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Algorithm for Finding the Norm of the Error Functional of a Hermite Type Cubature Formula in the Sobolev Space $\tilde{W}_2^{(m)}(T_n)$

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Abstract. The modern formulation of the problem of optimization of approximate integration formulas consists in minimizing the norm of the formula of the error functional on chosen normed spaces. When studying the best formulas for approximate integration, the first question that arises: "Is there such formulas exist?". These research methods have not yet been effectively applied in the study of cubature formulas in $\tilde{W}_2^{(m)}(T_n)$, which are multidimensional analogue of the space $W_2^{(m)}(T_1)$.

INTRODUCTION

In [1, 2, 3, 4, 5], the problem of optimality of cubature formulas with respect to some specific space was studied. Most of them are considered in the Sobolev space [1]. Multidimensional cubature formulas differ from one-dimensional ones in two ways:

1) the forms of multidimensional domains of integration are infinitely diverse;

2) the number of integration nodes grows rapidly with increasing space dimension.

Problem 2) requires special attention to the construction of the most economical formulas.

There are various principles for constructing cubature formulas - the classical principle, given in [6, 7, 8], and the theoretical-functional principle in the theory of approximate integration (see [9, 10, 11, 12, 13, 14, 15]), and in variational methods for approximation functions (see [16, 17, 18]).

The second principle was first considered for quadrature formulas by A.Sard [19], S.M.Nikolskii [20], and cubature formulas - by S.L.Sobolev [1].

In this article we apply the theoretical-functional approach, so below the necessary information on this approach is given.

Consider a cubature formula of the following form

$$\int_{\Omega} f(x)\,dx \approx \sum_{\lambda=1}^{N} c_\lambda f\left(x^{(\lambda)}\right), \tag{1}$$

where Ω is a certain domain in Euclidean space R^n, c_λ are the coefficients (or weights), and $x^{(\lambda)} = \left(x_1^{(\lambda)}, x_2^{(\lambda)}, ..., x_n^{(\lambda)}\right)$ are the nodes of the cubature formula (1). The *error* of the cubature formula (1) is the remainder

$$< \ell_N, f >= \int_{\Omega} f(x)\,dx - \sum_{\lambda=1}^{N} c_\lambda f\left(x^{(\lambda)}\right) = \int_{R^n} \ell_N(x) f(x)\,dx, \tag{2}$$

where

$$\ell_N(x) = \varepsilon_\Omega(x) - \sum_{\lambda=1}^{N} c_\lambda \delta\left(x - x^{(\lambda)}\right), \tag{3}$$

$\varepsilon_\Omega(x) = \begin{cases} 1, x \in \Omega \\ 0, x \notin \Omega \end{cases}$, $\delta(x)$ is the Dirac delta function, N is the number of nodes. In (2) and (3), $\ell_N(x)$ is called the *error functional* of the cubature formula (1).

Let the function $f(x)$ belong to some Banach space B, then $\ell_N(x)$ is a functional from the conjugate space B^*. It is assumed that this space is compactly embedded in the space of continuous functions defined in domain Ω:

$$B \to C(\Omega). \tag{4}$$

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The functional $\ell_N(x)$ given in B^* is linear and continuous, and due to condition (4) it is bounded, i.e.

$$|<\ell_N, f>| \leq \|\ell_N|B^*\| \cdot \|f|B\|. \tag{5}$$

Estimate (5) shows that the quality of the cubature formula is characterized by the norm of the error functional, determined by the following formula

$$\|\ell_N|B^*\| = \sup_{f,\ \|f|B\| \neq 0} \frac{|<\ell_N, f>|}{\|f|B\|}, \tag{6}$$

and is the function of unknown coefficients and nodes. Therefore, for computational practice, it is appropriate to be able to calculate the norm of the error functional (6) and estimate it. Calculating the minimum of the norm of the error functional with respect to c_λ and $x^{(\lambda)}$ is the task of investigating a multivariable function on an extremum. The values c_λ and $x^{(\lambda)}$ realizing this minimum, determine the optimal formula. Thus, the *optimal cubature formula* is the formula in which, for a given number of nodes, the error functional has the least norm.

