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# NUMERICAL ANALYSIS OF INVERSE PROBLEMS FOR THE DIFFUSION EQUATION WITH INITIAL-BOUNDARY AND OVERDETERMINATION CONDITIONS

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**Abstract**
The paper proposes both numerical and analytical methods for solving the inverse problem of identifying the coefficient on the left-hand side of the time-dependent fractional diffusion equation, with initial-periodic boundary and over-determination conditions. First, we investigate a theoretical approach to clarify the existence and uniqueness of the inverse problem. In the numerical process, the finite difference method and numerical techniques for fractional integrals and derivatives are employed. Numerical results for several test examples are presented and discussed to illustrate the accuracy and stability of the numerical inversion.

**Keywords** Time-fractional diffusion equation · Periodic boundary condition · Integral equation · Finite difference method

## Introduction

In many cases, diffusion as a physical process is more accurately described by a fractional differential equation. Examples are viscoelasticity, fluid flows, control theory, food science, electromagnetic, mathematical modeling of real-life problems, and diffusion process (see [1–4]).

Theoretical results and numerical methods for solving inverse initial-boundary value problems for differential equations are generalized in the monographs of Isakov, Prilepko, and Cannon [5–7]. Different theories have been proposed and developed by researchers to investigate the existence and uniqueness of solutions to inverse problems for integro-differential equations (see [18–24]). The study of inverse problems for differential equations with fractional derivatives is in rapid development, both in theoretical terms and in their applications. A more detailed bibliography and a classification of the problem are found in [8–17].

The development of a numerical algorithm for finding the lowest-order coefficient in a parabolic equation is often based on the idea of transforming the equation by introducing new unknowns and reducing it to a linear inverse problem. This article discusses this problem and approach as applied to anomalous diffusion. In previous works [25–27], another direct non-iterative method for determining the coefficient was developed. A generalization of this approach and a numerical implementation of the computational algorithm were presented in [28], where the inverse problem of identifying the

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time-dependent source coefficient was solved. In [8], the work proposed a non-iteration method for numerically solving the inverse problem of identifying the coefficient of the right-hand side of the time-dependent fractional diffusion equation. The results of the numerical implementation of the proposed method on test problems with exact solutions are presented. The paper [10] investigated the inverse problem of finding a time-dependent diffusion coefficient in a parabolic equation with periodic boundary and integral overdetermination conditions. Under certain assumptions on the data, the existence, uniqueness, and continuous dependence of the solution on the data are demonstrated. The accuracy and computational efficiency of the proposed method are verified through numerical examples.

In the present work, a time-fractional diffusion equation is used with initial, periodic boundary conditions for the determination of coefficients. The existence and uniqueness of the classical solution of the problem (1)–(4) are reduced to fixed point principles by applying the Fourier method. The numerical procedure for the solution of the inverse problem using the finite difference scheme combined with an iteration method is given. Finally, numerical experiments are presented and discussed.

## Formulation of the problem

We consider the initial-periodic boundary problem for the time-fractional diffusion equation

$$\partial_t^\alpha u - u_{xx} + a(t)u = f(x,t)\ \ (x,t)\epsilon D_T\ , \tag{1}$$

$$u(x,0) = \varphi(x), x \in [0,l], \tag{2}$$

$$u(0,t) = u(l,t),\ u_x(0,t) = u_x(l,t), \varphi(0) = \varphi(l),\ \varphi\prime(0) = \varphi\prime(l), t \in [0,T], \tag{3}$$

where $a(t), t > 0$ are the source control terms, $f(x,t)$ is known source term, $\varphi(x)$ is the initial temperature, $T$ is an arbitrary positive number and $D_T := \{(x,t) : 0 < x < l, 0 < t \le T\})$. The Caputo fractional derivative of order $\alpha$ is determined by the formula

$$\partial_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\frac{\partial u(x,\tau)}{\partial\tau}d\tau,\ \ \partial_t^1 u(x,t) = \frac{\partial u(x,t)}{\partial t},$$

where $\alpha \in (0,1), \Gamma(\cdot)$ is the Euler's Gamma function.

The problem of determining a function $u(x,t), (x,t) \in D_T$ that satisfies (1)–(3) with known functions $a(t), f(x,t)$, and $\varphi(x)$ will be called the direct problem.

In the inverse problem, it is assumed that the coefficient $a(t), t > 0$ in (1) is unknown and it is required to determine it using additional information about the solution of the direct problem:

$$u_x(0,t) = h(t), \quad x \in [0,l], \tag{4}$$

or

$$\int_0^l \omega(x)u(x,t)dx = h(t),\ x \in [0,l], \tag{5}$$

where $\omega(x), h(t)$ are given functions.

In the sequel, we will call the problem of determining functions $u(x,t), a(t)$ from Eqs. (1-4) as inverse problem 1 and the problem of determining functions $u(x,t), a(t)$ from Eqs. (1-3) and (5) as inverse problem 2.

Let $C^{2,\alpha}\left(D_T\right)$ be the class of the 2 times continuously differentiable functions with respect to $x \in [0,l]$ variable, continuous in $t$ and let its fractional integral of the order $\alpha$ be continuously differentiable in $t$ on $[0,T]$.

**Definition 1**. The double of functions $\{u(x,t), a(t)\}$ from the class $C^{2,\alpha}\left(D_T\right) \cap C^{1,0}\left(\overline{D}_T\right) \times C[0,T]$ are said to be a classical solution of problem (1–3), if the functions $u(x,t), a(t)$ satisfy the following conditions:

(1) The function $u(x,t)$ and its derivatives $\partial_t^\alpha u(x,t), u_{xx}(x,t)$ are continuous in the domain $D_T$;

(2) The function $a(t)$ is continuous on the interval $[0, T]$;
(3) Eq. (1) and conditions (2–3) are satisfied in the classical sense.

We have the following assumptions on $\varphi$, $f$, and $h$:

$$(\boldsymbol{A}1)\ \ \varphi(x) \in C^4[0,l];\ \ \varphi^{(5)}(x) \in L_2(0,l);\ \ \varphi(0) = \varphi(l);\ \ \varphi'(0) = \varphi'(l);\ \ \varphi''(0) = \varphi''(l);$$

$$\varphi^{(3)}(0) = \varphi^{(3)}(l);\ \ \varphi^{(4)}(0) = \varphi^{(4)}(l);$$

$$(\boldsymbol{A}2)\ f(x,t) \in C\left(\overline{D}_T\right) \cap C^{4,1}_{x,t}\left(D_T\right);\ f^{(5)}_{xxxxx}(x,t) \in L_2(0,l);\ \ f(0,t) = f(l,t);$$

$$f'(0,\mathrm{t}) = f'(l,t); f''(0,t) = f''(l,t);\ f^{(3)}(0,t) = f^{(3)}(l,t);\ f^{(4)}(0,t) = f^{(4)}(l,t);$$

($\boldsymbol{A}3$) $h(t) \in C^1[0,T]$ and $|h(t)| \geq h_0 = const > 0, h_0$ is a given number, $\varphi' = h(0), \varphi'''\ (0) = q(0)h(0) + \partial_t^\alpha h(t) - f_x(0,0)$.

$$(B1)\ \ \varphi(x)\epsilon C^2(0,l);\ \ \varphi^{(3)}(x)\epsilon L_2(0,l);\ \varphi(0) = \varphi(l);\ \ \varphi''(0) = \varphi''(l);\ \varphi''(0) = \varphi''(0)$$

$$(\boldsymbol{B}2)\ f(x,t) \in C\left(\overline{D}_T\right) \cap C^{2,1}_{x,t}\left(D_T\right);\ f^{(3)}_{xxx}(x,t) \in L_2(0,l);\ f(0,t) = f(l,t);$$

$$f'(0,\mathrm{t}) = f'(l,t);\ f''(0,t) = f''(l,t);$$

($\boldsymbol{B}3$) $h(t) \in C^1[0,T]$ and $|h(t)| \geq h_0 = const > 0, h_0$ is a given number.

