Received: 24 September 2020
DOI: 10.1002/mma.7133

SPECIAL ISSUE PAPER

WILEY

Memory kernel reconstruction problems in the integro-differential equation of rigid heat conductor

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Communicated by: Y. Qin

Funding information
Ministry of Innovation Development of the Republic of Uzbekistan, Grant/Award Number: F-4-02

The inverse problems of determining the energy–temperature relation $\alpha(t)$ and the heat conduction relation $k(t)$ functions in the one-dimensional integro-differential heat equation are investigated. The direct problem is the initial-boundary problem for this equation. The integral terms have the time convolution form of unknown kernels and direct problem solution. As additional information for solving inverse problems, the solution of the direct problem for $x = x_0$ is given. At the beginning, an auxiliary problem, which is equivalent to the original problem, is introduced. Then the auxiliary problem is reduced to an equivalent closed system of Volterra-type integral equations with respect to unknown functions. Applying the method of contraction mappings to this system in the continuous class of functions with weighted norms, we prove the main result of the article, which is a global existence and uniqueness theorem of inverse problem solutions.

KEYWORDS
Green function, initial-boundary problem, integro-differential equation, inverse problem, thermal memory

MSC CLASSIFICATION
35A01; 35A02; 35L02; 35L03; 35R03

1 | INTRODUCTION

Integro-differential equations arise in many fields of physics and applied mathematics for modeling the processes of heat transfer with finite propagation speed, systems with thermal memory, viscoelasticity problems, and acoustic waves in composite media. In their study,[1] Gurtin and Pipkin derived the integro-differential equation

$$u_{tt} = \triangle u(x,\tau) + \int_0^t K'(t-\tau) \triangle u(x,\tau)d\tau + h(x,t), \tag{1}$$

describing propagation of heat in media with memory at a finite speed. Here, Δ is the Laplace operator in the variables $x = (x_1, \dots, x_n)$. Along with Equation (1), in the literatures, it is considered the equation

$$u_t(x,t) = \int_0^t K(t-\tau) \triangle u(x,\tau)d\tau + g(x,t) \tag{2}$$

of the first order in the time variable t. Nowadays, Equations (1) and (2) are referred to as the Gurtin–Pipkin equations. It can readily be seen that Equation (1) is derived from (2) by differentiating with respect to variable t if we set $K(0) = 1$ and $h(x, t) = g_t(x, t)$.

In a previous study,[2] Miller studied existence, uniqueness, and continuous dependence on parameters for solutions of the certain initial-boundary value problem for following system of integro-differential equations:

$$\begin{aligned} e(t,x) &= e_0 + \alpha(0)\theta(t,x) + \int_0^t \alpha'(t-\tau)\theta(\tau,x)d\tau, \\ q(t,x) &= -\, k(0)\theta_x(t,x) - \int_0^t k'(t-\tau)\theta_x(\tau,x)d\tau, \\ e_t(t,x) &= -\, q_x(t,x) + r(t,x), \end{aligned} \tag{3}$$

where $0 \le t < \infty$, $x \in (0; l)$, $e_t = (\partial/\partial t)\, e$, $q_x = (\partial/\partial x)\, q$. In (3), $\alpha(t)$ and $k(t)$ are relaxation functions of internal energy and heat flow, respectively. Moreover, $\theta(t,x)$ is a function of temperature, and $r(t,x)$ is an external heat source function. The first and second equalities in Equation (3) are linearized (with respect to certain constant e_0 energy) constitutive equations for internal energy and heat flow, respectively. And the third relation in (3) expresses the fundamental law of thermal conductivity—Fourier's law. For $k(0) = 0$, these equations represent the linearized theory for heat flow in a rigid, isotropic, homogeneous material as proposed by Gurtin and Pipkin (see e.g., Gurtin and Pipkin[1,3]). For $k(0) > 0$, the equations represent an alternate linearized theory proposed by Coleman and Gurtin.[4] For the direct problem consisting in determining the distribution of heat from some initial-boundary value problem for Equation (3), Grabmueller[5] gave a very general uniqueness proof for generalized solutions in a Sobolev space and proved existence theorems in certain special situations.

The determination of the integral operator from the observable information about the solutions of the corresponding equations is a new class of inverse problems that has not yet been sufficiently studied. In view of a wide range of applications, the theory of inverse problems for integro-differential equations is one of the most urgent and rapidly developing fields of world science.

The problem of determining the kernel $K(t)$ of the integral term in Equation (1) was studied in many publications[6-31] (see also references in them) in which both one- and multidimensional inverse problems were investigated. In these works, the questions of correctness of the considered problems were studied. The numerical solutions for this problems were considered in the works.[32-35] Direct and inverse problems for an anomalous diffusion equation were investigated in Bondarenko et al.[36,37]

In the present paper, we study the inverse problems about determining the kernels of an integral convolution-type terms in the system of integro-differential Equation (3) by the single observation at the point $x = x_0$ from below Equations (5–7).

Among the works which are close to the problem under study below we note.[38-42] In Durdiev,[38] there was proven the uniqueness theorem for solution of kernel determination problem for one-dimensional heat conduction equation. The papers[39-42] deal with the inverse problems of determining the kernel depending on a time variable t and $(n-1)$-dimensional spatial variable $x' = (x_1, \ldots, x_{n-1})$. While the main part of the considered integro-differential equation is n-dimensional heat conduction operator and the integral term has a convolution type form with respect to unknown functions: the solutions of direct and inverse problem. In these works, the theorems of existence and uniqueness of problems solutions were obtained.

2 | SETTING UP THE PROBLEM

It is supposed the rigid body will occupy a fixed open interval $(0, l)$ (one-dimensional case). The energy–temperature relation function $\alpha(t)$ and the heat conduction relation $k(t)$ are both assumed sufficiently continuously differentiable functions.