STATEMENT OF THE PROBLEM

Many publications, for example [21, 22, 23], are devoted to cubature formulas, which include the values of derivatives of integrable functions. When besides the values of the function at the nodes of the cubature formulas, the values of its derivatives of certain orders are known, then it is natural that with the correct use of these data, a more accurate result can be expected than in the case of using only the values of the functions. In this regard, consider a cubature formula of the following form

$$\int_{T_n} f(x)\,dx \approx \sum_{|\alpha| \leq q} \sum_{\lambda=1}^{N} (-1)^{\alpha} c_\lambda^{\alpha} f^{(\alpha)}\left(x^{(\lambda)}\right), \tag{7}$$

in the S.L.Sobolev space $\tilde{W}_2^{(m)}(T_n)$. Here, c_λ^α and $x^{(\lambda)}$ are the *coefficients* and *nodes* of the cubature formula, respectively (7), $f(x) \in \tilde{W}_2^{(m)}(T_n)$, T_n is the n-dimensional torus and α is the order of generalized derivatives and $0 \leq q \leq m-1$.

Definition 1. The set $T_n = \{x = (x_1, x_2, ... x_n); x_k = \{t_k\}, t_k \in R\}$, where $\{t_k\} = t_k - [t_k]$, i.e. fractional part of t_k, is called an *n-dimensional torus* T_n.

Definition 2. Space $\tilde{W}_2^{(m)}(T_n)$ is defined as the space of functions given on n- dimensional torus T_n and having all generalized derivatives of order m summable with a square in norm [1]

$$\left\| f(x) | \tilde{W}_2^{(m)}(T_n) \right\|^2 = \left(\int_{T_n} f(x)\,dx \right)^2 + \sum_{k \neq 0} |2\pi k|^{2m} \left| \widehat{f}_k \right|^2, \tag{8}$$

with the inner product

$$< f(x), \phi(x) >_{\tilde{W}_2^{(m)}(T_n)} = \int_{T_n} \sum_{|\alpha| \leq q} D^\alpha f(x) D^\alpha \phi(x)\,dx + \left(\int_{T_n} f(x)\,dx \right) \left(\int_{T_n} \varphi(x)\,dx \right), \tag{9}$$

where $\widehat{f}_k$ are the Fourier coefficients, i.e. $\widehat{f}_k = \int_{T_n} f(x) e^{2\pi i (k,x)} dx$.

Since in this article we consider the space of functions defined on the n- dimensional torus T_n, i.e. on a manifold that is not Euclidean space (we assume that the Euclidean structure is locally maintained), it is necessary to discuss the question of the invariance of the introduced norm under orthogonal transformations. The difficulties that arise are related to the fact that the space of periodic functions on R^n is not invariant with respect to rotations. Therefore, it is necessary to use some information from the differential geometry of manifolds. It is known from [24] , that the set of derivatives of some function at a given point $x \in T_n$ forms a tangent space $T(x)$. The space of derivatives of order m at point x forms m symmetric power of the introduced tangent space and is denoted by $S(T)^m$.

It is required that the introduced norm at each point be invariant under orthogonal transformations of the tangent space $T(x)$. It is easy to see that norm (8) satisfies this requirement.

The remainder of the integral and the cubature sum, i.e.

$$\int\limits_{T_n} f(x)\,dx - \sum_{|\alpha|\leq q}\sum_{\lambda=1}^{N}(-1)^{\alpha}c_{\lambda}^{\alpha}f^{(\alpha)}(x^{(\lambda)})$$

$$=\int\limits_{T_n}\left[\varepsilon_{T_n}(x) - \sum_{|\alpha|\leq q}\sum_{\lambda=1}^{N}c_{\lambda}^{\alpha}\delta^{(\alpha)}(x-x^{(\lambda)})\right]f(x)\,dx = <\ell_N^{(\alpha)}(x), f(x)>$$

is called the *error of the cubature formula* (7), and this remainder corresponds to the generalized function

$$\ell_N^{(\alpha)}(x) = \varepsilon_{T_n}(x) - \sum_{|\alpha|\leq q}\sum_{\lambda=1}^{N}c_{\lambda}^{\alpha}\delta^{(\alpha)}(x-x^{(\lambda)}), \tag{10}$$

called the *error functional of the cubature formula* (7). Here $\varepsilon_{T_n}(x)$ is the characteristic function T_n.

The task of constructing the best cubature formulas over Sobolev space $\tilde{W}_2^{(m)}(T_n)$ is the calculation of the following quantity:

$$\left\|\ell_N^{(\alpha)}(x)\,|\tilde{W}_2^{(m)*}(T_n)\right\| = \inf_{c_{\lambda}^{\alpha},x^{(\lambda)}}\sup_{\|f(x)\|\neq 0}\frac{\left|<\ell_N^{(\alpha)}(x), f(x)>\right|}{\left\|f(x)\,|\tilde{W}_2^{(m)}(T_n)\right\|}, \tag{11}$$

where $\tilde{W}_2^{(m)*}(T_n)$ is the conjugate space to space $\tilde{W}_2^{(m)}(T_n)$. To estimate the error of the cubature formula, it is necessary to solve the following problem.