## Preliminaries

In this section, we briefly state some definitions and results needed to prove the main result.

**Two parameter Mittag–Leffler function** [31, pp. 40–42] The two-parameter Mittag–Leffler function $E_{\alpha,\beta}(z)$ is defined by the following series:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

where $\alpha, \beta, z \in \mathbb{C}$ with $\mathfrak{R}(\alpha) > 0$, $\mathfrak{R}(\alpha)$ denotes the real part of the complex number $\alpha$.

**Theorem 1** [31, *pp*. 135–144] *The solution $T(t) \in AC[0,T]$ of the linear nonhomogeneous fractional problem.*

$$\partial^\alpha_{0+,t} T(t) + \lambda T(t) = f(t),\ \ t \in (0,T],\ \ \lambda > 0,$$

$$T(0) = c,$$

*where $f \in L^1[0,T]$, is given by the integral expression*

$$T(t) = cE_{\alpha,1}(-\lambda t^\alpha) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-\tau)^\alpha) f(\tau) d\tau.$$

In the next two sections, the obtained theorems and lemmas are proved similar to the results in [11], so we do not write the proving states.

## Mathematical formulation of the inverse problems

### Investigation of inverse problem 1

Let $u(x,t)$ be a classical solution to the problem (1)–(3) and $f, \varphi, h$ be enough smooth functions. We carry out the next converting of the inverse problem (1)–(4). For this purpose, denote the second derivative of $u(x,t)$ with respect to $x$, by $\vartheta(x,t)$, i.e., $\vartheta(x,t) := u_{xx}(x,t)$. Differentiating (1) and (2) twice in $x$, we get

$$\partial_t^\alpha \vartheta - \vartheta_{xx} + a(t)\vartheta(x,t) = f_{xx}(x,t),\ (x,t) \in D_T, \tag{6}$$

$$\vartheta(x,0) = \varphi''(x),\ x \in [0,l]. \tag{7}$$

To obtain boundary conditions for the function $\vartheta(x,t)$, we note that the second term in (1) is $\vartheta(x,t)$. Suppose $f(0,t) = f(l,t), f'(0,t) = f'(l,t)$. Then we have the following boundary condition

$$\vartheta(0,t) = \vartheta(l,t);\ \vartheta_x(0,t) = \vartheta_x(l,t). \tag{8}$$

To obtain an additional condition for the function $\vartheta(x,t)$, we differentiate Eq. (1) with respect to $x$ and using equality $u_{xx}(x,t) = \vartheta(x,t)$ and additional condition (4), we get

$$\vartheta_x(0,t) = a(t)h(t) + \partial_t^\alpha h(t) - f_x(0,t). \tag{9}$$

When the matching condition $\varphi'(0) = h(0)$ is satisfied, it is easy to derive from (6)–(9) Eqs. (1-4).
We shall seek the $u(x,t)$ of classical solution of the problem (6–8) in the form

$$\vartheta(x,t) = \sum_{n=0}^{\infty} \vartheta_{1n}(t)\cos\lambda_n x + \sum_{n=1}^{\infty} \vartheta_{2n}(t)\sin\lambda_n x,\ \lambda_n = \frac{2\pi n}{l}, \tag{10}$$

where

$$\vartheta_{10}(t) = \frac{1}{\sqrt{l}}\int_0^l \vartheta(x,t)dx,\ \ \vartheta_{1n}(t) = \sqrt{\frac{2}{l}}\int_0^l \vartheta(x,t)\cos\lambda_n x dx,\ \ \vartheta_{2n}(t) = \sqrt{\frac{2}{l}}\int_0^l \vartheta(x,t)\sin\lambda_n x dx.$$

$$f_{10}(t) = \frac{1}{\sqrt{l}}\int_0^1 f_{xx}(x,t)dx,\ f_{1n}(t) = \sqrt{\frac{2}{l}}\int_0^l f_{xx}(x,t)\cos\lambda_n x dx,\ \ f_{2n}(t) = \sqrt{\frac{2}{l}}\int_0^l f_{xx}(x,t)\sin\lambda_n x dx.$$

Then, applying the formal scheme of the Fourier method for determining of unknown coefficients $\vartheta_{10}(t)$ and $\vartheta_{in}(t)$ $(i := 1,2; n = 1,2,...)$ of function $\vartheta(x,t)$ from (6) and (7), we have

$$\partial^\alpha \vartheta_{10}(t) = -a(t)\vartheta_{10}(t) + f_{10}(t), \tag{11}$$

$$\vartheta_{10}(t)|_{t=0} = \varphi_{10}, \tag{12}$$

$$\partial^\alpha \vartheta_{in}(t) + \lambda_n^2 \vartheta_{in}(t) = -a(t)\vartheta_{in}(t) + f_{in}(t), \tag{13}$$

$$\vartheta_{in}(t)|_{t=0} = \varphi_{in},\ i = 1,2,\ n = 1,2,..., \tag{14}$$

where

$$\varphi_{10} = \frac{1}{\sqrt{l}}\int_0^l \varphi''(x)dx,\ \ \varphi_{1n} = \sqrt{\frac{2}{l}}\int_0^l \varphi''(x)\cos\lambda_n x dx,\ \ \varphi_{2n} = \sqrt{\frac{2}{l}}\int_0^l \varphi''(x)\sin\lambda_n x dx.$$

According to Theorem 1, the solutions of problems (11), (12) and (13), (14) satisfy the following integral equations:

$$\vartheta_{10}(t) = \varphi_{10} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\big(f_{10}(\tau) - a(\tau)\vartheta_{10}(\tau)\big)d\tau, \tag{15}$$

and

$$\vartheta_{in}(t) = \varphi_{in}E_\alpha\left(-\lambda^2 t^\alpha\right) + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda^2(t-\tau)^\alpha\right)\big(f_{in}(\tau) - a(\tau)\vartheta_{in}(\tau)\big)d\tau. \tag{16}$$

