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EXISTENCE AND SINGULARITY WITH FRACTIONAL DERIVATIVES FOR THE TELEGRAPH EQUATION.

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Abstracts. *In this paper, we present an analytical method for solving the fractional telegraph problem corresponding to the fractional time derivative. Many physical processes are represented by a system of first and second order hyperbolic type equations. Telegraph equation of electromagnetic oscillations, dynamic equations of elasticity theory and others. So that second-order equations are derived from them by means of a number of additional constraints.*

Methods. *This method is based on the Fourier Method and the properties of the corresponding fractional calculus.*

Results obtained. *The problem considered in the paper is a nonlocal boundary value problem for this system on a finite interval [0, T]. At full fulfilment of some data conditions the problem reduces to the solution of Volterra type integral equations with respect to unknowns. In addition, the theorem on the local exists the only solution of the direct problem.*

Keywords: *conformable fractional, Fourier Method, telegraph equation, eigenfunctions.*

Introduction.

This equation is a second order linear hyperbolic equation and models several phenomena in many different fields such as signal analysis [1], wave propagation [2], and random walk theory [3].

In the field of $D = \{(x,t) |\ 0 < x < 1,\ 0 < t \leq T\}$ consider the telegraph equation of the conformable fractional order

$$D_t^{(2\alpha)} u(x,t) + 2a D_t^{(\alpha)} u(x,t) - u_{xx}(x,t) = f(x,t), \ \ (x,t) \in D, \qquad (1.1)$$

with the initial conditions

$$u(x,0) = \varphi(x), \ \ D_t^{(\alpha)} u(x,0) = \psi(x), \quad 0 \leq x \leq 1, \qquad (1.2)$$

and non-local boundary conditions

$$u_x(0,t) = u_x(1,t), \qquad u(0,t) = 0, \qquad 0 \leq t \leq T, \qquad (1.3)$$

$D_t^{(\alpha)}$ is called the conformable fractional derivative of order $\alpha(0 < \alpha \leq 1)$, by t and $D_t^{(2\alpha)}: D_t^{(\alpha)}\left(D_t^{(\alpha)}\right)$, $f(x,t), \varphi(x), \psi(x)$ given functions.

The problem (1.1), (1.3) is conjugate to the problem considered in [4]. The peculiarity of the problem (1.1), (1.3) is its non-self-conjugacy generated by the boundary conditions (1.3). For equation (1.1) with classical boundary conditions, the qualitative characteristics of the eigenvalues, such as the area of variation, are the same as for equation (1.1) with classical boundary conditions. This kind of problems arise in plasma physics and in engineering. In [5], the existence of a classical in the sense of [6] solution of the problem (1.1) - (1.3) was proved and a number of a priori estimates expressing the stability of the solution on the initial data and the right-hand side of the equation were obtained.

PRELIMINARY

We'll be writing $D_t^{(\alpha)}$ to denote the operator, which is called the corresponding fractional derivative of order α.

Khalil et al. [7] introduced a completely new definition of fractional calculus, which is more natural and efficient than previous definitions of ordering $\alpha \in (0, 1)$. In addition, this definition can be generalised to any α. However, the case $\alpha \in (0, 1)$ is the most important, and the other cases become simple when it is established.

Definition 2.1. [7] Given a function: f: $[0, \infty) \to \mathbb{R}$. Then from f orders α defined as follows

$$D_t^{(\alpha)}(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}.$$

For everyone t >0, $\alpha \in (0.1)$, if f is α-differentiable in some $(0, \alpha)$. $\alpha > 0$ and, $\lim_{t\to 0+} f^{(\alpha)}(t)$ exists, then we define $f^{(\alpha)}(0) = \lim_{t\to 0+} f^{(\alpha)}(t)$.

Definition 2.2. [7]. Let's have $\alpha \in]0,1]$ и $\varphi: [0, +\infty[\to \mathbb{R} -$ real function. The left corresponding fractional integral from φ of order α from zero to t is defined by the formula:

$$I_\alpha \varphi(t) := \int_0^t s^{\alpha-1} \varphi(s) ds, t \geqslant 0.$$

Definition 2.3. [8]. Let $0 < \alpha \leqslant 1$ и $\varphi: [0, +\infty[\to \mathbb{R}$ is a real function. Then the fractional Laplace transform of order α of the function φ with origin at zero is defined as follows:

$$\mathcal{L}_\alpha[\varphi(t)](s) = \int_0^{+\infty} t^{\alpha-1} \varphi(t) e^{-s\frac{t^\alpha}{\alpha}} dt.$$

Theorem 1. [8]. Let $0 < \alpha \leqslant 1$ and $\varphi: [0, +\infty[\to \mathbb{R} -$ is a differentiable real function. The following equality holds:

$$\mathcal{L}_\alpha \left[T_t^{(\alpha)} \varphi(t) \right](s) = s\mathcal{L}_\alpha[\varphi(t)](s) - \varphi(0).$$

The following theorem will be useful in the study of problems.