Problem 1. Find the norm of the error functional (10) of the given cubature formula.

First, we must calculate norm $\left\|\ell_N^{(\alpha)}(x)\,|\tilde{W}_2^{(m)*}(T_n)\right\|$ of the error functional $\ell_N^{(\alpha)}(x)$ in the space $\tilde{W}_2^{(m)}(T_n)$, and then, if necessary, to construct the best cubature formula, by varying c_{λ}^{α} and $x^{(\lambda)}$ $\lambda=\overline{1,N}$, and solving the following problem.

Problem 2. Find values c_{λ}^{α} and $x^{(\lambda)}$, such that equality (11) is satisfied.

NORM OF THE ERROR FUNCTIONAL OF CUBATURE FORMULAS IN THE SOBOLEV SPACE $\tilde{W}_2^{(m)}(T_n)$

In this article we solve Problem 1 for a cubature formula of the form (7), i.e., calculation of the norm $\left\|\ell_N^{(\alpha)}(x)\,|\tilde{W}_2^{(m)*}(T_n)\right\|$ of the error functional $\ell_N^{(\alpha)}(x)$ of the cubature formula (7) with the setting of derivatives. Its extremal function is used to calculate the norm of the error functional (10) in the space $\tilde{W}_2^{(m)*}(T_n)$.

Definition 3. Function $\psi_{\ell}(x)$ is called an *extremal function* of functional $\ell_N^{(\alpha)}$, if the following equality holds

$$<\ell_N^{(\alpha)}(x), \psi_{\ell}(x)> = \left\|\ell_N^{(\alpha)}(x)\,|\tilde{W}_2^{(m)*}(T_n)\right\|\cdot\left\|\psi_{\ell}|\tilde{W}_2^{(m)}(T_n)\right\|.$$

Since the space $\tilde{W}_2^{(m)}(T_n)$ with the inner product (9) becomes Hilbert, then based on the Riesz theorem on the general form of a linear continuous functional [1], there is a unique function $\psi_{\ell}(x)\in\tilde{W}_2^{(m)}(T_n)$ for which

$$<\ell_N^{(\alpha)}(x), f(x)> = <\psi_{\ell}(x), f(x)>_{\tilde{W}_2^{(m)}(T_n)}$$

and

$$\left\|\ell_N^{(\alpha)}(x)\,|\tilde{W}_2^{(m)*}(T_n)\right\| = \left\|\psi_{\ell}(x)\,|\tilde{W}_2^{(m)}(T_n)\right\|.$$

Hence, in particular, for $f(x)=\psi_\ell(x)$, we have

$$<\ell_N^{(\alpha)}(x),\psi_\ell(x)>=<\psi_\ell(x),\psi_\ell(x)>_{\tilde W_2^{(m)}(T_n)}=\left\|\psi_\ell(x)\,|\tilde W_2^{(m)}(T_n)\right\|^2$$

$$=\left\|\psi_\ell(x)\,|\tilde W_2^{(m)}(T_n)\right\|\cdot\left\|\ell_N^{(\alpha)}(x)\,|\tilde W_2^{(m)*}(T_n)\right\|=\left\|\ell_N^{(\alpha)}(x)\,|\tilde W_2^{(m)*}(T_n)\right\|^2. \tag{12}$$

The following theorem is true.

Theorem 1. *The square of the norm of the error functional (10) of a cubature formula of form (7) over space* $\tilde W_2^{(m)}(T_n)$ *is*

$$\left\|\ell_N^{(\alpha)}(x)\,|\tilde W_2^{(m)*}(T_n)\right\|^2=\left|1-\sum_{|\alpha|\le q}\sum_{\lambda=1}^{N}c_\lambda^\alpha\right|^2$$

$$+\frac{1}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\left|\sum\limits_{|\alpha|\le q}\sum\limits_{\lambda=1}^{N}c_\lambda^\alpha(2\pi i)^\alpha(\prod\limits_{j=1}^{n}k_j)^\alpha e^{2\pi i(k,x^{(\lambda)})}\right|^2}{k^{2m}}, \tag{13}$$

where c_λ^α are the coefficients, $x^{(\lambda)}$ are the nodes of the cubature formula (7).