**Lemma 1** *For any* $t\in[0;T]$, *the following estimates are valid*:

$$\left|\vartheta_{10}(t)\right| \le \left(\left|\varphi_{10}\right| + \frac{T^\alpha}{\Gamma(\alpha+1)}\parallel f_{10}\parallel\right)E_\alpha(\parallel a\parallel T^\alpha),$$

$$\left|\vartheta_{in}(t)\right| \le \left(\left|\varphi_{in}\right| + \frac{T^\alpha}{\Gamma(\alpha+1)}\parallel f_{in}\parallel\right)E_\alpha(\parallel a\parallel T^\alpha),$$

$$\left|\partial^\alpha\vartheta_{10}(t)\right| \le \parallel a\parallel\left(\left|\varphi_{10}\right| + \frac{T^\alpha}{\Gamma(\alpha+1)}\parallel f_{10}\parallel\right)E_\alpha(\parallel a\parallel T^\alpha) + \parallel f_{10}\parallel,$$

$$\left|\partial^\alpha\vartheta_{in}(t)\right| \le \left(\lambda^2 + \parallel a\parallel\right)\left(\left|\varphi_{in}\right| + \frac{T^\alpha}{\Gamma(\alpha+1)}\parallel f_{in}\parallel\right)E_\alpha(\parallel a\parallel T^\alpha) + \parallel f_{in}\parallel .$$

Formally, from (10), by term-by-term differentiation, we compose the series

$$\partial^\alpha_{t,0+}\vartheta(x,t) = \sum_{n=0}^\infty \partial^\alpha_{0+}\vartheta_{1n}(t)\cos\lambda_n x + \sum_{n=1}^\infty \partial^\alpha_{0+}\vartheta_{2n}(t)\sin\lambda_n x, \tag{17}$$

$$\vartheta_{xx}(x,t) = -\sum_{n=0}^\infty \lambda_n^2\vartheta_{1n}(t)\cos\lambda_n x - \sum_{n=1}^\infty \lambda_n^2\vartheta_{2n}(t)\sin\lambda_n x. \tag{18}$$

In view of Lemma 1, if the following series converge, then the series (10), (17), and (18) converge for any $(x,t)\in D_T$

$$C_4\sum_{n=1}^\infty\left(\lambda_n^2\left|\varphi_{in}\right| + \lambda_n^2\parallel f_{in}\parallel\right),$$

where the constant $C_4$ depends only on $T, \alpha, \parallel a\parallel$ .
We hold the following auxiliary lemma:

**Lemma 2** *If the conditions* $(A1)$ *and* $(A2)$ *are valid, then there are equalities.*

$$\varphi_{in} = \frac{1}{\lambda_n^3}\varphi_{in}^{(3)}, f_{in}(t) = \frac{1}{\lambda_n^3}f_{in}^{(3)}, (i=1{,}2) \tag{19}$$

*where*

$$\varphi_{1n}^{(3)} = \sqrt{\frac{2}{l}}\int_0^l \varphi^{(5)}(x)\sin\lambda_n x dx,\quad \varphi_{2n}^{(3)} = \sqrt{\frac{2}{l}}\int_0^l \varphi^{(5)}(x)\cos\lambda_n x dx,$$

$$f_{1n}^{(3)}(t) = \sqrt{\frac{2}{l}} \int_0^l f_{xxxxx}^{(5)}(x,t)\sin\lambda_n x dx, \ \ f_{2n}^{(3)}(t) = \sqrt{\frac{2}{l}} \int_0^l f_{xxxxx}^{(5)}(x,t)\cos\lambda_n x dx$$

*with the following estimate*:

$$\sum_{n=1}^{\infty} \left|\varphi_{in}^{(3)}\right|^2 \leq |\varphi^{(3)}|^2{}_{L_2(0,1)}, \sum_{n=1}^{\infty} \left|f_{in}^{(3)}(t)\right|^2 \leq |f^{(3)}|^2{}_{L_2(0,1)\times C[0,T]}. \tag{20}$$

If the functions $\varphi(x), f(x,t)$ satisfy the conditions of Lemma 2, then due to representations (19) and (20) series (10), (17) and (18) converge uniformly in the rectangle $D_T$; therefore, function $\vartheta(x,t)$ satisfies relations (6-8).
Using the above results, we obtain the following assertion.

**Lemma 3** *Let* $a(t) \in C[0,T], (A1) - (A2)$ *be satisfied, then there exists a unique solution of the direct problem* (6)–(8) $\vartheta(x,t) \in C^{2,\alpha}\left(D_T\right) \cap C^{1,0}\left(\overline{D}_T\right)$.

Firstly, by differentiating (10) with respect to $x$, we get the following equality

$$\vartheta_x(x,t) = -\sum_{n=0}^{\infty} \lambda_n \vartheta_{1n}(t)\sin\lambda_n x + \sum_{n=1}^{\infty} \lambda_n \vartheta_{2n}(t)\cos\lambda_n x. \tag{21}$$

Setting in (21) $x = 0$ and using additional condition (9), after simple converting, we get the following integral equation for determining $a(t)$ :

$$a(t) = a_0(t) - \frac{1}{h(t)} \sum_{n=1}^{\infty} \lambda_{2n} \vartheta_{2n}(t;a), \tag{22}$$

where

$$a_0(t) = \frac{1}{h(t)} \left[f_x(0,t) - \partial^\alpha h(t)\right],$$

and $\vartheta_{2n}(t;a)$ means that the solution of integral Eq. (16) depends on $a(t)$.
The main result of this section is presented as follows:

**Theorem 2** *Let* $(A1)$-$(A3)$ *be satisfied. Then there exists a number* $T^* \in (0,T)$, *such that there exists a unique solution* $a(t)$ *of the inverse problem* (6)–(9).

## Investigation of inverse problem 2

We shall seek the $u(x,t)$ of classical solution of the problem (1)–(3) in the form

$$u(x,t) = \sum_{n=0}^{\infty} u_{1n}(t)\cos\lambda_n x + \sum_{n=1}^{\infty} u_{2n}(t)\sin\lambda_n x, \ \lambda_n = \frac{2\pi n}{l}, \tag{23}$$

where

$$u_{10}(t) = \frac{1}{\sqrt{l}} \int_0^l u(x,t)dx, \ \ u_{1n}(t) = \sqrt{\frac{2}{l}} \int_0^l u(x,t)\cos\lambda_n x dx, \ \ u_{2n}(t) = \sqrt{\frac{2}{l}} \int_0^l u(x,t)\sin\lambda_n x dx.$$

$$f_{10}(t) = \frac{1}{\sqrt{l}}\int_0^1 f(x,t)dx,\ \ f_{1n}(t) = \sqrt{\frac{2}{l}}\int_0^l f(x,t)\cos\lambda_n x dx,\ \ f_{2n}(t) = \sqrt{\frac{2}{l}}\int_0^l f(x,t)\sin\lambda_n x dx.$$

**Lemma 4** *If the conditions* $(B1)$, $(B2)$ *are valid, then there are equalities.*

$$\varphi_{in} = \tfrac{1}{\lambda_n^3}\varphi_{in}^{(3)},\ f_{in}(t) = \tfrac{1}{\lambda_n^3}f_{in}^{(3)},\ (i = 1,2) \tag{24}$$