Theorem 2. [9]. Let $g: [0, +\infty[\to \mathbb{R} -$ continuous function, and $\eta, \gamma \in \mathbb{R}_+$ such that $\eta < \gamma$. Consider the following problem at $0 < \alpha \leqslant 1$

$$\begin{cases} T_t^{(2\alpha)} y(t) + 2\eta T_t^{(\alpha)} y(t) + \gamma^2 y(t) = g(t), \\ y(0) = y_0, T_t^{(\alpha)} y(0) = y_\alpha. \end{cases}$$

It has a single solution given by the formula

$$y(t) = y_0 e^{-\eta \frac{t^\alpha}{\alpha}} \cos\left(\sqrt{\gamma^2 - \eta^2} \frac{t^\alpha}{\alpha}\right) + \frac{y_0 \eta + y_\alpha}{\sqrt{\gamma^2 - \eta^2}} e^{-\eta \frac{t^\alpha}{\alpha}} \sin\left(\sqrt{\gamma^2 - \eta^2} \frac{t^\alpha}{\alpha}\right) +$$

$$+ \frac{1}{\sqrt{\gamma^2 - \eta^2}} \int_0^t g\left((t^\alpha - \tau^\alpha)^{1/\alpha}\right) e^{-\eta \frac{\tau^\alpha}{\alpha}} \sin\left(\sqrt{\gamma^2 - \eta^2} \frac{\tau^\alpha}{\alpha}\right) \frac{d\tau}{\tau^{1-\alpha}}$$

Lemma 1. [10],c.189] Let r - continuous non-negative function on the interval $[a, b]$, and δ, p - non-negative constants such that

$$r(t) \leqslant \delta + p \int_a^t r(s)(s - a)^{\alpha - 1} ds$$

then for all $t \in [a, b]$

$$r(t) \leqslant \delta e^{p \frac{(t-a)^\alpha}{\alpha}}.$$

PROBLEM SOLVING (1.1)-(1.3)

Let us consider the simplest variant of the problem (1.1)-(1.3) when $f(x, t) = 0$ and write the eigenvalue problem for it. We will search for a partial solution of the problem (1.1)-(1.3) in the form

$$u(x, t) = X(x) T(t), \quad (3.1)$$

Substituting this form of solution into equation (1.1) and boundary conditions (1.3) and separating the variables, we obtain the problem for finding the eigenfunctions and eigenvalues. Then we obtain an ordinary linear differential equation for $X(x)$:

$$\begin{cases} X''(x) + \lambda X(x) = 0, \\ X'(0) = X'(1), \qquad X(0) = 0, \end{cases} \tag{3.2}$$

The boundary value problem (3.2) is non-self-conjugate. The problem (3.3) - (3.4) will be conjugate to it

$$\bar{Y}''(x) + \bar{\lambda}\bar{Y}(x) = 0,\ 0 < x < 1, \tag{3.3}$$

$$\bar{Y}(0) = \bar{Y}(1), \qquad \bar{Y}'(1) = 0, \tag{3.4}$$

Indeed,

$$\int_0^1 X''(x)\bar{Y}(x)dx - \int_0^1 \bar{Y}'X(x)dx = X'(1)(\bar{Y}(1) - \bar{Y}(0)) - \bar{Y}'(1)X(1).$$

Hence we see that the right part of the obtained relation is equal to zero at $\bar{Y}(0) = \bar{Y}(1)$ and $\bar{Y}' = 0$.

The problem (3.2) has eigenvalues

$$\lambda_k = (2\pi k)^2,\ k = 0,1, \ldots,$$

and eigenfunctions

$$X_0(x) = x,\ X_k(x) = \sin(2\pi k x).$$

By analogy with the adjoint function $\tilde{X}_k(x)$, answering the same λ_k, that the eigenfunction $X_k(x)$, we define as the solution of the boundary value problem

$$\tilde{X}_k''(x) + \lambda_k \tilde{X}_k(x) = P_k X_k(x),\ \tilde{X}_k(0) = 0,\ \tilde{X}_k'(0) = \tilde{X}_k'(1),\ k = 0,1, \ldots, \tag{3.5}$$

where $P_k \neq 0$ - arbitrary constant.