From (3), it follows that

$$\begin{aligned} \theta_t(t,x) = &-\frac{\alpha'(0)}{\alpha(0)}\theta(t,x) + \frac{k(0)}{\alpha(0)}\theta_{xx}(t,x) + \\ &+ \int_0^t \left[\frac{k'(t-\tau)}{\alpha(0)}\theta_{xx}(\tau,x) - \frac{\alpha''(t-\tau)}{\alpha(0)}\theta(\tau,x)\right] d\tau + \frac{r(t,x)}{\alpha(0)}. \end{aligned} \tag{4}$$

Everywhere in this paper, it is supposed $\alpha(0)$ and $k(0)$ are given numbers such that $k(0) > 0$, $\alpha(0) \neq 0$. Rewrite the Equation (4) in the compact form:

$$\theta_t(t,x) = f(t,x) + C\theta_{xx}(t,x) - a(0)\theta(t,x) + \int_0^t \left[Cb(t-\tau)\theta_{xx}(\tau,x) - a'(t-\tau)\theta(\tau,x)\right] d\tau \tag{5}$$

for all $t \geq 0, x \in (0;l)$ and consider the initial-boundary value problem with

$$\theta(0,x) = \theta_0(x), \tag{6}$$

$$\theta(t,0) = \mu_1(t);\ \theta(t,l) = \mu_2(t);\ \theta_0(0) = \mu_1(0);\ \theta_0(l) = \mu_2(0); \tag{7}$$

the initial and boundary conditions, where

$$C = \frac{k(0)}{\alpha(0)},\ a(t) = \frac{\alpha'(t)}{\alpha(0)},\ b(t) = \frac{k'(t)}{k(0)},\ f(t,x) = \frac{r(t,x)}{\alpha(0)}.$$

In equalities (6) and (7), $\theta_0(x)$, $\mu_1(t)$, and $\mu_1(t)$ are given functions. If $r(t,x)$, $\theta_0(x)$, $\alpha(t)$, $k(t)$, $\mu_1(t)$, $\mu_2(t)$ are given functions, then finding the function $\theta(t,x)$ from (5) to (7) is called as a direct problem.

We pose the inverse problems:

Inverse problem 1. For given functions $r(t,x)$, $\theta_0(x)$, $k(t)$, $\mu_1(t)$, $\mu_2(t)$, it is required to determine the function $\alpha(t), t > 0$ of the integral term in (5) using additional information about the solution of the direct problems (5–7):

$$\theta|_{x=x_0} = \psi(t),\ x_0 \in (0,l),\ t > 0 \tag{8}$$

In this case, $\psi(t), t > 0$ are assumed to be given functions.

Inverse problem 2. For given functions $r(t,x)$, $\theta_0(x)$, $\alpha(t)$, $\mu_1(t)$, $\mu_2(t)$, it is required to determine the function $k(t), t > 0$ of the integral term in (5) using additional information (8) on the solution of the direct problems (5–7).

Since the method for studying the inverse problems allow to find simultaneously the solution to the inverse problem and the solution to the direct problem, then in the sequel, we will call the inverse problem 1 as a problem of determining functions $\theta(t,x)$, $\alpha(t)$ from Equations (5–8).

3 | PRELIMINARIES

Let $C^m\ (0;l)$ be the class of m times continuously differentiable with all derivatives up to the m-th order (inclusive) in $(0;l)$ functions. In the case $m = 0$, this space coincides with the class of continuous functions. $C^{m,k}(D_T)$ is the class of m times continuously differentiable with respect to t and k times continuously differentiable with respect to x all derivatives in the domain D_T functions.

We need the following assertion:

Lemma 1 (see Miller[2]). *Suppose $\alpha(0) > 0$, $\alpha \in C^3[0,T]$, $k \in C^2[0,T]$, $T > 0$ is an arbitrary fixed number, are true with $k(0) > 0$. Then Equation (3) is equivalent to the following integro-differential equation:*

$$\frac{\partial\theta}{\partial t}(t,x) = F(t,x) + C\triangle\theta(t,x) + y(0)\theta(t,x) + \int_0^t y'(t-\tau)\theta(\tau,x)d\tau, \tag{9}$$

where F is defined as

$$F(t,x) = f(t,x) - \int_0^t D(t-\tau)f(\tau,x)d\tau + D(t)\theta(0,x),$$

and where $D(t)$ and $y(t)$ satisfy the scalar equations

$$D(t) = b(t) - \int_0^t b(t-\tau)D(\tau)d\tau, \tag{10}$$

$$y(t) = b(t) - a(t) - \int_0^t b(t-\tau)y(\tau)d\tau. \tag{11}$$

If $b(t)$ function is continuously for $t > 0$, then the solution to the integral Equation (10) exists and unique. Note that for given Equation (11), it can be considered to be an integral Volterra equation of the second kind with respect to $y(t)$ with the kernel $b(t)$,

$$y(t) = -\int_0^t b(t-\tau)y(\tau)d\tau + [b(t) - a(t)]. \tag{12}$$

It follows from the general theory of integral equations (see, e.g., Kilbas[43, pp.39–44]) that the solution of this equation is expressed by the formula

$$y(t) = b(t) - a(t) + \int_0^t R(t-\tau)\left[b(\tau) - a(\tau)\right] d\tau, \tag{13}$$

where the kernels $R(t)$ and $b(t)$ are related by

$$b(t) = -R(t) - \int_0^t R(t-\tau)b(\tau)d\tau. \tag{14}$$

If $b(0)$ is a known number, from relation (14), we find $R(0) = -b(0)$.

4 | PROBLEM OF DETERMINING THE FUNCTIONS $\theta(t,x), \alpha(t)$

In this section, existence and uniqueness for the inverse problem (9) and (6–8) are proved using the contraction mapping principle.[44, pp. 87–97] The idea is to write the integral equations for unknown functions $\theta(x,t),\ a(t)$ as a system with a non-linear operator and prove that this operator is a contraction mapping operator. The existence and uniqueness then follow immediately.