*where*

$$\varphi_{1n}^{(3)} = -\sqrt{\frac{2}{l}}\int_0^l \varphi^{(3)}(x)\sin\lambda_n x dx,\ \ \varphi_{2n}^{(3)} = -\sqrt{\frac{2}{l}}\int_0^l \varphi^{(3)}(x)\cos\lambda_n x dx,$$

$$f_{1n}^{(3)}(t) = -\sqrt{\frac{2}{l}}\int_0^l f_{xxx}^{(3)}(x,t)\sin\lambda_n x dx,\ \ f_{2n}^{(3)}(t) = -\sqrt{\frac{2}{l}}\int_0^l f_{xxx}^{(3)}(x,t)\cos\lambda_n x dx$$

*with the following estimate*:

$$\textstyle\sum_{n=1}^{\infty}\left|\varphi_{in}^{(3)}\right|^2 \le |\varphi^{(3)}|^2{}_{L_2(0,1)},\ \ \sum_{n=1}^{\infty}\left|f_{in}^{(3)}\right|^2 \le |f^{(3)}|^2{}_{L_2(0,1)\times C[0,T]},\ (i = 1,2). \tag{25}$$

If the functions $\varphi(x), f(x,t)$ satisfy the conditions of Lemma 4, then due to representations (24) and (25) series (23), (18) and (19) converge uniformly in the rectangle $D_T$; therefore, function $u(x,t)$ satisfies relations (1-3).

**Lemma 5** *Let* $a(t) \in C[0,T], (B1), (B2)$ *be satisfied, then there exists a unique solution of the problem* (1)-(3) $u(x,t) \in C^{2,\alpha}\left(D_T\right) \cap C^{1,0}\left(\overline{D}_T\right)$.

Let us multiply (1) by $\omega(x)$ and integrate over $x$ from 0 to $l$:

$$\int_0^l \omega(x)\partial_t^\alpha u(x,t)dx - \int_0^l \omega(x)u_{xx}dx + a(t)\int_0^l \omega(x)u(x,t)dx =$$

$$= \int_0^l \omega(x)f(x,t)dx, \quad (x,t) \in D_T.$$

After integrating by parts, in view of conditions (2), (5), and (B4), we obtain the equality

$$\partial_{0+,t}^\alpha h(t) - \int_0^l \omega''(x)u(x,t)dx + a(t)h(t) = \int_0^l \omega(x)f(x,t)dx.$$

We obtain the following integral equation with respect to the unknown function $a(t)$:

$$a(t) = \frac{1}{h(t)}\left(-\partial_{0+,t}^\alpha h(t) + \int_0^l \omega''(x)u(x,t)dx + \int_0^l \omega(x)f(x,t)dx\right). \tag{26}$$

The main result of this subsection is presented as follows:

**Theorem 3** *Let* $(B1)$-$(B3)$ *be satisfied. Then there exists a number* $T^* \in (0,T)$, *such that there exists a unique solution* $a(t) \in C[0,T^*]$ *of the inverse problem* (1)–(3),(5).

## Numerical procedure

Let us highlight the basic features of the numerical solution of the identification problem. The algorithm is based on the approximate solution of an initial/boundary value problem for the loaded equation. Unfortunately, numerical methods for these nonclassical boundary value problems are not sufficiently developed at present time.

### Algorithm development and finite-difference analog of the inverse problem 1

Let us move on to constructing a difference analogue of the inverse problem (1)–(4). Assume that for variable $x$, we have a uniform grid with spacing $\Delta x$, with $x_i = i\Delta x, i = 0, 1, ..., N, N\Delta x = 1$ designating the grid points and $\vartheta = \vartheta_i = \vartheta(x_i)$. We will use standard index-free notations of the theory of difference schemes [29],

$$y_x = \frac{y_{i+1} - y_i}{\Delta x}, \quad y_{xx} = \frac{y_{i+1} - 2y_i + y_{i-1}}{\Delta x^2}.$$

To approximate the fractional Caputo derivative of order $\alpha$ in time on a uniform time grid with a step $\tau = \frac{T}{N_t}$, we use the discrete analogue of P. Zhuang and F. Liu [30]:

$$\partial^{\alpha}_{0+,t} u(x, t_j) = \sigma_{\tau\alpha} \sum_{k=1}^{j} s_k \left( u_i^{j-k+1} - u_i^{j-k} \right), \tag{27}$$

where

$$s_1 = 1, s_k = k^{1-\alpha} - (k-1)^{1-\alpha}, k = 2, 3, ..., j. \sigma_{\tau\alpha} = \frac{1}{\Gamma(2-\alpha)\tau^{\alpha}}.$$

The same work shows that the error in the approximation of the fractional derivative is of the order of $O(\tau^{2-\alpha})$. Let us write down the sum for selecting layers:

$$\partial^{\alpha}_{0+,t} u(x, t_j) = \sigma_{\tau\alpha} \sum_{k=1}^{j} s_k \left( u_i^{j-k+1} - u_i^{j-k} \right) = \sigma_{\tau\alpha} \left( u_i^j - u_i^{j-1} + \sum_{k=2}^{j} s_k \left( u_i^{j-k+1} - u_i^{j-k} \right) \right). \tag{28}$$

Let us introduce the notation $\Phi_i^{j-1}$ for the lower layers:

$$\Phi_i^{j-1} = -u_i^{j-1} + \sum_{k=2}^{j} s_k \left( u_i^{j-k+1} - u_i^{j-k} \right).$$

We write the approximation of the second derivative in the form:

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{u_{j-1}^{i-1} - 2u_j^{i-1} + u_{j+1}^{i-1}}{\Delta x^2} = \Lambda u^{i-1}.$$

Taking into account the introduced notations, the main equation of problem (1)–(3) will take the form:

$$\sigma_{\tau\alpha} \left( u_i^j + \Phi_i^{j-1} \right) - \Lambda u_i^{j-1} + a^{j-1} u_i^{j-1} = f_i^j.$$

$$u_i^0 = \varphi_i,$$

$$u_0^j = u_{N_x}^j, \quad u_1^j = u_{N_x+1}^j,$$

where $1 \le i \le N_x$ and $0 \le j \le N_t$ are the indices for the spatial and time steps. Then, $u_i^j$ at grid point $i$ for the time step $j$ results in

$$u_i^j = -\Phi_i^{j-1} + \frac{1}{\sigma_{\tau\alpha}}\left(\Lambda u_i^{j-1} - a^{j-1}u_i^{j-1} + f_i^j\right). \tag{29}$$

The unknown coefficient $a^{j-1}$ can be determined by applying the forward finite difference to the condition (4) at the grid point $i = 1$,

$$\frac{u_1^j - u_0^j}{\Delta x} = h^j.$$

Components $u_0^j$ and $u_1^j$ can be obtained at the discretized points $i = 0$ and $i = 1$ from (29). Then, by substituting $u_0^j$ and $u_1^j$ the unknown coefficient $a^{j-1}$ can be evaluated as

$$a^{j-1} = \frac{\Phi_0^{j-1} - \Phi_1^{j-1} + \frac{1}{\sigma_{\tau\alpha}}\left(\Lambda u_1^{j-1} - \Lambda u_0^{j-1} + f_1^{j-1} - f_0^{j-1}\right) - \Delta x h^j}{u_1^{j-1} - u_0^{j-1}}. \tag{30}$$

For $j = 1$, the values of $\varphi_i$ help us to start our computation. In numerical computation, since the time step is very small, $a^{j-1}, u_i^j$. At each $j$ th iteration step, we first determine $a^{j-1}$ from the formula (30). Then from (29), we obtain $u_i^j$. In virtue of this iteration, we can move from level $j-1$ to level $j$.