When $k = 0$ the problem (3.5) has no solution. If we put $P_k = -2\sqrt{\lambda_k}$, then for $k = 1,2,\ldots$ we get

$$\tilde{X}_k(x) = x\cos(2\pi kx).$$

Let us redefine the system of eigenfunctions and adjoint functions of problem (3.2) as follows:

$$X_0(x) = x,\ X_{2k-1}(x) = x\cos(2\pi kx),\ X_{2k}(x) = \sin(2\pi kx),\ k = 1,2,\ldots \quad (3.6)$$

For every eigenvalue λ_k when $k > 0$ eigenfunction $X_{2k}(x)$ and attached $X_{2k-1}(x)$. Solving the problem (3.3), (3.4), we find eigenfunctions

$$\bar{Y}_0(x) = S_0,\ \bar{Y}_k(x) = S_k \cdot \cos(2\pi kx),\ k = 1,2,\ldots,$$

where $S_k \neq 0$ - arbitrary constants, and eigenvalues

$$\bar{\lambda}_k = \lambda_k = (2\pi k)^2,\ k = 0,1,\ldots$$

For bi-orthonormalisation reasons, let us assume $S_0 = 2,\ S_k = 4,\ k = 1,2,\ldots$

The adjoint functions of the conjugate problem (3.3), (3.4), are defined from the problem

$$\tilde{Y}_k''(x) + \bar{\lambda}_k\tilde{Y}_k(x) = P_k\bar{Y}_k(x),\ \tilde{Y}_k(0) = \tilde{Y}_k(1),\ \tilde{Y}_k'(1) = 0,\ k = 0,1,\ldots \quad (3.7)$$

They are of the form

$$\tilde{Y}_k(x) = 4(1-x)\sin(2\pi kx),\ k = 1,2,\ldots$$

When $k = 0$ the problem (3.6) has no solution.

Let us redefine the system of eigenfunctions and adjoint functions of the conjugate problem as follows:

$$Y_0(x)=2,\ Y_{2k-1}(x)=4\cos(2\pi kx),\ Y_{2k}(x)=4(1-x)\sin(2\pi kx),\quad k=1,2,\dots \tag{3.8}$$

In doing so, each $\lambda_k=(2\pi k)^2$ when $k>0$ corresponds to the eigenfunction $Y_{2k-1}(x)$ and attached $Y_{2k}(x)$.

Since the set of functions (3.6) is a basis, an arbitrary function $\varphi(x)\in L_2(0,1)$ is represented as a biorthogonal series

$$u(x,t)=\vartheta_0X_0+\sum_{k=1}^{\infty}\left(\vartheta_{2k-1}X_{2k-1}(x)+\vartheta_{2k}X_{2k}(x)\right), \tag{3.9}$$

where $\vartheta_0=(u(x,t),Y_0),\vartheta_{2k-1}=(u(x,t),Y_{2k-1}(x)),\vartheta_{2k}=(u(x,t),Y_{2k}(x))$.

We obtain a fractional ordinary linear differential equation with an appropriate derivative for the function $\vartheta_0(t),\vartheta_{2k-1}(t),\vartheta_{2k}(t)$:

$$\begin{cases} D_t^{(2\alpha)}\vartheta_0(t)+2aD_t^{(\alpha)}\vartheta_0(t)=f_0(t),\\ \vartheta_0(0)=\varphi_0,\quad D_t^{(\alpha)}\vartheta_0(0)=\psi_0. \end{cases} \tag{3.10}$$

$$\begin{cases} D_t^{(2\alpha)}\vartheta_{2k-1}(t)+2aD_t^{(\alpha)}\vartheta_{2k-1}(t)+\lambda_k^2\vartheta_{2k-1}(t)+2\lambda_k\vartheta_{2k}(t)=f_{2k-1}(t),\\ \vartheta_{2k-1}(0)=\varphi_{2k-1},\quad D_t^{(\alpha)}\vartheta_{2k-1}(0)=\psi_{2k-1}. \end{cases} \tag{3.11}$$

$$\begin{cases} D_t^{(2\alpha)}\vartheta_{2k}(t)+2aD_t^{(\alpha)}\vartheta_{2k}(t)+\lambda_k^2\vartheta_{2k}(t)=f_{2k}(t),\\ \vartheta_{2k}(0)=\varphi_{2k},\quad D_t^{(\alpha)}\vartheta_{2k}(0)=\psi_{2k}. \end{cases} \tag{3.12}$$