The solution of the initial-boundary problems (9), (6), and (7) satisfy the integral equation:[45, pp. 200–221]

$$\begin{aligned}\theta(t,x) = \Psi(t,x) + \int_0^t \int_0^l G(t-\tau,x,\xi)\left(y(0)\theta(\tau,\xi) + \int_0^\tau y'(\tau-\alpha)\theta(\alpha,\xi)d\alpha\right) d\xi d\tau = \\ = \Psi(t,x) + \int_0^t \int_0^l G(t-\tau,x,\xi)\theta_0(\xi)y(\tau)d\xi d\tau + \\ + \int_0^t \int_0^l G(t-\tau,x,\xi)\int_0^\tau y(\alpha)\theta_\alpha(\tau-\alpha,\xi)d\alpha d\xi d\tau,\end{aligned} \tag{15}$$

where

$$\begin{aligned}\Psi(t,x) = \int_0^l G(t,x,\xi)\theta_0(\xi)d\xi + \int_0^t \int_0^l G(t-\tau,x,\xi)F(\tau,\xi)d\xi d\tau + \\ + \sum_{n=1}^{\infty}\int_0^t \frac{2\pi n}{l^2}\left[\mu_1(\tau) - (-1)^n\mu_2(\tau)\right] e^{-\left(\frac{\pi n}{l}\right)^2(t-\tau)} sin\left(\frac{\pi n}{l}x\right) d\tau; \\ G(t-\tau,x,\xi) = \frac{2}{l}\sum_{n=1}^{\infty} e^{-\left(\frac{\pi an}{l}\right)^2(t-\tau)} sin\left(\frac{\pi an}{l}\xi\right) sin\left(\frac{\pi an}{l}x\right)\end{aligned}$$

is the Green function of the initial-boundary problem for one-dimensional heat equation.

We differentiate the Equation (15) with respect to t. Introducing the notation $\vartheta(t,x) := \theta_t(t,x)$ and taking into account the following relations:

$$\lim_{t\to 0} G(t,\xi,x) = \delta(x-\xi),\ \lim_{t\to 0}\int_0^l G(t,x,\xi)\theta_0(\xi)d\xi = \theta_0(x),$$

where $\delta(\cdot)$ is the Dirac's delta function, we rewrite the result in the form

$$\vartheta(t,x) = \Psi_t(t,x) + \theta_0(x)y(t) + \int_0^t y(\alpha)\vartheta(t-\alpha,x)d\alpha + \\ + \int_0^t \int_0^l G_t(t-\tau,x,\xi)\theta_0(\xi)y(\tau)d\xi d\tau + \int_0^t \int_0^l G_t(t-\tau,x,\xi) \int_0^\tau y(\alpha)\vartheta(\tau-\alpha,\xi)d\alpha d\xi d\tau. \tag{16}$$

Further, using the condition (8), we obtain:

$$\psi'(t) = \Psi_t(t,x_0) + \theta_0(x_0)y(t) + \int_0^t y(\alpha)\vartheta(t-\alpha,x_0)d\alpha + \\ + \int_0^t \int_0^l G_t(t-\tau,x_0,\xi)\theta_0(\xi)y(\tau)d\xi d\tau + \int_0^t \int_0^l G_t(t-\tau,x_0,\xi) \int_0^\tau y(\alpha)\vartheta(\tau-\alpha,\xi)d\alpha d\xi d\tau.$$

Next we write this equality as the integral equation of the second order with respect to unknown function $y(t)$

$$y(t) = -\frac{1}{\theta_0(x_0)} \left[\Psi_t(t,x_0) - \psi'(t) + \int_0^t y(\alpha)\vartheta(t-\alpha,x_0)d\alpha + \right. \\ \left. + \int_0^t \int_0^l G_t(t-\tau,x_0,\xi)\theta_0(\xi)y(\tau)d\xi d\tau + \int_0^t \int_0^l G_t(t-\tau,x_0,\xi) \int_0^\tau y(\alpha)\vartheta(\tau-\alpha,\xi)d\alpha d\xi d\tau \right]. \tag{17}$$

Replacing $t = 0$ in integral Equation (17), the unknown function $y(0)$ is found as follows:

$$y(0) = \frac{\psi'(0) - \Psi_t(0,x_0)}{\theta_0(x_0)}.$$

In what follows, we assume $\theta_0(x_0) \neq 0$.

We represent the system of Equations (16) and (17) in the form

$$Ag = g, \tag{18}$$

where $g = (g_1, g_2) = (\vartheta(x,t) - \theta_0(x)y(t),\ y(t))$ is the vector-function, and unknown functions are represented by g_1, g_2 functions as follows:

$$\vartheta(t,x) = \theta_t(t,x) = g_1(t,x) + \theta_0(x)g_2(t);$$
$$y(t) = g_2(t).$$

$A = (A_1,\ A_2)$ is defined by the right sides of Equations (16) and (17):

$$A_1 g = g_{01} + \int_0^t g_2(\alpha)\left(g_1(t-\alpha,x) + \theta_0(x)g_2(t-\alpha)\right)d\alpha + \\ + \int_0^t \int_0^l G_t(t-\tau,x,\xi)\theta_0(\xi)g_2(\tau)d\xi d\tau + \int_0^t \int_0^l G_t(t-\tau,x,\xi) \times \\ \times \int_0^\tau g_2(\alpha)\left(g_1(\tau-\alpha,\xi) + \theta_0(\xi)g_2(\tau-\alpha)\right)d\alpha d\xi d\tau; \tag{19}$$

$$A_2 g = g_{02} - \frac{1}{\psi(0)} \int_0^t g_2(\alpha)\left(g_1(t-\alpha,x_0) + \theta_0(x_0)g_2(t-\alpha)\right)d\alpha - \\ - \frac{1}{\psi(0)} \int_0^t \int_0^l G_t(t-\tau,x_0,\xi)\theta_0(\xi)g_2(\tau)d\xi d\tau - \\ - \frac{1}{\psi(0)} \int_0^t \int_0^l G_t(t-\tau,x_0,\xi) \int_0^\tau g_2(\alpha)\left(g_1(\tau-\alpha,\xi) + \theta_0(\xi)g_2(\tau-\alpha)\right)d\alpha d\xi d\tau. \tag{20}$$