### Numerical example and discussions

We will carry out a numerical implementation of computational algorithms on a model problem with different conditions and different values of the fractional time derivative exponent $\alpha$ and compare the obtained calculation results with the exact solution.

***Example*** We consider the inverse problem with a smooth initial condition (1)–(3) on the domains $x \in [0, l]$ and $t \in [0, T]$ with different $\alpha = 0.5, 0.75, 1$. Design parameters for space: $N_x = 32$; $l = 2\pi$, for time: $T = 1; N_t = 1000$. All input and output functions are known:

$$\varphi(x) = \Gamma(2-\alpha)\sin x, \quad h(t) = \Gamma(2-\alpha)(1+t), \quad f(x,t) = \left(t^{1-\alpha} + \Gamma(2-\alpha)(1+t) + 1\right)\sin x.$$

$$u(x,t) = \Gamma(2-\alpha)(1+t)\sin x, \quad a(t) = \frac{1}{\Gamma(2-\alpha)(1+t)}.$$

Figures 1, 2, and 3 present the results of the calculation according to the proposed algorithm to illustrate the effect in space and time. As it turned out, the accuracy is more influenced in time. At least for this example, the impact is minimal.

## Algorithm development and finite-difference analog of the inverse problem 2

We will consider the examples of numerical solution of the inverse problem (1)–(3), (5). For the convenience of discussion of the numerical method, we will rewrite (1)–(3), (5) as follows:

$$\partial_t^\alpha u - u_{xx} + a(t)u = f(x,t), \quad (x,t) \in D_T, \tag{31}$$

$$u(x,0) = \varphi(x), x \in [0,l], \tag{32}$$

$$u(0,t) = u(l,t), \quad u_x(0,t) = u_x(l,t), t \in [0,T], \tag{33}$$

$$\int_0^l \omega(x)u(x,t)dx = h(t), \quad x \in [0,l]. \tag{34}$$

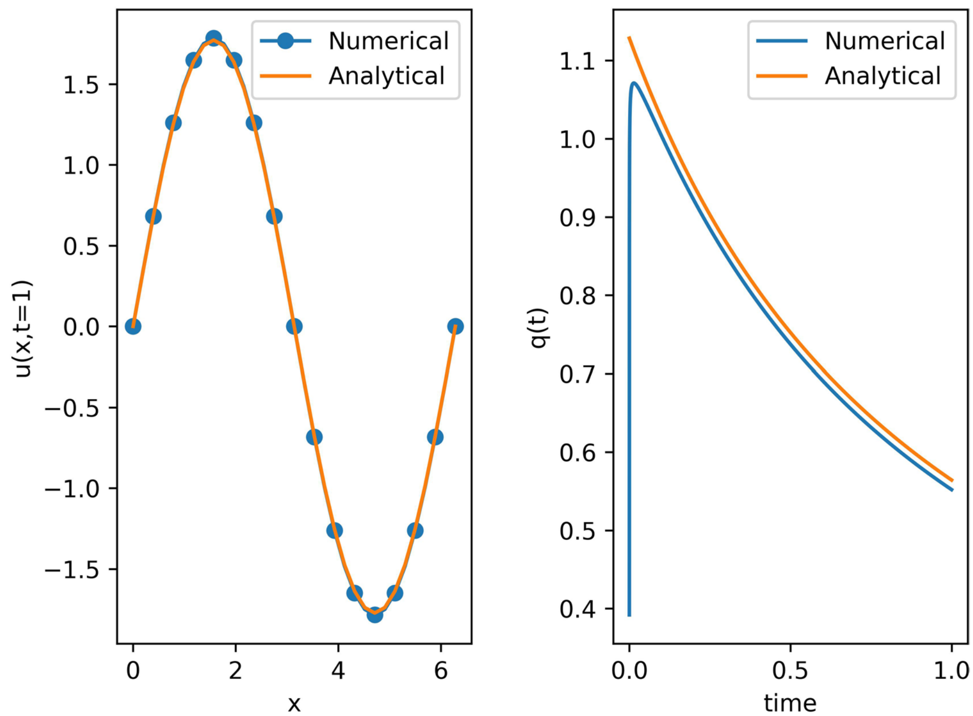


**Fig. 1** Exact and approximate graphs of functions $u(x, t)$ and $a(t)$ for $\alpha = 0.5$

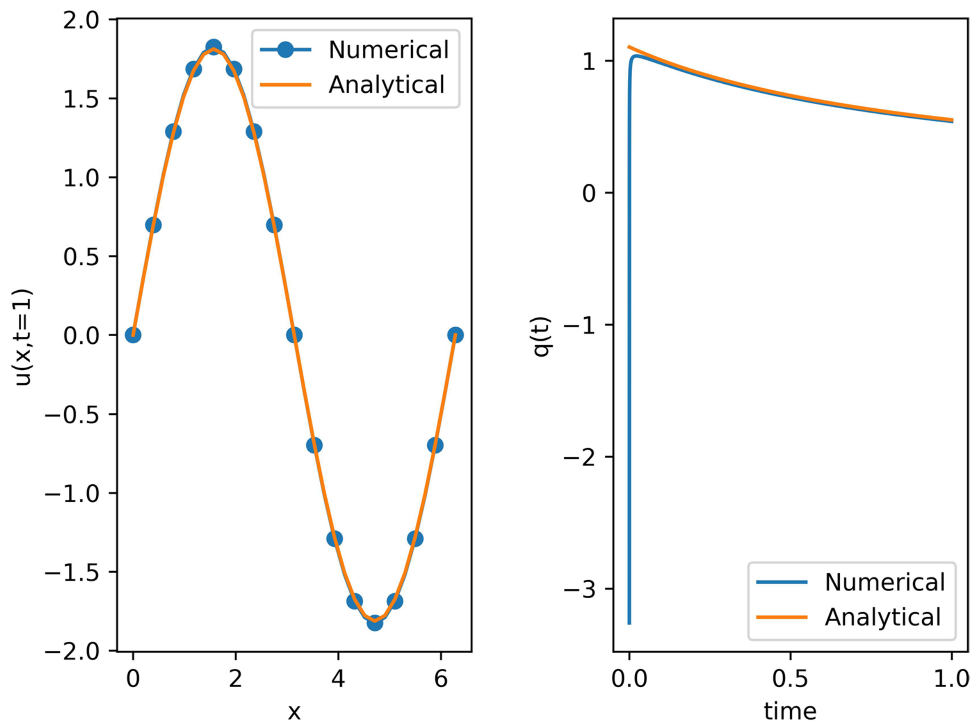


**Fig. 2** Exact and approximate graphs of functions $u(x, t)$ and $a(t)$ for $\alpha = 0.75$

We subdivide the intervals $[0, l]$ and $[0, T]$ into $N_x$ and $N_t$ subintervals of equal lengths $h = \frac{l}{N_x}$ and $\tau = \frac{T}{N_t}$, respectively. Then, we add a line $x = \left(N_x + 1\right)h$ to generate the fictitious point needed for the boundary condition. We choose the forward scheme. The scheme for (31)–(34) is as follows:

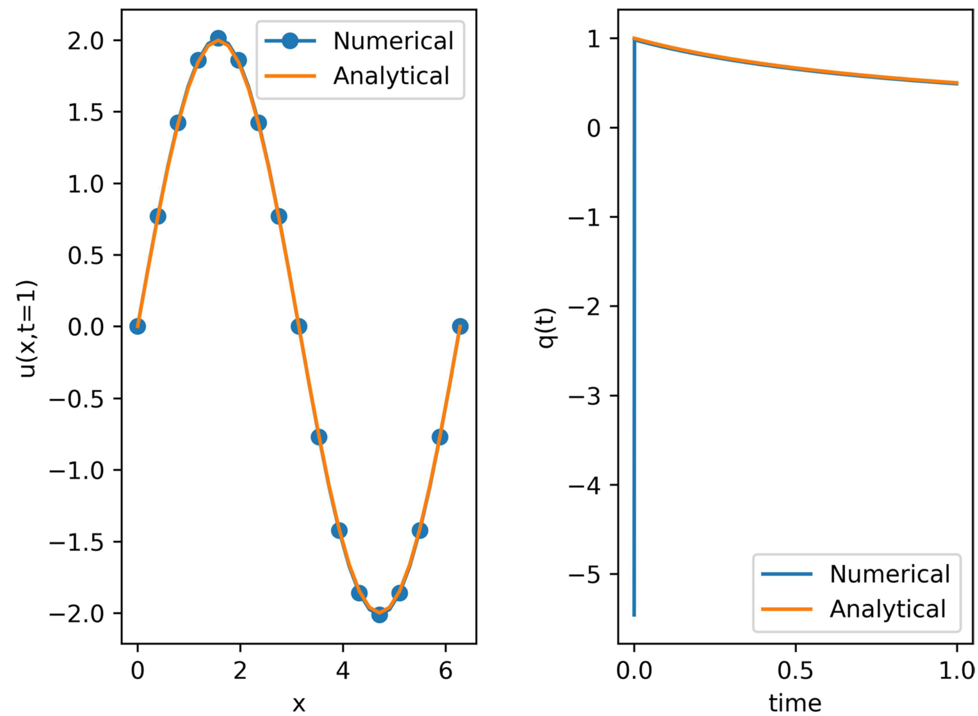


**Fig. 3** Exact and approximate graphs of functions $u(x,t)$ and $a(t)$ for $\alpha = 1$

$$\begin{cases} \sigma_{\tau\alpha}\left(u_j^{n+1} - u_j^{n-1} + \sum_{k=1}^{n} s_k\left(u_j^{n-k+1} - u_j^{n-k}\right)\right) - \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{h^2} + a^n u_j^n = f_j^n; \\ u_j^0 = \varphi_j; \\ u_0^n = u_{N_x}^n; \\ u_1^n = u_{N_{x+1}}; \end{cases} \tag{35}$$

where $0 \le j \le N_x$ and $0 \le n \le N_t$ are the indices for the spatial and time steps, respectively, and $u_j^n$ is the approximation to $u\left(x_j, t_n\right), a^n = a\left(t_n\right), f_j^n = f\left(x_j, t_n\right), \varphi_j = \varphi\left(x_j\right), x_j = jh, t_n = n\tau.$

Now, we approximate $\int_0^l \omega(x)u(x,t)dx$ formally by the trapezoidal formula

$$\int_0^l \omega(x)u(x,t)dx = h\left(\frac{(\omega u)_0}{2} + (\omega u)_1 + \ldots + (\omega u)_{N_x-1} + \frac{(\omega u)_{N_x}}{2}\right), \tag{36}$$

where $(\omega u)_j = \omega\left(x_j\right)u\left(x_j, t\right), \quad 0 \le j \le N_x.$

Substituting (34), with $\int_0^l \omega(x)u(x,t)dx$ into (13) and rewriting the resulting integral equation for unknown function $a(t)$ :

$$a(t) = \frac{1}{h(t)}\left(-\partial_{0+,t}^{\alpha} h(t) + \int_0^l \omega''(x)u(x,t)dx + \int_0^l \omega(x)f(x,t)dx\right). \tag{37}$$

Now, we rewrite to approximate by the trapezoidal formula integral Eq. (36) and obtain the resulting system

$$a^n = \frac{1}{h^n}\left(-Dh^n + (\omega u)^j + (\omega f)^j\right), \tag{38}$$

where

$$(\omega u)^n = \int_0^l \omega''(x)u\left(x, t_n\right)dx \approx \frac{1}{2h^2}\sum_{j=1}^{N_x-1}\left(\omega_{j-1} - 2\omega_j + \omega_{j+1}\right)u_j^n, n = 0, 1, 2, \ldots, N_t,$$

$$Dh^n = \partial^{\alpha}_{0+,t}h(t), (\omega f)^n = \int_0^l \omega(x)f\left(x, t_n\right)dx, Dh^n \approx \partial^{\alpha}_{0+,t}h(t).$$

For $n = 0$, the values of $\varphi_j$ help us to start our computation. In numerical computation, since the time step is very small, $a^n, u_i^{j+1}$. At each $n$ th iteration step, we first determine $a^n$ from the formula (38). Then from (35), we obtain $u_j^{n+1}$. In virtue of this iteration, we can move from level $n$ to level $n+1$.

## Numerical examples and discussions

***Example*** Consider the inverse problem (31)–(34), with.

$$\varphi(x) = \sin x, \quad h(t) = \pi\left(1 + t^2\right),$$

$$\omega(x) = \sin x, \quad f(x,t) = \left(\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + \left(1+t^2\right)^2\right)\sin x, \quad x \in [0, 2\pi], \quad t \in [0, T].$$

It is easy to check that the analytical solution of the problem (31)–(34) is

$$\{a(t), u(x,t)\} = \{t^2; \left(1+t^2\right)\sin x\}. \tag{39}$$

Let us apply the scheme which was explained in the previous section.
In the case when $T = 1$, the comparisons between the analytical solution (39) and the numerical finite difference solution are shown in Figs. 4, 5, and 6. For the writing of definitions, theorems, lemmas, and their proofs use the following variables. If necessary, you can add your variables to the sample.