Proof. It is known that the following equality is true for the function $f(x)\in\tilde W_2^{(m)}(T_n)$:

$$f(x)=\sum_k\widehat{f}_k e^{-2\pi i(k,x)},$$

where $\widehat{f}_k=<f(x),e^{2\pi i(k,x)}>=\int\limits_{T_n}f(x)\,e^{2\pi i(k,x)}dx$, i.e. Fourier coefficients.

Thus, we have

$$<\ell_N^{(\alpha)},f(x)>=<\ell_N^{(\alpha)}(x),\sum_k\widehat{f}_k e^{-2\pi i(k,x)}>$$

$$=\sum_k\widehat{f}_k<\ell_N^{(\alpha)}(x),e^{-2\pi i(k,x)}>=\sum_k\widehat{f}_k\widehat{\ell}_{-k}^{(\alpha)}=\widehat{f}_0\widehat{\ell}_0^{(\alpha)}+\sum_{k\neq0}\widehat{f}_k\widehat{\ell}_{-k}^{(\alpha)}. \tag{14}$$

Here $\widehat{\ell}_0^{(\alpha)}=\int\limits_{T_n}\ell_N^{(\alpha)}(x)\,dx$, $\widehat{\ell}_{-k}^{(\alpha)}=\int\limits_{T_n}\ell_N^{(\alpha)}(x)\,e^{-2\pi i(k,x)}dx$.

Applying the Cauchy-Schwarz inequality to the right side of (14) and taking into account (8), we obtain the following estimate

$$\left|<\ell_N^{(\alpha)},f(x)>\right|=\left|\widehat{f}_0\widehat{\ell}_0^{(\alpha)}+\sum_{k\neq0}\widehat{f}_k\widehat{\ell}_{-k}^{(\alpha)}\right|\le|\widehat{f}_0\widehat{\ell}_0^{(\alpha)}|$$

$$+\left|\sum_{k\neq0}\widehat{f}_k\widehat{\ell}_{-k}^{(\alpha)}|2\pi k|^m\cdot\frac{1}{|2\pi k|^m}\right|\le|\widehat{f}_0\widehat{\ell}_0^{(\alpha)}|+\sum_{k\neq0}\left|\widehat{f}_k\right|\left|\widehat{\ell}_{-k}^{(\alpha)}\right||2\pi k|^m\frac{1}{|2\pi k|^m}$$

$$\le\left\{\left|\widehat{f}_0\right|^2+\sum_{k\neq0}\left|\widehat{f}_k\right|^2|2\pi k|^{2m}\right\}^{1/2}\cdot\left\{\left|\widehat{\ell}_0^{(\alpha)}\right|^2+\sum_{k\neq0}\frac{\left|\widehat{\ell}_{-k}^{(\alpha)}\right|^2}{|2\pi k|^{2m}}\right\}^{1/2}$$

$$= \left\| f(x) \,|\tilde{W}_2^{(m)}(T_n) \right\| \cdot \left\{ \left| \widehat{\ell}_0^{(\alpha)} \right|^2 + \frac{1}{(2\pi)^{2m}} \sum_{k \neq 0} \frac{\left| \widehat{\ell}_k^{(\alpha)} \right|^2}{|k|^{2m}} \right\}^{1/2}. \tag{15}$$

With (8), (6), and (15), we obtain

$$\left\| \ell_N^{(\alpha)}(x) \,|\tilde{W}_2^{(m)*}(T_n) \right\| \leq \left\{ \left| \widehat{\ell}_0^{(\alpha)} \right|^2 + \frac{1}{(2\pi)^{2m}} \sum_{k \neq 0} \frac{\left| \widehat{\ell}_k^{(\alpha)} \right|^2}{|k|^{2m}} \right\}^{1/2}, \tag{16}$$

We calculate $\widehat{\ell}_k^{(\alpha)}$:

$$\widehat{\ell}_k^{(\alpha)} = \int_{T_n} \ell_N^{(\alpha)}(x) e^{2\pi i(k,x)} dx = < \ell_N^{(\alpha)}(x), e^{2\pi i(k,x)} >$$

$$= < \varepsilon_{T_n}(x) - \sum_{|\alpha| \leq q} \sum_{\lambda=1}^{N} c_\lambda^\alpha \delta^{(\alpha)}(x - x^{(\lambda)}), e^{2\pi i(k,x)} >$$

$$= \int_{T_n} e^{2\pi i(k,x)} dx - \sum_{|\alpha| \leq q} \sum_{\lambda=1}^{N} c_\lambda^\alpha < \delta^{(\alpha)}(x - x^{(\lambda)}), e^{2\pi i(k,x)} >$$

$$= \sum_{|\alpha| \leq q} \sum_{\lambda=1}^{N} c_\lambda^\alpha (2\pi i)^\alpha (\prod_{j=1}^{n} k_j)^\alpha e^{2\pi i(k,x^{(\lambda)})},$$

i.e.