The following notations were introduced in the equalities (15) and (16):

$$g_0(t,x)=(g_{01}(t,x),\ g_{02}(t))=\left(\Psi_t(t,x),\ -\frac{1}{\psi(0)}\left(\Psi_t(t,x_0)-\psi'(t)\right)\right).$$

Theorem 1 (existence and uniqueness). *Assume the conditions $\theta_0(x)\in C(0,l)$, $\psi(t)\in C[0;T]$, $r(t,x)\in C(D_T)$, $\mu_i(t)\in C[0,T]$, $i=1,2$, $k(t)\in C^2[0,T]$, $\theta_0(x_0)=\psi(0)$, $\theta_0(x_0)\neq 0$, $\theta_0(0)=\mu_1(0)$, $\theta_0(l)=\mu_2(0)$ are hold. Then there exists sufficiently small number $T^*\in(0,T)$ that the solution to the integral Equations (15) and (16) in the class of functions $\vartheta(t,x)\in C^{1,2}(D_{T^*})$, $y(t)\in C[0;T^*]$ exist and unique, where $D_{T^*}=\{(x,t)|x\in(0,l),t\in[0,T^*]\}$.*

To **prove** the Theorem 1, we define for the unknown vector-function $g(x,t)\in C(D_T)$ the following weight norm:

$$\|g\|_\sigma=\max\left\{\sup_{(x,t)\in D_T}\left|g_1(x,t)e^{-\sigma t}\right|,\ \sup_{t\in[0,T]}\left|g_2(t)e^{-\sigma t}\right|\right\}=$$
$$=\max\left\{\|g_1\|_\sigma,\|g_2\|_\sigma\right\},\ \sigma\geq 0.$$

At $\sigma=0$, this norm coincides with the usual norm

$$\|g\|=\max\left\{\sup_{(x,t)\in D_T}|g_1(x,t)|,\ \sup_{t\in[0,T]}|g_2(t)|\right\}.$$

The number $\sigma>0$ will be chosen later. Denote by $S(g_0,\rho)$ the ball of vector-functions g with center at the point g_0 and radius $\rho>0$, that is, $S(g_0,\rho)=\{g:\|g-g_0\|_\sigma\leq\rho\}$. The number $\rho>0$ will be also chosen later.

Obviously, $\|g\|\leq\rho+\|g_0\|$ for $g(x,t)\in S(g_0,\rho)$. We prove that the operator A is contracting in the Banach space $S(g_0,\rho)$ if the numbers σ and ρ will be chosen in suitable way.

Note that the weight norm $\|\cdot\|_\sigma$ is equivalent to the usual norm $\|\cdot\|$:

$$\|\cdot\|_\sigma\leq\|\cdot\|\leq e^{\sigma T}\|\cdot\|_\sigma,\ \sigma\geq 0. \tag{21}$$

The convolution operator is commutative and invariant with respect to multiplication by $e^{-\sigma t}$:

$$(h_1*h_2)(t)=\int_0^t h_1(t-s)h_2(s)ds=\int_0^t h_1(s)h_2(t-s)ds=(h_2*h_1)(t), \tag{22}$$

$$e^{-\sigma t}(h_1*h_2)(t)=(e^{-\sigma t}h_1(t))*(e^{-\sigma t}h_2(t)). \tag{23}$$

The last formula implies the estimation

$$\|h_1*h_2\|_\sigma\leq\|h_1\|_\sigma\|h_2\|_\sigma T. \tag{24}$$

Moreover, since

$$\int_0^t e^{-\sigma s}ds=\int_0^t e^{-\sigma(t-s)}ds\leq\frac{1}{\sigma},\ \sigma\geq 0, \tag{25}$$

we have

$$\|h_1*h_2\|_\sigma\leq\frac{1}{\sigma}\|h_1\|\|h_2\|_\sigma\leq\frac{1}{\sigma}\|h_1\|\|h_2\|,\ \sigma\geq 0 \tag{26}$$

using (21) and the results of Janno and Wolfersdorf.[10]

Now we write two properties of Green function (see Tikhonov and Samarsky,[45, pp. 200–221]), which will be needed in the future.

Remark 1. The integral of the Green function does not exceed 1:

$$\int_0^l G(x,\xi,t)d\xi\leq 1,x\in(0,l),t\in(0,T].$$

Remark 2. The function $G(x, \xi, t)$ is infinitely continuously differentiable with respect to x, ξ, t, and $G_t(x, \xi, t)$ is bounded for $0 < x < l$, $0 < \xi < l$, $0 < t \leq T$, that is,

$$|G_t(x, \xi, t-\tau)| \leq \frac{2}{l}.$$

Now we check the first condition of contractive mapping[44, pp. 87–97] for operator A .
We introduce the notations

$$\theta_0 := \max_{x \in (0;l)} |\theta_0(x)|\,,\ \psi_0 := \max\left\{ \max_{t \in [0,T]} |\psi(t)|\,,\ \max_{t \in [0,T]} \left|\psi'(t)\right| \right\}.$$

Let $g(x, t)$ be an element of $S(g_0, \rho)$, that is, $g \in S(g_0, \rho)$. Then for $(x, t) \in D_T$, we have

$$\|A_1 g - g_{01}\|_\sigma = \sup_{(x,t)\in D_T} |(A_1 g - g_{01}) e^{-\sigma t}| \leq \sup_{(x,t)\in D_T} \left| \int_0^t g_2(\alpha)(g_1(t-\alpha, x) + \theta_0(x) g_2(t-\alpha)) e^{-\sigma t} d\alpha \right| +$$
$$+ \sup_{(x,t)\in D_T} \left| \int_0^t \int_0^l G_t(t-\tau, x, \xi)\theta_0(\xi) g_2(\tau) e^{-\sigma t} d\xi d\tau \right| + \sup_{(x,t)\in D_T} \left| \int_0^t \int_0^l G_t(t-\tau, x, \xi) \times \right.$$
$$\left. \times \int_0^\tau g_2(\alpha)\left(g_1(\tau-\alpha, \xi) + \theta_0(\xi) g_2(\tau-\alpha)\right) e^{-\sigma t} d\alpha d\xi d\tau \right| =: I_1 + I_2 + I_3.$$