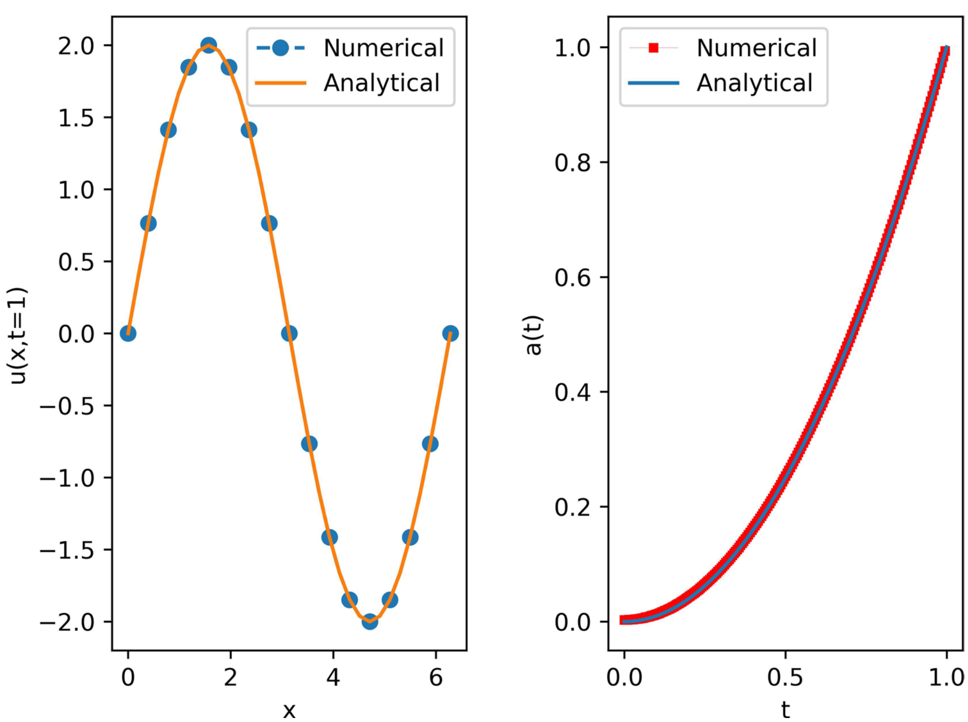


**Fig. 4** The analytical and numerical solutions of $u(x,t)$ and $a(t)$ when $T = 1$, $\alpha = 0.5$

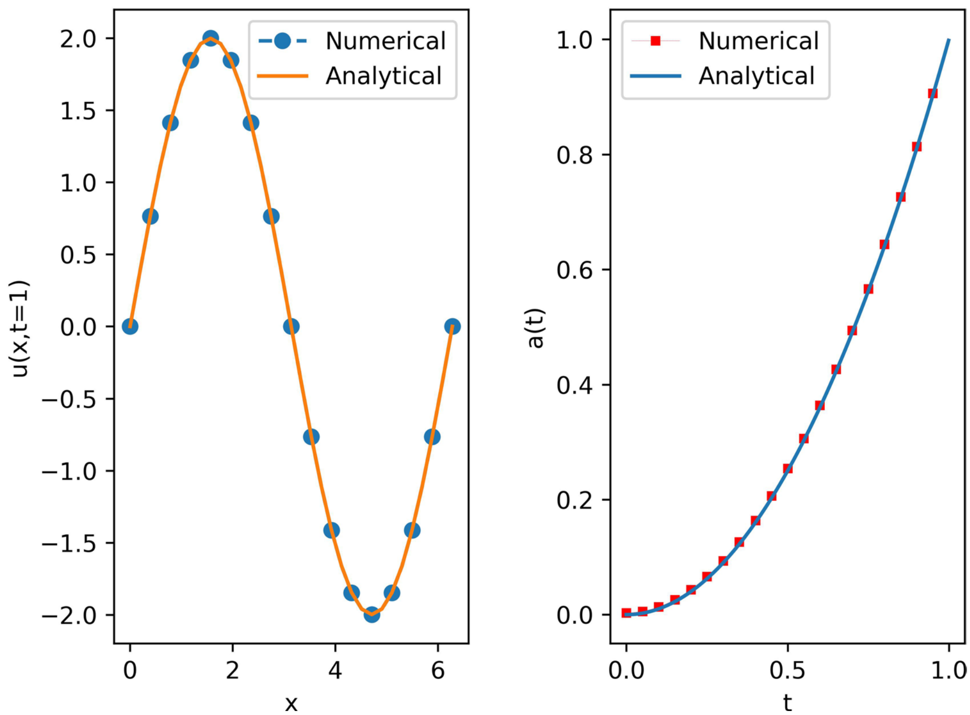


**Fig. 5** The analytical and numerical solutions of $u(x, t)$ and $a(t)$ when $T = 1$, $\alpha = 0.75$

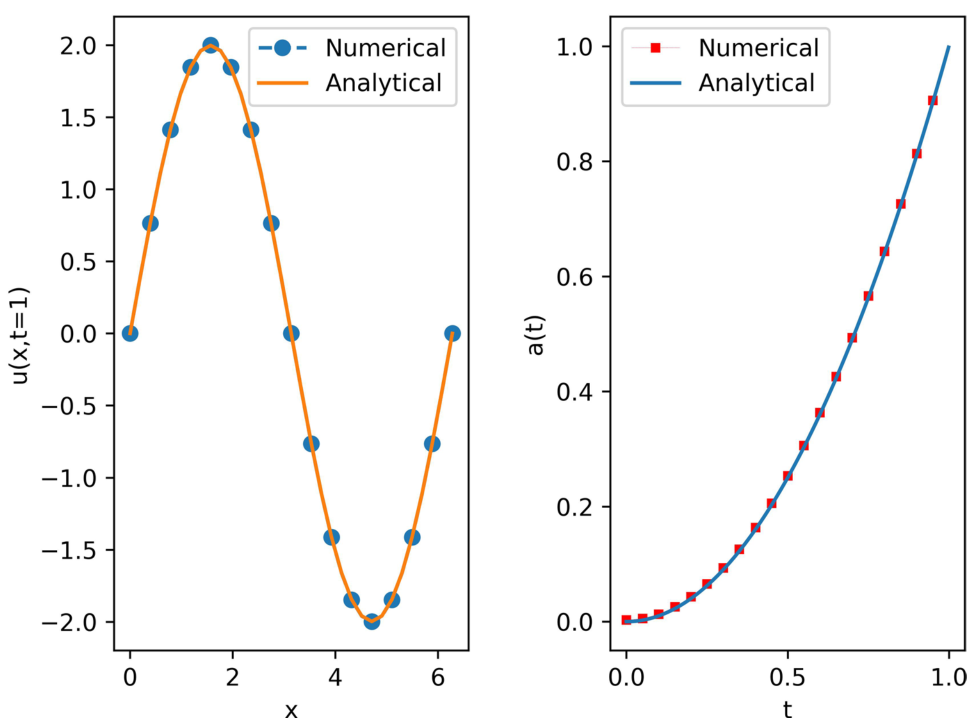


**Fig. 6** The analytical and numerical solutions of $u(x, t)$ and $a(t)$ when $T = 1$, $\alpha = 1$

## Conclusion

The inverse problem regarding the simultaneous identification of the time-dependent coefficient $a(t)$ and the function $u(x, t)$ in one-dimensional diffusion equation with periodic boundary and integral overdetermination conditions has been considered.

