_N(x), 1 - \sum_{\lambda=1}^{N} C_\lambda(z) + \frac{1}{(2\pi)^{2m}} \sum_{k\neq 0} \frac{\widehat{\ell}_{-k} e^{-2\pi i(k,x)}}{|k|^{2m}}>$$

$$= <\ell_N(x), 1 - \sum_{\lambda=1}^{N} C_\lambda(z)> + <\ell_N(x), \frac{1}{(2\pi)^{2m}} \sum_{k\neq 0} \frac{\widehat{\ell}_{-k} e^{-2\pi i(k,x)}}{|k|^{2m}}>$$

$$= \left[1 - \sum_{\lambda=1}^{N} C_\lambda(z)\right] \int\limits_{T_n} \ell_N(x)\, dx + \frac{1}{(2\pi)^{2m}} \sum_{k\neq 0} \frac{\widehat{\ell}_{-k} <\ell_N^{(\alpha)}(x), e^{-2\pi i(k,x)}>}{|k|^{2m}}$$

$$= \left[1 - \sum_{\lambda=1}^{N} C_\lambda(z)\right] \widehat{\ell}_0 + \frac{1}{(2\pi)^{2m}} \sum_{k\neq 0} \frac{\widehat{\ell}_{-k} \widehat{\ell}_{-k}}{|k|^{2m}}$$

$$=\left|1-\sum_{\lambda=1}^{N} C_{\lambda}(z)\right|^{2}+\frac{1}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\left|\widehat{\ell}_{-k}\right|^{2}}{|k|^{2m}}=\left|1-\sum_{\lambda=1}^{N} C_{\lambda}(z)\right|^{2}$$

$$+\frac{1}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\left|\cos 2\pi(k,z)-\sum\limits_{\lambda=1}^{N} C_{\lambda}(z)\, e^{2\pi i(k,x^{(\lambda)})}\right|^{2}}{k^{2m}}=\left\|u(x)\,|\tilde{W}_{2}^{(m)}(T_{n})\right\|^{2}. \tag{22}$$

Let us prove the following lemma.

Lemma 1. *The square of the norm of function $u(x)$ in space $\tilde{W}_{2}^{(m)}(T_n)$ is:*

$$\left\|u(x)\,|\tilde{W}_{2}^{(m)}(T_{n})\right\|^{2}=\left|1-\sum_{\lambda=1}^{N} C_{\lambda}(z)\right|^{2}+\frac{1}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\left|\cos 2\pi(k,z)-\sum\limits_{\lambda=1}^{N} C_{\lambda}(z)\, e^{2\pi i(k,x^{(\lambda)})}\right|^{2}}{k^{2m}}. \tag{23}$$

Proof. Since equality (2) holds for all functions $f(x)\in\tilde{W}_{2}^{(m)}(T_n)$, it follows that the following equality holds for the norm of the function $u(x)$

$$\left\|u(x)\,|\tilde{W}_{2}^{(m)}(T_{n})\right\|^{2}=\left(\int_{T_n} u(x)\,dx\right)^{2}+\sum_{k_1\neq 0}|2\pi k_1|^{2m}\left|\hat{u}_{k_1}\right|^{2}, \tag{24}$$

where $k_1\in z$ and $\widehat{u}_{k_1}$ are the Fourier coefficients.
Thus, we calculate the norm of the following function

$$u(x)=1-\sum_{\lambda=1}^{N} C_{\lambda}(z)+\frac{1}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\widehat{\ell}_{-k}e^{-2\pi i(k,x)}}{|k|^{2m}}, \tag{25}$$

in the space $\tilde{W}_{2}^{(m)}(T_n)$ according to formula (24).

In (24) for each term we perform a separate calculation:

$$1.\left(\int_{T_n} u(x)\,dx\right)^{2}=\left(\int_{T_n}\left[1-\sum_{\lambda=1}^{N} C_{\lambda}(z)+\frac{1}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\widehat{\ell}_{-k}e^{-2\pi i(k,x)}}{|k|^{2m}}\right]dx\right)^{2}=$$

$$=\left(\left[1-\sum_{\lambda=1}^{N} C_{\lambda}(z)\right]\int_{T_n} dx+\frac{1}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\widehat{\ell}_{-k}\int_{T_N} e^{-2\pi i(k,x)}dx}{|k|^{2m}}\right)^{2}, \tag{26}$$

where $k=(k_1,k_2,\ldots,k_n)$ and $(k,x)=k_1x_1+k_2x_2+\cdots+k_nx_n$.
From (26) $\int_{T_n} e^{-2\pi i(k,x)}dx=\int_0^1\int_0^1\ldots\int_0^1 e^{-2\pi i(k,x)}dx$.
Since