$$\widehat{\ell}_k^{(\alpha)} = \varepsilon_{T_n}(x) - \sum_{|\alpha| \leq q} \sum_{\lambda=1}^{N} c_\lambda^\alpha (2\pi i)^\alpha (\prod_{j=1}^{n} k_j)^\alpha e^{2\pi i(k,x^{(\lambda)})} \tag{17}$$

Hence, for $k = 0$ we have

$$\widehat{\ell}_0^{(\alpha)} = 1 - \sum_{|\alpha| \leq q} \sum_{\lambda=1}^{N} c_\lambda^\alpha, \tag{18}$$

So, with (17) and (18) from (16), we obtain

$$\left\| \ell_N^{(\alpha)}(x) \,|\tilde{W}_2^{(m)*}(T_n) \right\|^2 \leq \left| 1 - \sum_{|\alpha| \leq q} \sum_{\lambda=1}^{N} c_\lambda^\alpha \right|^2$$

$$+ \frac{1}{(2\pi)^{2m}} \sum_{k \neq 0} \frac{\left| \sum\limits_{|\alpha| \leq q} \sum\limits_{\lambda=1}^{N} c_\lambda^\alpha (2\pi i)^\alpha (\prod\limits_{j=1}^{n} k_j)^\alpha e^{2\pi i(k,x^{(\lambda)})} \right|^2}{k^{2m}}. \tag{19}$$

There is a function from $\tilde{W}_2^{(m)}(T_n)$ such that inequality (19) reaches equality.

Indeed, consider the following function $u(x)$:

$$u(x) = 1 - \sum_{|\alpha| \leq q} \sum_{\lambda=1}^{N} c_\lambda^\alpha + \frac{1}{(2\pi)^{2m}} \sum_{k \neq 0} \frac{\widehat{\ell}_{-k}^{(\alpha)} e^{-2\pi i(k,x)}}{|k|^{2m}}.$$

Calculating the value of the functional $\ell_N^{(\alpha)}(x)$ on function $u(x)$, we obtain

$$< \ell(x), u(x) >=< \ell_N^{(\alpha)}(x), 1 - \sum_{|\alpha|\le q}\sum_{\lambda=1}^{N} c_\lambda^\alpha + \frac{1}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\widehat{\ell}_{-k}^{(\alpha)} e^{-2\pi i(k,x)}}{|k|^{2m}} >$$

$$=< \ell_N^{(\alpha)}(x), 1 - \sum_{|\alpha|\le q}\sum_{\lambda=1}^{N} c_\lambda^\alpha > + < \ell_N^{(\alpha)}(x), \frac{1}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\widehat{\ell}_{-k}^{(\alpha)} e^{-2\pi i(k,x)}}{|k|^{2m}} >$$

$$= \left[1 - \sum_{|\alpha|\le q}\sum_{\lambda=1}^{N} c_\lambda^\alpha\right]\int_{T_n}\ell_N^{(\alpha)}(x)\,dx + \frac{1}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\widehat{\ell}_{-k}^{(\alpha)} < \ell_N^{(\alpha)}(x), e^{-2\pi i(k,x)} >}{|k|^{2m}}$$

$$= \left[1 - \sum_{|\alpha|\le q}\sum_{\lambda=1}^{N} c_\lambda^\alpha\right]\widehat{\ell}_0^{(\alpha)} + \frac{1}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\widehat{\ell}_{-k}^{(\alpha)}\widehat{\ell}_{-k}^{(\alpha)}}{|k|^{2m}}$$

$$= \left|1 - \sum_{|\alpha|\le q}\sum_{\lambda=1}^{N} c_\lambda^\alpha\right|^2 + \frac{1}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\left|\widehat{\ell}_{-k}^{(\alpha)}\right|^2}{|k|^{2m}} = \left|1 - \sum_{|\alpha|\le q}\sum_{\lambda=1}^{N} c_\lambda^\alpha\right|^2$$

$$+\frac{1}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\left|\sum\limits_{|\alpha|\le q}\sum\limits_{\lambda=1}^{N} c_\lambda^\alpha (2\pi i)^\alpha (\prod\limits_{j=1}^{n} k_j)^\alpha e^{2\pi i(k,x^{(\lambda)})}\right|^2}{k^{2m}} = \left\|u(x)\,|\tilde{W}_2^{(m)}(T_n)\right\|^2. \tag{20}$$

Let us prove the following lemma.