We estimate each I_i, $i = 1, 2, 3$, separately:

$$I_1 := \sup_{(x,t)\in D_T} \left| \int_0^t g_2(\alpha)(g_1(t-\alpha, x) + \theta_0(x) g_2(t-\alpha)) e^{-\sigma t} d\alpha \right| \leq$$
$$\leq \sup_{(x,t)\in D_T} \left| \int_0^t g_2(\alpha) g_1(t-\alpha, x) e^{-\sigma t} d\alpha \right| + \sup_{(x,t)\in D_T} \left| \int_0^t g_2(\alpha)\theta_0(x) g_2(t-\alpha) e^{-\sigma t} d\alpha \right| \leq$$
$$\leq \sup_{(x,t)\in D_T} \left|(g_2 * g_1)(t) e^{-\sigma t}\right| + \theta_0 \sup_{(x,t)\in D_T} \left|(g_2 * g_2)(t) e^{-\sigma t}\right| \leq$$
$$\leq \sup_{(x,t)\in D_T} \left| \left\{ \left[(g_2 - g_{02}) * (g_1 - g_{01})\right](t) + (g_2 * g_{01})(t) + (g_1 * g_{02})(t) - (g_{02} * g_{01})(t) \right\} e^{-\sigma t} \right| +$$
$$+ \theta_0 \sup_{(x,t)\in D_T} \left| \left\{ \left[(g_2 - g_{02}) * (g_2 - g_{02})\right](t) + (g_2 * g_{02})(t) + (g_2 * g_{02})(t) - (g_{02} * g_{02})(t) \right\} e^{-\sigma t} \right| \leq$$
$$\leq \left(\|g_2 - g_{02}\|_\sigma \|g_1 - g_{01}\|_\sigma T + \frac{1}{\sigma}\|g_2\|_\sigma \|g_{01}\| + \frac{1}{\sigma}\|g_1\|_\sigma \|g_{02}\| + \frac{1}{\sigma}\|g_{01}\|_\sigma \|g_{02}\| \right) +$$
$$+ \theta_0 \left(\|g_2 - g_{02}\|_\sigma \|g_2 - g_{02}\|_\sigma T + \frac{1}{\sigma}\|g_2\|_\sigma \|g_{02}\| + \frac{1}{\sigma}\|g_2\|_\sigma \|g_{02}\| + \frac{1}{\sigma}\|g_{02}\|_\sigma \|g_{02}\| \right) \leq$$
$$\leq (1 + \theta_0) \left(\rho^2 T + \frac{2}{\sigma}(\rho + \|g_0\|)\|g_0\| + \frac{1}{\sigma}\|g_0\|^2 \right);$$
$$I_2 := \sup_{(x,t)\in D_T} \left| \int_0^t \int_0^l G_t(t-\tau, x, \xi)\theta_0(\xi) g_2(\tau) e^{-\sigma t} d\xi d\tau \right| \leq \frac{2\theta_0(\rho + \|g_0\|)}{\sigma};$$
$$I_3 := \sup_{(x,t)\in D_T} \left| \int_0^t \int_0^l G_t(t-\tau, x, \xi) \int_0^\tau g_2(\alpha)\left(g_1(\tau-\alpha, \xi) + \theta_0(\xi) g_2(\tau-\alpha)\right) e^{-\sigma t} d\alpha d\xi d\tau \right| \leq$$
$$\leq \sup_{(x,t)\in D_T} \left| \int_0^t \int_0^l G_t(t-\tau, x, \xi) \int_0^\tau g_2(\alpha) g_1(\tau-\alpha, \xi) e^{-\sigma t} d\alpha d\xi d\tau \right| +$$
$$+ \sup_{(x,t)\in D_T} \left| \int_0^t \int_0^l G_t(t-\tau, x, \xi) \int_0^\tau g_2(\tau-\alpha)\theta_0(\xi) g_2(\alpha) e^{-\sigma t} d\alpha d\xi d\tau \right| \leq$$
$$\leq \frac{2(\rho + \|g_0\|)^2 T}{\sigma} + \frac{2\theta_0(\rho + \|g_0\|)^2 T}{\sigma} = (1 + \theta_0)\frac{2(\rho + \|g_0\|)^2 T}{\sigma}.$$

Accordingly, we get

$$\|A_1 g - g_{01}\|_\sigma \leq (1+\theta_0)\left(\rho^2 T + \frac{2}{\sigma}(\rho + \|g_0\|)\|g_0\| + \frac{1}{\sigma}\|g_0\|^2\right) + \frac{2\theta_0(\rho + \|g_0\|)}{\sigma} + $$
$$+ (1+\theta_0)\frac{2(\rho + \|g_0\|)^2 T}{\sigma} = (1+\theta_0)T\rho^2 + (2\|g_0\|(1+\theta_0) + 2\theta_0)\frac{(\rho + \|g_0\|)}{\sigma} +$$
$$+ (1+\theta_0)\frac{1}{\sigma}\|g_0\|^2 + (1+\theta_0)\frac{2(\rho + \|g_0\|)^2 T}{\sigma}.$$