$$\int_0^1 e^{-2\pi i k_1 x_1}dx_1=\frac{1}{-2\pi i k_1}e^{-2\pi i k_1 x_1}\Big|_0^1=\frac{1}{-2\pi i k_1}(1-1)=0,$$

then

$$\int_{T_n} e^{-2\pi i(k,x)}dx=0. \tag{27}$$

With (27) from (26) we obtain

$$\left(\int_{T_n} u(x)\,dx\right)^2 = \left[1 - \sum_{\lambda=1}^{N} C_\lambda(z)\right]^2 \tag{28}$$

2. Now we calculate the value of $\widehat{u}_{k_1}$:

$$\widehat{u}_{k_1} = \int_{T_n} u(x) e^{2\pi i(k_1,x)} dx = \int_{T_n} \left[1 - \sum_{\lambda=1}^{N} C_\lambda(z) + \frac{1}{(2\pi)^{2m}} \sum_{k\neq 0} \frac{\widehat{\ell}_{-k} e^{-2\pi i(k,x)}}{|k|^{2m}}\right] e^{2\pi i(k_1,x)} dx$$

$$= \left[1 - \sum_{\lambda=1}^{N} C_\lambda(z)\right] \int_{T_n} e^{2\pi i(k_1,x)} dx + \frac{1}{(2\pi)^{2m}} \sum_{k\neq 0} \frac{\widehat{\ell}_k \int_{T_n} e^{-2\pi i(k,x)} e^{2\pi i(k_1,x)} dx}{|k_1|^{2m}}, \tag{29}$$

where $k_1 = \left(k_1^{(1)}, k_1^{(2)}, \ldots, k_1^{(n)}\right)$ and $(k_1,x) = k_1^{(1)} x_1 + k_1^{(2)} x_2 + \cdots + k_1^{(n)} x_n$.

We calculate if

$$\int_{T_n} e^{-2\pi i(k,x)} e^{2\pi i(k_1,x)} dx = \int_{T_n} e^{2\pi i(k_1-k,x)} dx = \begin{cases} 1, & k_1 = k, \\ 0 & k_1 \neq k. \end{cases} \tag{30}$$

Bearing in mind (27) and (30), from (29) we obtain

$$\widehat{u}_k = \widehat{u}_{k_1} = \frac{1}{(2\pi)^{2m}} \frac{\widehat{\ell}_{k_1}}{|k_1|^{2m}}. \tag{31}$$

Substituting (28) and (31) into the right side of (24), we have

$$\left\| u(x) \,|\tilde{W}_2^{(m)}(T_n) \right\|^2 = \left|1 - \sum_{\lambda=1}^{N} C_\lambda(z)\right|^2 + \sum_{k\neq 0} (2\pi)^{2m} |k|^{2m} \frac{1}{(2\pi)^{4m}} \frac{\left|\widehat{\ell}_k\right|^2}{|k|^{4m}}. \tag{32}$$

Thus, after some cancellations, it follows from (32) that

$$\left\| u(x) \,|\tilde{W}_2^{(m)}(T_n) \right\|^2 = \left|1 - \sum_{\lambda=1}^{N} C_\lambda(z)\right|^2 + \frac{1}{(2\pi)^{2m}} \sum_{k\neq 0} \frac{\left|\widehat{\ell}_k\right|^2}{|k|^{2m}}. \tag{33}$$

The proof of the lemma follows from (33) considering (19).

Comparing the right parts of (21) and (33), we obtain

$$\left\| \ell_N(x) |\tilde{W}_2^{(m)^*}(T_n) \right\| \leq \left\| u(x) |\tilde{W}_2^{(m)}(T_n) \right\|. \tag{34}$$

Taking into account (23) for the right part of (22), we obtain

$$< \ell_N(x), u(x) > = \left\| u(x) |\tilde{W}_2^{(m)}(T_n) \right\| \cdot \left\| u(x) |\tilde{W}_2^{(m)}(T_n) \right\|. \tag{35}$$

For the error of the interpolation formula (1) on function $u(x)$, we have:

$$|< \ell_N(x), u(x) >| \leq \left\| \ell_N(x) |\tilde{W}_2^{(m)^*}(T_n) \right\| \cdot \left\| u(x) |\tilde{W}_2^{(m)}(T_n) \right\|. \tag{36}$$