Lemma 1. *The square of the norm of function $u(x)$ in space $\tilde{W}_2^{(m)}(T_n)$ is:*

$$\left\|u(x)\,|\tilde{W}_2^{(m)}(T_n)\right\|^2 = \left|1 - \sum_{|\alpha|\le q}\sum_{\lambda=1}^{N} c_\lambda^\alpha\right|^2 +$$

$$+\frac{1}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\left|\sum\limits_{|\alpha|\le \ell}\sum\limits_{\lambda=1}^{N} c_\lambda^\alpha (2\pi i)^\alpha (\prod\limits_{j=1}^{n} K_j)^\alpha e^{2\pi i(k,x^{(\lambda)})}\right|^2}{|k|^{2m}}. \tag{21}$$

Proof. Since equality (8) holds for all functions $f(x) \in \tilde{W}_2^{(m)}(T_n)$, it follows that the norm of the function $u(x)$ also satisfies the following equality

$$\left\|u(x)\,|\tilde{W}_2^{(m)}(T_n)\right\|^2 = \left(\int_{T_n} u(x)\,dx\right)^2 + \sum_{k_1\neq 0}|2\pi k_1|^{2m}\left|\hat{u}_{k_1}\right|^2, \tag{22}$$

where $k_1 \in z$ and $\widehat{u}_{k_1}$ are the Fourier coefficients.
Thus, we calculate the norm of the function

$$u(x) = \left[1 - \sum_{|\alpha|\le q}\sum_{\lambda=1}^{N} c_\lambda^\alpha\right] + \frac{1}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\widehat{\ell}_k^{(\alpha)} e^{-2\pi i(k,x)}}{|k|^{2m}}, \tag{23}$$

in the space $\tilde{W}_2^{(m)}(T_n)$ according to formula (22).
In (22), we perform a separate calculation for each term:

$$1.\left(\int\limits_{T_n} u(x)\,dx\right)^2 = \left(\int\limits_{T_n}\left[1-\sum_{|\alpha|\le q}\sum_{\lambda=1}^{N} c_\lambda^\alpha + \frac{1}{(2\pi)^{2m}}\sum_{k\ne 0}\frac{\widehat{\ell}_k^{(\alpha)}e^{-2\pi i(k,x)}}{|k|^{2m}}\right]dx\right)^2$$

$$=\left(\left[1-\sum_{|\alpha|\le q}\sum_{\lambda=1}^{N} c_\lambda^\alpha\right]\int\limits_{T_n} dx + \frac{1}{(2\pi)^{2m}}\sum_{k\ne 0}\frac{\widehat{\ell}_k^{(\alpha)}\int\limits_{T_n} e^{-2\pi i(k,x)}dx}{|k|^{2m}}\right)^2, \tag{24}$$

where $k=(k_1,k_2,\ldots,k_n)$ and $(k,x)=k_1x_1+k_2x_2+...+k_nx_n$.
From (24) $\int\limits_{T_n} e^{-2\pi ikx}dx = \int\limits_0^1\int\limits_0^1\ldots\int\limits_0^1 e^{-2\pi ik_1x_1}dx.$
since
$\int\limits_0^1 e^{-2\pi ik_1x_1}dx_1 = \frac{1}{-2\pi ik_1}e^{-2\pi ik_1x_1}\Big|_0^1 = \frac{1}{-2\pi ik_1}(1-1)=0,$
then

$$\int\limits_{T_n} e^{-2\pi i(k,x)}dx = 0. \tag{25}$$

With (25) from (24), we obtain

$$\left(\int\limits_{T_n} u(x)\,dx\right)^2 = \left[1-\sum_{|\alpha|\le q}\sum_{\lambda=1}^{N} c_\lambda^\alpha\right]^2. \tag{26}$$