Now we can choose ρ, σ such that there hold the inequalities:

$$\begin{cases} (1+\theta_0)T\rho^2 < \frac{1}{4}\rho, \\ (2\|g_0\|(1+\theta_0) + 2\theta_0)\frac{(\rho+\|g_0\|)}{\sigma} < \frac{1}{4}\rho, \\ (1+\theta_0)\frac{1}{\sigma}\|g_0\|^2 < \frac{1}{4}\rho, \\ (1+\theta_0)\frac{2(\rho+\|g_0\|)^2 T}{\sigma}) < \frac{1}{4}\rho. \end{cases}$$

It follows that if

$$\begin{cases} \rho < \frac{1}{4(1+\theta_0)T} = \rho_1, \\ \beta_1 = 4(2\|g_0\|(1+\theta_0) + 2\theta_0)\frac{(\rho_1+\|g_0\|)}{\rho_1} < \sigma, \\ \beta_2 = (1+\theta_0)\frac{4}{\rho_1}\|g_0\|^2 < \sigma, \\ \beta_3 = (1+\theta_0)\frac{8(\rho_1+\|g_0\|)^2 T}{\rho_1} < \sigma. \end{cases}$$

then $A_1 g \in S(g_0, \rho)$.

So, if the inequality

$$\sigma > \sigma_1 = \max\{\beta_1, \beta_2, \beta_3\}$$

and $\rho \in (0, \rho_1)$ holds, then the operator A_1 maps $S(g_0, \rho)$ into itself, that is, $A_1 g \in S(g_0, \rho)$.

$$\|A_2 g - g_{02}\|_\sigma = \sup_{t\in[0,T]} \left|(A_2 g - g_{02})e^{-\sigma t}\right| \leq \sup_{t\in[0,T]} \left| \frac{1}{\theta_0(x_0)} \int_0^t g_2(\alpha)(g_1(t-\alpha, x_0) + \right.$$
$$\left. + \theta_0(x_0)g_2(t-\alpha))e^{-\sigma t} d\alpha \right| + \sup_{t\in[0,T]} \left| \frac{1}{\theta_0(x_0)} \int_0^t \int_0^l G_t(t-\tau, x_0, \xi)\theta_0(\xi)g_2(\tau)e^{-\sigma t} d\xi d\tau \right| +$$
$$+ \sup_{t\in[0,T]} \left| \frac{1}{\theta_0(x_0)} \int_0^t \int_0^l G_t(t-\tau, x_0, \xi) \int_0^\tau g_2(\alpha)\left(g_1(\tau-\alpha, \xi) + \theta_0(\xi)g_2(\tau-\alpha)\right) e^{-\sigma t} d\alpha d\xi d\tau \right| \leq$$
$$\leq \frac{(1+\theta_0)}{\theta_0(x_0)}\left(\rho^2 T + \frac{2}{\sigma}(\rho + \|g_0\|)\|g_0\| + \frac{1}{\sigma}\|g_0\|^2\right) + \frac{2\theta_0(\rho + \|g_0\|)}{\theta_0(x_0)\sigma} + (1+\theta_0)\frac{2(\rho + \|g_0\|)^2 T}{\theta_0(x_0)\sigma} =$$
$$= \frac{(1+\theta_0)T\rho^2}{\theta_0(x_0)} + (2\|g_0\|(1+\theta_0) + 2\theta_0)\frac{(\rho + \|g_0\|)}{\theta_0(x_0)\sigma} + (1+\theta_0)\frac{1}{\theta_0(x_0)\sigma}\|g_0\|^2 + (1+\theta_0)\frac{2(\rho + \|g_0\|)^2 T}{\theta_0(x_0)\sigma}.$$

Now we can choose ρ, σ such that there hold the inequalities:

$$\begin{cases} \frac{(1+\theta_0)T\rho^2}{\theta_0(x_0)} < \frac{1}{4}\rho, \\ (2\|g_0\|(1+\theta_0) + 2\theta_0)\frac{(\rho+\|g_0\|)}{\theta_0(x_0)\sigma} < \frac{1}{4}\rho, \\ (1+\theta_0)\frac{1}{\theta_0(x_0)\sigma}\|g_0\|^2 < \frac{1}{4}\rho, \\ (1+\theta_0)\frac{2(\rho+\|g_0\|)^2 T}{\theta_0(x_0)\sigma} < \frac{1}{4}\rho. \end{cases}$$

It follows that if

$$\begin{cases} \rho < \frac{\theta_0(x_0)}{4(1+\theta_0)T} = \rho_2, \\ \beta_4 = 4(2\|g_0\|(1+\theta_0) + 2\theta_0)\frac{(\rho_1+\|g_0\|)}{\theta_0(x_0)\rho_2} < \sigma, \\ \beta_5 = (1+\theta_0)\frac{4}{\theta_0(x_0)\rho_2}\|g_0\|^2 < \sigma, \\ \beta_6 = (1+\theta_0)\frac{8(\rho_1+\|g_0\|)^2 T}{\theta_0(x_0)\rho_2} < \sigma. \end{cases}$$

then $A_2 g \in S(g_0, \rho)$.

So, if the inequality

$$\sigma > \sigma_2 = \max\{\beta_4, \beta_5, \beta_6\}$$

and $\rho \in (0, \rho_1)$ holds, then the operator A_2 maps $S(g_0, \rho)$ into itself, that is, $A_2 g \in S(g_0, \rho)$.

As a result, we conclude that if σ, ρ satisfy the conditions $\sigma > \max\{\sigma_1, \sigma_2\}$, $\rho \in (0, \rho_2)$, then operator A maps $S(g_0, \rho)$ into itself, that is, $Ag \in S(g_0, \rho)$.