Using Laplace transformations for the problem (3.10), we obtain the following integral equations

$$\vartheta_0(t) = \varphi_0 e^{-2a\frac{t^\alpha}{\alpha}} + \frac{\psi_0 + 2a\varphi_0}{2a}\left(1 - e^{-2a\frac{t^\alpha}{\alpha}}\right) +$$

$$+\int_0^t \left(1 - e^{-2a\frac{t^\alpha - \tau^\alpha}{\alpha}}\right) f_0(\tau)d\tau. \qquad (3.13)$$

Based on Theorem 2, we have that the original problem (3.10) is equivalent in the space C[0, T] to the Volterra integral equation of the second kind in the following form

$$\vartheta_{2k-1}(t) = \varphi_{2\mathrm{k}-1} e^{-a\frac{t^\alpha}{\alpha}} \cos\left(\gamma_k \frac{t^\alpha}{\alpha}\right) +$$

$$+\frac{2a\varphi_{2\mathrm{k}-1} + \psi_{2\mathrm{k}-1}}{\gamma_k} e^{-a\frac{t^\alpha}{\alpha}} \sin\left(\gamma_k \frac{t^\alpha}{\alpha}\right) +$$

$$+\frac{1}{\sqrt{\gamma_n^2 - a^2}} \int_0^t f_{2k-1}(\tau) e^{-a\frac{t^\alpha - \tau^\alpha}{\alpha}} \sin\left(\gamma_k \frac{t^\alpha - \tau^\alpha}{\alpha}\right) \frac{d\tau}{\tau^{1-\alpha}} -$$

$$-\frac{2\lambda_k}{\gamma_k} \int_0^t \vartheta_{2k}(\tau) e^{-a\frac{t^\alpha - \tau^\alpha}{\alpha}} \sin\left(\gamma_k \frac{t^\alpha - \tau^\alpha}{\alpha}\right) \frac{d\tau}{\tau^{1-\alpha}}, \qquad (3.14)$$

$$\vartheta_{2k}(t) = \varphi_{2\mathrm{k}} e^{-a\frac{t^\alpha}{\alpha}} \cos\left(\gamma_k \frac{t^\alpha}{\alpha}\right) + \frac{2a\varphi_{2\mathrm{k}} + \psi_{2\mathrm{k}}}{\gamma_k} e^{-a\frac{t^\alpha}{\alpha}} \sin\left(\gamma_k \frac{t^\alpha}{\alpha}\right) +$$

$$+\frac{1}{\gamma_k} \int_0^t f_{2k}(\tau) e^{-\eta\frac{t^\alpha - \tau^\alpha}{\alpha}} \sin\left(\gamma_k \frac{t^\alpha - \tau^\alpha}{\alpha}\right) \frac{d\tau}{\tau^{1-\alpha}}. \qquad (3.15)$$

where $\gamma_k^2 = \lambda_k^2 - a^2$

Lemma 3.1. Problems (3.13), (3.14), (3.15) have the following estimates

$$|\vartheta_0(t)| \leq C_1|\varphi_0| + C_2\frac{2a|\varphi_0| + |\psi_0|}{2a}$$

$$+ C_3\frac{\|f_0\|_{C[0,T]}}{2a}\frac{T^\alpha}{\alpha}. \qquad (3.16)$$

$$|\vartheta_{2k-1}(t)| \leq C_1|\varphi_{2k-1}| + C_2\frac{2a|\varphi_{2k-1}| + |\psi_{2k-1}|}{\gamma_k}$$

$$+$$

$$+C_3\frac{\|f_{2k-1}\|_{C[0,T]}}{\gamma_k}\frac{T^\alpha}{\alpha} + 2\lambda_k\frac{\|\vartheta_{2k}\|_{C[0,T]}}{\gamma_k}\frac{T^\alpha}{\alpha}. \qquad (3.17)$$

$$|\vartheta_{2k}(t)| \leq C_1|\varphi_{2k}| + C_2\frac{2a|\varphi_{2k}| + |\psi_{2k}|}{\gamma_k} + C_3\frac{\|f_{2k}\|_{C[0,T]}}{\gamma_k}\frac{T^\alpha}{\alpha}. \qquad (3.18)$$