2. Now we calculate $\widehat{u}_{k_1}$:

$$\widehat{u}_{k_1} = \int\limits_{T_n} u(x)e^{2\pi i(k_1,x)}dx = \int\limits_{T_n}\left[1-\sum_{|\alpha|\le q}\sum_{\lambda=1}^{N} c_\lambda^\alpha + \frac{1}{(2\pi)^{2m}}\sum_{k\ne 0}\frac{\widehat{\ell}_k^{(\alpha)}e^{-2\pi i(k,x)}}{|k|^{2m}}\right]e^{2\pi i(k_1,x)}dx$$

$$=\left[1-\sum_{|\alpha|\le q}\sum_{\lambda=1}^{N} c_\lambda^\alpha\right]\int\limits_{T_n} e^{2\pi i(k_1,x)}dx + \frac{1}{(2\pi)^{2m}}\sum_{k\ne 0}\frac{\widehat{\ell}_k^{(\alpha)}\int\limits_{T_n} e^{-2\pi i(k,x)}e^{2\pi i(k_1,x)}dx}{|k_1|^{2m}}, \tag{27}$$

where $k_1=\left(k_1^{(1)},k_1^{(2)},\ldots,k_1^{(n)}\right)$ and $(k_1,x)=k_1^{(1)}x_1+k_1^{(2)}x_2+\cdots+k_1^{(n)}x_n$.
We calculate - if

$$\int\limits_{T_n} e^{-2\pi i(k,x)}e^{2\pi i(k_1,x)}dx = \int\limits_{T_n} e^{2\pi i(k_1-k,x)}dx = \begin{cases}1, & k_1=k\\ 0 & k_1\ne k\end{cases}. \tag{28}$$

With (25) and (28) from (27), we obtain

$$\widehat{u}_k = \widehat{u}_{k_1} = \frac{1}{(2\pi)^{2m}}\frac{\widehat{\ell}_{k_1}^{(\alpha)}}{|k_1|^{2m}}. \tag{29}$$

Substituting (26) and (29) into the right side of (22), we have

$$\left\|u(x)\,|\tilde{W}_2^{(m)}(T_n)\right\|^2 = \left|1-\sum_{|\alpha|\le q}\sum_{\lambda=1}^{N} c_\lambda^\alpha\right|^2 + \sum_{k\ne 0}(2\pi)^{2m}|k|^{2m}\frac{1}{(2\pi)^{4m}}\frac{\left|\widehat{\ell}_k^{(\alpha)}\right|^2}{|k|^{4m}}. \tag{30}$$

Thus, after some cancellations, it follows from (30) that

$$\left\| u(x)\,|\tilde{W}_2^{(m)}(T_n) \right\|^2 = \left| 1 - \sum_{|\alpha|\le q} \sum_{\lambda=1}^{N} c_\lambda^\alpha \right|^2 + \frac{1}{(2\pi)^{2m}} \sum_{k\neq 0} \frac{\left| \widehat{\ell}_k^{(\alpha)} \right|^2}{|k|^{2m}}. \tag{31}$$

With (17) from (31) follows the proof of Lemma 1.

Lemma 2. *The following equality is true*

$$\left\| \ell_N^{(\alpha)}(x)\,|\tilde{W}_2^{(m)^*}(T_n) \right\| = \left\| u(x)\,|\tilde{W}_2^{(m)}(T_n) \right\|.$$

Proof. Comparing the right sides of (19) and (31), we obtain

$$\left\| \ell_N^{(\alpha)}(x)\,|\tilde{W}_2^{(m)^*}(T_n) \right\| \le \left\| u(x)\,|\tilde{W}_2^{(m)}(T_n) \right\|. \tag{32}$$

Taking into account (21) for the right parts of (20), we obtain

$$< \ell_N^{(\alpha)}(x)\,, u(x) >= \left\| u(x)\,|\tilde{W}_2^{(m)}(T_n) \right\| \cdot \left\| u(x)\,|\tilde{W}_2^{(m)}(T_n) \right\|. \tag{33}$$

For the error of the cubature formula (7) on function u(x), the following is true:

$$\left| < \ell_N^{(\alpha)}(x)\,, u(x) > \right| \le \left\| \ell_N^{(\alpha)}(x)\,|\tilde{W}_2^{(m)^*}(T_n) \right\| \cdot \left\| u(x)\,|\tilde{W}_2^{(m)}(T_n) \right\|. \tag{34}$$

Substituting the right side of (33) into the left side of (34) we obtain

$$\left\| u(x)\,|\tilde{W}_2^{(m)}(T_n) \right\| \cdot \left\| u(x)\,|\tilde{W}_2^{(m)}(T_n) \right\| \le \left\| \ell_N^{(\alpha)}(x)\,|\tilde{W}_2^{(m)^*}(T_n) \right\| \cdot \left\| u(x)\,|\tilde{W}_2^{(m)}(T_n) \right\|. \tag{35}$$

After cancellations, it follows from (35) that

$$\left\| \ell_N^{(\alpha)}(x)\,|\tilde{W}_2^{(m)^*}(T_n) \right\| \ge \left\| u(x)\,|\tilde{W}_2^{(m)}(T_n) \right\|. \tag{36}$$

From (32) and (36) we obtain

$$\left\| \ell_N^{(\alpha)}(x)\,|\tilde{W}_2^{(m)^*}(T_n) \right\| = \left\| u(x)\,|\tilde{W}_2^{(m)}(T_n) \right\|, \tag{37}$$

which is what was required to be proved.