Further, we check the second condition of contractive mapping. In accordance with (19) for the first component of operator A, we get

$$\begin{aligned} \|(Ag^1 - Ag^2)_1\|_\sigma = & \sup_{(x,t)\in D_T} \left| \int_0^t \left[g_2^1(\alpha) g_1^1(t-\alpha, x) - g_2^2(\alpha) g_1^2(t-\alpha, x) \right] d\alpha e^{-\sigma t} \right| + \\ & + \sup_{(x,t)\in D_T} \left| \int_0^t \theta_0(x) [g_2^1(\alpha) g_2^1(t-\alpha) - g_2^2(\alpha) g_2^2(t-\alpha)] d\alpha e^{-\sigma t} \right| + \\ & + \sup_{(x,t)\in D_T} \left| \int_0^t \int_0^l G_t(t-\tau, x, \xi)\theta_0(\xi) \left[g_2^1(\tau) - g_2^2(\tau) \right] d\xi d\tau e^{-\sigma t} \right| + \\ & + \sup_{(x,t)\in D_T} \left| \int_0^t \int_0^l G_t(t-\tau, x, \xi) \times \right. \\ & \left. \times \int_0^\tau \left[g_2^1(\alpha) g_1^1(\tau-\alpha, \xi) - g_2^2(\alpha) g_1^2(\tau-\alpha, \xi) \right] d\alpha d\xi d\tau e^{-\sigma t} \right| + \\ & + \sup_{(x,t)\in D_T} \left| \int_0^t \int_0^l G_t(t-\tau, x, \xi) \times \right. \\ & \left. \times \int_0^\tau \theta_0 \left[g_2^1(\alpha) g_2^1(\tau-\alpha) - g_2^2(\alpha) g_2^2(\tau-\alpha) \right] d\alpha d\xi d\tau e^{-\sigma t} \right| =: \sum_{i=1}^5 J_i. \end{aligned}$$

We denoted the summands in this equality by $J_i (i = \overline{1, 5})$, respectively, and carry out the estimates for them separately.

Taking into account the relation

$$\begin{aligned} g_2^1 * g_1^1 - g_2^2 * g_1^2 = & \left(g_2^1 - g_2^2 \right) * \left(g_1^1 - g_{01} \right) + \left(g_1^1 - g_1^2 \right) * \left(g_2^2 - g_{02} \right) + \\ & + g_{01} * \left(g_2^1 - g_2^2 \right) + g_{02} * \left(g_1^1 - g_1^2 \right), \end{aligned}$$

estimate the J_1, J_2 as follows:

$$\begin{aligned}
J_1 :=& \sup_{(x,t)\in D_T} \left| \int_0^t \left[g_2^1(\alpha) g_1^1(t-\alpha,x) - g_2^2(\alpha) g_1^2(t-\alpha,x) \right] d\alpha e^{-\sigma t} \right| = \\
=& \sup_{(x,t)\in D_T} \left| \int_0^t \left[g_2^1 * g_1^1 - g_2^2 * g_1^2 \right] d\alpha e^{-\sigma t} \right| \leq \\
\leq& \left[\left\| g_2^1 - g_2^2 \right\|_\sigma \| g_1^1 - g_{01} \|_\sigma T + \left\| g_1^1 - g_1^2 \right\|_\sigma \left\| g_2^2 - g_{02} \right\|_\sigma T + \| g_{01} \|_\sigma \left\| g_2^1 - g_2^2 \right\|_\sigma + \right. \\
& \left. + \| g_{02} \|_\sigma \left\| g_1^1 - g_1^2 \right\|_\sigma \right] \leq 2 \left(\rho T + \frac{1}{\sigma} \| g_0 \| \right) \left\| g^1 - g^2 \right\|_\sigma, \\
J_2 :=& \sup_{(x,t)\in D_T} \left| \int_0^t \theta_0(x) \left[g_2^1(\alpha) g_2^1(t-\alpha) - g_2^2(t-\alpha) g_2^2(\alpha) \right] d\alpha e^{-\sigma t} \right| = \\
=& \sup_{(x,t)\in D_T} \left| \int_0^t \theta_0(x) \left[g_2^1 * g_2^1 - g_2^2 * g_2^2 \right] d\alpha e^{-\sigma t} \right| \leq \\
\leq& \theta_0 \left[\left\| g_2^1 - g_2^2 \right\|_\sigma \| g_2^1 - g_{02} \|_\sigma T + \left\| g_2^1 - g_2^2 \right\|_\sigma \left\| g_2^2 - g_{02} \right\|_\sigma T + \| g_{02} \|_\sigma \left\| g_2^1 - g_2^2 \right\|_\sigma + \right. \\
& \left. + \| g_{02} \|_\sigma \left\| g_2^1 - g_2^2 \right\|_\sigma \right] \leq 2\theta_0 \left(\rho T + \frac{1}{\sigma} \| g_0 \| \right) \left\| g^1 - g^2 \right\|_\sigma, \\
J_3 :=& \sup_{(x,t)\in D_T} \left| \int_0^t \int_0^l G_t(t-\tau,x,\xi) \theta_0(\xi) \left[g_2^1(\tau) - g_2^2(\tau) \right] d\xi d\tau e^{-\sigma t} \right| \leq \\
\leq& \frac{2\theta_0}{\sigma} \left\| g^1 - g^2 \right\|_\sigma, \\
J_4 :=& \sup_{(x,t)\in D_T} \left| \int_0^t \int_0^l G_t(t-\tau,x,\xi) \int_0^\tau \left[g_2^1(\alpha) g_1^1(\tau-\alpha,\xi) - g_2^2(\alpha) g_1^2(\tau-\alpha,\xi) \right] d\alpha d\xi d\tau e^{-\sigma t} \right| \leq \\
\leq& \frac{4T}{\sigma} (\rho + \| g_0 \|) \left\| g^1 - g^2 \right\|_\sigma, \\
J_5 :=& \sup_{(x,t)\in D_T} \left| \int_0^t \int_0^l G_t(t-\tau,x,\xi) \int_0^\tau \theta_0 \left[g_2^1(\alpha) g_2^1(\tau-\alpha) - g_2^2(\alpha) g_2^2(\tau-\alpha) \right] d\alpha d\xi d\tau e^{-\sigma t} \right| \leq \\
\leq& \frac{4\theta_0 T}{\sigma} (\rho + \| g_0 \|) \left\| g^1 - g^2 \right\|_\sigma.
\end{aligned}$$