its first corresponding fractional derivative

$$\left|D_t^{(\alpha)}\vartheta_0(t)\right| \leq 2aC_1|\varphi_0| + C_2(2a|\varphi_0| + |\psi_0|) + C_3\|f_0\|_{C[0,T]}\frac{T^\alpha}{\alpha} = Z_0(T);$$

$$\left|D_t^{(\alpha)}\vartheta_{2k-1}(t)\right| \leq (a+\gamma_k)\left(C_1|\varphi_{2k-1}| + C_2\frac{2a|\varphi_{2k-1}| + |\psi_{2k-1}|}{\gamma_k} + \right.$$

$$\left. + C_3\frac{\|f_{2k-1}\|_{C[0,T]}}{\gamma_k}\frac{T^\alpha}{\alpha} + 2\lambda_k\frac{\|\vartheta_{2k}\|_{C[0,T]}}{\gamma_k}\frac{T^\alpha}{\alpha}\right) = Z_{2k-1}(T);$$

$$\left|D_t^{(\alpha)}\vartheta_{2k}(t)\right| \leq (a+\gamma_k)\times$$

$$\times\left(C_1|\varphi_{2k}| + C_2\frac{2a|\varphi_{2k}| + |\psi_{2k}|}{\gamma_k} + C_3\frac{\|f_{2k}\|_{C[0,T]}}{\gamma_k}\frac{T^\alpha}{\alpha}\right) = Z_{2k}(T).$$

From the problem (3.10), (3.11), (3.12) we obtain estimates for the function $D_t^{(2\alpha)}\vartheta_0(t), D_t^{(2\alpha)}\vartheta_{2k-1}(t), D_t^{(2\alpha)}\vartheta_{2k}(t)$:

$$\left|D_t^{(2\alpha)}\vartheta_0(t)\right| \leq 2a\mathrm{Z}_0(T) + \|f_0\|_{C[0,T]};$$

$$\left|D_t^{(2\alpha)}\vartheta_{2k-1}(t)\right| \le 2a\mathrm{Z}_{2k-1}(T) + \lambda_k^2\|\vartheta_{2k-1}\| + 2\lambda_k\|\vartheta_{2k}\| + \|f_{2k-1}\|_{C[0,T]};$$

$$\left|D_t^{(2\alpha)}\vartheta_{2k}(t)\right| \le 2a\mathrm{Z}_{2k}(T) + \lambda_k^2\|\vartheta_{2k}\| + \|f_{2k}\|_{C[0,T]}.$$

Lemma 3.2. Let $\varphi(x) \in C^{(3)}[0,1], \psi(x) \in C^{(2)}[0,1], f(x) \in C^{(2)}(\overline{D})$, conditions are fulfilled

$$\varphi(0) = \varphi(1) = 0, \psi(0) = \psi(1) = 0, \varphi''(0) = \varphi''(1) = 0, \psi''(0) = \psi''(1) = 0;$$

$$f(0,t) = f(1,t) = 0,\ f_{xx}(0,t) = f_{xx}(1,t) = 0,$$

then there is equality

$$\varphi_k = \frac{1}{\lambda_k^3}\varphi_k^{(3)}, \psi_k = \frac{1}{\lambda_k^2}\psi_k^{(2)}, f_k(\mathrm{t}) = \frac{1}{\lambda_k^2}f_k^{(2)}(t)\ .$$

with the following estimates:

$$\sum_{k=1}^{\infty}\left|\varphi_k^{(3)}\right|^2 \le \|\varphi^{(3)}\|_{L_2[0,l]}, \sum_{k=1}^{\infty}\left|\psi_k^{(2)}\right|^2 = \|\psi^{(2)}\|_{L_2[0,l]},$$

$$\sum_{k=1}^{\infty}\left|f_k^{(2)}\right|^2 = \|f^{(2)}\|_{L_2[0,T]}.$$

If the functions $\varphi(x), \psi(x)$ and $f(x,t)$ satisfy the conditions of Lemma 3.2, then the function $u(x,t)$ satisfies relations (1.1)-(1.3).

We obtain the following statement.

Theorem 3.1. If they are fulfilled, then there exists a single solution of the problem (1.1)–(1.3) $u(x,t) \in C[0,T]$.

Conclusion. In this paper we study the solvability for the telegraph equation with corresponding fractional time derivative with initial nonlocal boundaries. Local existence and global uniqueness of the solution of the problem are proved.

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