If (37) is taken into account, the following can be written:

$$< \ell_N^{(\alpha)}(x)\,, u(x) >=< u(x), f(x) > . \tag{38}$$

Equality (38) testifies to the existence of $u(x) \in \tilde{W}_2^{(m)}(T_n)$ and thus it is an extremal function for the cubature formula (7), i.e.

$$u(x) = \psi_\ell(x) \in \tilde{W}_2^{(m)}(T_n). \tag{39}$$

Then (38) takes the following form

$$< \ell_N^{(\alpha)}(x)\,, \psi_\ell(x) >=< \psi_\ell(x), f(x) > . \tag{40}$$

This means that all the conditions of the Riesz theorem [1] are met.

Thus, taking into account (20), (37), (39) and the conditions of the lemma, we obtain

$$\left\| \ell_N^{(\alpha)}(x)\,|\tilde{W}_2^{(m)^*}(T_n) \right\|^2 = \left| 1 - \sum_{|\alpha|\le \ell} \sum_{\lambda=1}^{N} c_\lambda^\alpha \right|^2$$

$$+\frac{1}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\left|1-\sum\limits_{|\alpha|\leq \ell}\sum\limits_{\lambda=1}^{N}c_{\lambda}^{\alpha}(2\pi i)^{\alpha}(\prod\limits_{j=1}^{n}K_{j})^{\alpha}e^{2\pi i(k,x^{(\lambda)})}\right|^{2}}{|k|^{2m}}, \tag{41}$$

which is what was required to be proved.

The following theorem is true.

Theorem 2. *Equalities (20), (21), and (38) confirm that*

$$u(x)=1-\sum_{|\alpha|\leq q}\sum_{\lambda=1}^{N}c_{\lambda}^{\alpha}+\frac{1}{(2\pi)^{2m}}\sum_{k\neq 0}\frac{\widehat{\ell}_{k}^{(\alpha)}e^{2\pi i(k,x)}}{|k|^{2m}}$$

is an extremal function for the cubature formula (7) and $u(x)=\psi_{\ell}(x)\in\tilde{W}_{2}^{(m)}(T_n)$.

Based on Theorem 1, the error functional (10) of the cubature formula (7) for functions from class $W_{2}^{(m)}(T_n)$ has the following estimate:

$$\left|<\ell_{N}^{(\alpha)},f(x)>\right|\leq\left\{\left|\widehat{f}_{0}\right|^{2}+\sum_{k\neq 0}\left|\widehat{f}_{k}\right|^{2}|2\pi k|^{2m}\right\}^{1/2}\cdot\left\{\left|\widehat{\ell}_{0}^{(\alpha)}\right|^{2}+\sum_{k\neq 0}\frac{\left|\widehat{\ell}_{k}^{(\alpha)}\right|^{2}}{|2\pi k|^{2m}}\right\}^{1/2}.$$

CONCLUSION

The quality of the cubature formula is characterized by the norm of the error functional. It is a function of unknown coefficients and nodes. Therefore, for computational practice, it is appropriate to be able to calculate the norm of the error functional and estimate it. To find the minimum of the norm of the error functional with respect to c_{λ} and $x^{(\lambda)}$ is the task of investigating a multivariable function on an extremum.

The values of c_{λ}^{α} and $x^{(\lambda)}$, realizing this minimum, determine the optimal formula. Thus, we consider the optimal cubature formula to be the one in which, for a given number of nodes N, the error functional has the least norm.

The following problems should be solved for the practical implementation:

1. Calculation of the norm of the error functional of cubature formulas over space B.
2. Construction of the optimal cubature formula, i.e. cubature formula with the least norm of the error functional over B.

In this paper, the authors considered the first Problem for a cubature formula of the form (7), i.e., calculation of norm $\left\|\ell_{N}^{(\alpha)}(x)\,|\tilde{W}_{2}^{(m)*}(T_n)\right\|$ of the error functional $\ell_{N}^{(\alpha)}(x)$ of the cubature formula (7) with the setting of derivatives, i.e., cubature formula of the Hermite type. Its extremal function is used to find the norm of the error functional (10) of the Hermite-type cubature formula of the form (7) in space $\tilde{W}_{2}^{(m)*}(T_n)$.

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