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## Declarations

**Conflict of interest** The author declares no competing interests.

## REFERENCES

1. R. Magin, "Fractional Calculus in bioengeenering", Crit.Rev. Biomed. Eng., **1**(32), 1–104(2004).
2. F. Mainardi, *Fractal and Fractional in Continuum Mechanics,* (Springer, New York, 1997).
3. K. Nishimoto, *Fractional Calculus and Its Applications,* (Nihon University, Koriyama, 1990).
4. K. B. Oldham, "Fractional Differential Equations in electrochemistry", Adv.Eng.Softw., **1**(41), 9–14(2010).
5. V. Isakov, *Inverse Problems for Partial Differential Equations,* (Berlin, Springer, 1998).
6. A. Prilepko, D. Orlovsky, I. Vasin, *Methods for Solving Inverse Problems in Mathematical Physics*, (New York: Marcel Dekker, 2000).
7. J. R. Cannon, Yanping Lin, and Shuzhan Xu, "Numerical procedures for the determination of an unknown coefficient in semi-linear parabolic differential equations," Inverse problems, **10**, 227–243 (1994). https://doi.org/10.1088/0266-5611/10/2/004.
8. A. M. Kardashevsky, V. V. Popov, Z. Guo, "Recovery of the non-stationary component of the right-hand side of the subdiffusion equation," Eurasian Journal of Mathematical and computer applications, **12**(1), 83–93 (2024).
9. M. I. Ismailova, F. Kanca, "An inverse coefficient problem for a parabolic equation in the case of nonlocal boundary and overdetermination conditions," Math. Meth. Appl. Sci., (2010). https://doi.org/10.1002/mma.1396
10. F. Kanca, "Inverse coefficient problem of the parabolic equation with periodic boundary and integral overdetermination conditions," Generalized Differential and Integral Equations, **2013**. https://doi.org/10.1155/2013/659804.
11. D. K. Durdiev, J. J. Jumaev, "Inverse Coefficient Problem for a Time-Fractional Diffusion Equation in the Bounded Domain," Lobachevskii journal of mathematics, **2**(44), 548–557 (2023).
12. D. K. Durdiev and A. A. Rahmonov, "A multidimensional diffusion coefficient determination problem for the time-fractional equation," Turkish J. of Mathematics, **6**(46), 2250–2263 (2021).
13. D. K. Durdiev, A. A. Rahmonov, Z. R. Bozorov, "A two-dimensional diffusion coefficient determination problem for the time-fractional equation,"Math. Meth. Appl. Sci., **44**, 10753–10761 (2021).
14. H. H. Turdiev, "Inverse coefficient problems for a time-fractional wave equation with the generalized Riemann-Liouville time derivative," Russian Mathematics (Izvestiya VUZ. Matematika), (10), 46–59 (2023).
15. D.K. Durdiev, H. H. Turdiev, "Inverse coefficient problem for the fractional wave equation with the generalized Riemann–Liouville time derivative," Indian J. Pure Appl. Math., (2023). https://doi.org/10.1007/s13226-023-00517-9
16. D.K. Durdiev, H. H. Turdiev, "Inverse coefficient problem for a time-fractional wave equation with initial-boundary and integral type overdetermination conditions," Math. Meth. Appl. Sci., (2023). https://doi.org/10.1002/mma.9867
17. Z.A. Subhonova, A. A. Rahmonov, "Problem of determining the time-dependent coefficient in the fractional diffusion-wave equation", Lobachevskii Journal of Mathematics, **15**(42), 3747-3760 (2021).
18. D.K. Durdiev, Zh. Zh. Zhumaev, "Problem of determining a multidimensional thermal memory in a heat conductivity equation", Meth. Funct. Anal. and Topology, **25** (3), 219–226 (2019).
19. D.K. Durdiev, Zh.Zh. Zhumaev, "Memory kernel reconstruction problems in the integro-differential equation of rigid heat conductor," Math. Meth. Appl. Sci., **45** (14), 8374–8388 (2022).
20. D. K. Durdiev, Zh. Zh. Zhumaev, "One-Dimensional Inverse Problems of Finding the Kernel of Integro- differential Heat Equation in a Bounded Domain", Ukrain. Math. J., **73** (3), 1723–1740 (2022).
21. D. K. Durdiev, Zh. Zh. Zhumaev, "On determination of the coefficient and kernel in an integro-differential equation of parabolic type", Eurasian Journal of Mathematical and computer applications, **11**(1), 49–65 (2023).
22. D.K. Durdiev and J. J. Jumaev, "Inverse problem of determining the kernel of the integro-differential fractional diffusion equation in the bounded domain", Russian mathematics,**10**, 22–35 (2023).
23. D. K. Durdiev, J. J. Jumaev, D. D. Atoev, "Inverse problem on determining two kernels in integro-differential equation of heat flow," Ufa Mathematical Journal, **15**(2), 119-134 (2023).
24. D. K. Durdiev, J. J. Jumaev, D. D. Atoev,"Convolution kernel determining problem for an integro-differential heat equation with nonlocal initial-boundary and overdetermination conditions, "Journal of Mathematical Sciences, (2023). https://doi.org/10.1007/s10958-023-06263-x
25. Z.R. Bozorov, "Numerical determining a memory function of a horizontally-stratified elastic medium with aftereffect ", Eurasian Journal of Mathematical and Computer Applications, **8**(2), 28–40 (2020).
26. P. Vabishchevich, V. Vasilev, "Computational Determination of the Lowest Order Coefficient in a Parabolic Equation", Dokl. Math., **89**(2), 179-181 (2014). https://doi.org/10.1134/S1064562414020161
27. D. K. Durdiev, D. D. Durdiev, "An inverse problem of finding a time-dependent coefficient in a fractional diffusion equation", Turkish Journal of Mathematics, **47**(5), (2023). https://doi.org/10.55730/1300-0098.3439

28. P. N. Vabishchevich, V. I. Vasilev, M. V. Vasileva, "Computational Identification of the Right-hand Side of a Parabolic Equation", Comput. Math. Math. Phys.,**55**(6), 1015-1021 (2015). https://doi.org/10.1134/S0965542515030185
29. A. A. Samarskii, *Theory of Difference Schemes*, (Nauka, Moscow, 1989) [in Russian].
30. P. Zhuang, F. Liu, "Implicit Difference Approximation for the Time Fractional Diffusion Equation", J. Appl. Math. Computing., **22**(3), 87–99 (2006). https://doi.org/10.1007/BF02832039
31. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, (Elsevier, Amsterdam, 2006).