Substituting the right side of (34) into the left side of (35), we have

$$\left\| u(x)|\tilde{W}_2^{(m)}(T_n) \right\| \cdot \left\| u(x)|\tilde{W}_2^{(m)}(T_n) \right\| \leq \left\| \ell_N(x)|\tilde{W}_2^{(m)^*}(T_n) \right\| \cdot \left\| u(x)|\tilde{W}_2^{(m)}(T_n) \right\|. \quad (37)$$

After cancellations, it follows from (37) that

$$\left\| \ell_N(x)|\tilde{W}_2^{(m)^*}(T_n) \right\| \geq \left\| u(x)|\tilde{W}_2^{(m)}(T_n) \right\|. \quad (38)$$

From (34) and (38) we obtain

$$\left\| \ell_N(x)|\tilde{W}_2^{(m)^*}(T_n) \right\| = \left\| u(x)|\tilde{W}_2^{(m)}(T_n) \right\|. \quad (39)$$

With (39), we can write the following:

$$< \ell_N(x), u(x) >=< u(x), f(x) > . \quad (40)$$

Equality (40) testifies to the existence of $u(x) \in \tilde{W}_2^{(m)}(T_n)$ and thus it is an extremal function for the interpolation formula (1), i.e.

$$u(x) = \psi_\ell(x) \in \tilde{W}_2^{(m)}(T_n). \quad (41)$$

Then (40) takes the following form

$$< \ell_N(x), \psi_\ell(x) >=< \psi_\ell(x), f(x) > . \quad (42)$$

This means that all conditions of the Riesz theorem [12] are satisfied.
Thus, with (22), (39), (41), and conditions of the lemma, we obtain

$$\left\| \ell_N(x)|\tilde{W}_2^{(m)^*}(T_n) \right\|^2 = \left| 1 - \sum_{\lambda=1}^{N} C_\lambda(z) \right|^2 + \frac{1}{(2\pi)^{2m}} \sum_{k \neq 0} \frac{\left| \cos 2\pi(k,z) - \sum\limits_{\lambda=1}^{N} C_\lambda(z) e^{2\pi i (k, x^{(\lambda)})} \right|^2}{k^{2m}}, \quad (43)$$

which is what was required to be proved.

The following theorem is valid.

Theorem 2. *Equalities (22), (23), and (40) prove that*

$$u(x) = 1 - \sum_{\lambda=1}^{N} C_\lambda(z) + \frac{1}{(2\pi)^{2m}} \sum_{k \neq 0} \frac{\widehat{\ell}_{-k} e^{-2\pi i (k,x)}}{|k|^{2m}},$$

is an extremal function for the interpolation formula (1) and $u(x) = \psi_\ell(x) \in \tilde{W}_2^{(m)}(T_n)$.

Based on Theorem 1, the error functional (5) of the interpolation formula (1) for the functions from class $W_2^{(m)}(T_n)$ has the following estimate:

$$|< \ell_N, f(x) >| \leq \left\{ \left| \widehat{f}_0 \right|^2 + \sum_{k \neq 0} \left| \widehat{f}_k \right|^2 |2\pi k|^{2m} \right\}^{1/2} \cdot \left\{ \left| \widehat{\ell}_0 \right|^2 + \sum_{k \neq 0} \frac{\left| \widehat{\ell}_k \right|^2}{|2\pi k|^{2m}} \right\}^{1/2}. \quad (44)$$

CONCLUSION

The quality of the interpolation formula is characterized by the norm of the error functional.

It is a function of unknown coefficients and nodes. Therefore, for computational practice, it is appropriate to be able to calculate the norm of the error functional and evaluate it. Finding the minimum norm of the error functional with respect to $C_\lambda(z)$ and $x^{(\lambda)}$ is the task of investigating the extrema of multivariable functions.

The values of $C_\lambda(z)$ and $x^{(\lambda)}$, realizing this minimum, determine the optimal formula. Thus, the optimal interpolation formula is the formula in which, for a given number of nodes N, the error functional has the minimum norm.

For practice, we need to solve the following problems:

1. Calculation of the norm of the error functional of interpolation formulas over space B.

2. Construction of the optimal interpolation formula, i.e. interpolation formula with the minimum norm of the error functional over B. In this paper, we solve Problem 1 for an interpolation formula of the form (1), i.e., calculating norm $\left\|\ell_N^{(\alpha)}(x)\,|\tilde{W}_2^{(m)*}(T_n)\right\|$ of error functional $\ell_N^{(\alpha)}(x)$ of the interpolation formula (1). To find the norm of the error functional (5) of the interpolation formula of the form (1) in space $\tilde{W}_2^{(m)*}(T_n)$, its extremal function is used.

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