Here, the integrand in the last integral can be estimated as follows:

$$\left\| g_2^1 g_1^1 - g_2^2 g_1^2 \right\|_\sigma = \left\| (g_2^1 - g_2^2) g_1^1 + g_2^2 (g_1^1 - g_1^2) \right\|_\sigma \leq$$

$$\leq 2 \left\| g^1 - g^2 \right\|_\sigma \max \left(\left\| g_1^1 \right\|_\sigma, \left\| g_2^2 \right\|_\sigma \right) \leq 2(\| g_0 \| + \rho) \left\| g^1 - g^2 \right\|_\sigma.$$

Summing the obtained estimates for J_i, $i = 1, 2, .., 5$, we have that the first component of A can be estimated in the following form:

$$\begin{aligned}
& \| (Ag^1 - Ag^2)_1 \|_\sigma \leq 2 \left(\rho T + \frac{1}{\sigma} \| g_0 \| \right) \left\| g^1 - g^2 \right\|_\sigma + \\
& + 2\theta_0 \left(\rho T + \frac{1}{\sigma} \| g_0 \| \right) \left\| g^1 - g^2 \right\|_\sigma + \frac{2\theta_0}{\sigma} \left\| g^1 - g^2 \right\|_\sigma + \frac{4T}{\sigma} (\rho + \| g_0 \|) \left\| g^1 - g^2 \right\|_\sigma + \\
& + \frac{4\theta_0 T}{\sigma} (\rho + \| g_0 \|) \left\| g^1 - g^2 \right\|_\sigma = \\
= & \left((2 + 2\theta_0) \rho T + (2\| g_0 \| + 2\theta_0 \| g_0 \| + 2\theta_0) \frac{1}{\sigma} + (4\theta_0 T + 4T)(\rho + \| g_0 \|) \frac{1}{\sigma} \right) \left\| g^1 - g^2 \right\|_\sigma.
\end{aligned}$$

Now we choose numbers σ, ρ so that the expression at $\|g^1 - g^2\|_\sigma$ becomes less than 1, that is, the inequality

$$(2+2\theta_0)\rho T + (2\|g_0\| + 2\theta_0\|g_0\| + 2\theta_0)\frac{1}{\sigma} + (4\theta_0 T + 4T)(\rho + \|g_0\|)\frac{1}{\sigma} < 1$$

is fulfilled. This inequality is valid if numbers σ, ρ will be chosen from conditions

$$\begin{cases} (2+2\theta_0)\rho T < \frac{1}{3}, \\ (2\|g_0\| + 2\theta_0\|g_0\| + 2\theta_0)\frac{1}{\sigma} < \frac{1}{3}, \\ (4\theta_0 T + 4T)(\rho + \|g_0\|)\frac{1}{\sigma} < \frac{1}{3}. \end{cases}$$

Solving these inequalities with respect to σ, ρ, we obtain

$$\begin{cases} \rho < \frac{1}{3(2+2\theta_0)} = \rho_3, \\ \beta_7 = 3(2\|g_0\| + 2\theta_0\|g_0\| + 2\theta_0) < \sigma, \\ \beta_8 = 3(4\theta_0 T + 4T)(\rho_3 + \|g_0\|) < \sigma. \end{cases}$$

From these estimates, it is clear that if σ and ρ are chosen from condition $\sigma > \sigma_5 = max(\beta_7, \beta_8)$ and $\rho < (0, \rho_3)$, then the operator A_2 satisfies the second condition of contracting mapping.

The second component of A can be estimated in the following form:

$$\begin{aligned} \|(Ag^1 - Ag^2)_2\|_\sigma = {} & \frac{1}{\theta_0(x_0)} \sup_{(x,t)\in D_T} \left| \int_0^t \left[g_2^1(\alpha)g_1^1(t-\alpha, x_0) - g_2^2(\alpha)g_1^2(t-\alpha, x_0)\right] d\alpha e^{-\sigma t} \right| + \\ & + \frac{1}{\theta_0(x_0)} \sup_{(x,t)\in D_T} \left| \int_0^t \theta_0(x) \left[g_2^1(\alpha)g_2^1(t-\alpha) - g_2^2(\alpha)g_2^2(t-\alpha)\right] d\alpha e^{-\sigma t} \right| + \\ & + \frac{1}{\theta_0(x_0)} \sup_{(x,t)\in D_T} \left| \int_0^t \int_0^l G_t(t-\tau, x_0, \xi)\theta_0(\xi) \left[g_2^1(\tau) - g_2^2(\tau)\right] d\xi d\tau e^{-\sigma t} \right| + \\ & + \frac{1}{\theta_0(x_0)} \sup_{(x,t)\in D_T} \left| \int_0^t \int_0^l G_t(t-\tau, x_0, \xi) \times \right. \\ & \left. \times \int_0^\tau \left[g_2^1(\alpha)g_1^1(\tau-\alpha, \xi) - g_2^2(\alpha)g_1^2(\tau-\alpha, \xi)\right] d\alpha d\xi d\tau e^{-\sigma t} \right| + \\ & + \frac{1}{\theta_0(x_0)} \sup_{(x,t)\in D_T} \left| \int_0^t \int_0^l G_t(t-\tau, x_0, \xi) \times \right. \\ & \left. \times \int_0^\tau \theta_0 \left[g_2^1(\alpha)g_2^1(\tau-\alpha) - g_2^2(\alpha)g_2^2(\tau-\alpha)\right] d\alpha d\xi d\tau e^{-\sigma t} \right| \le \\ & \le \left((2+2\theta_0)\frac{\rho T}{\theta_0(x_0)} + (2\|g_0\| + 2\theta_0\|g_0\| + 2\theta_0)\frac{1}{\theta_0(x_0)\sigma} + \right. \\ & \left. + (4\theta_0 T + 4T)(\rho + \|g_0\|)\frac{1}{\theta_0(x_0)\sigma} \right) \left\|g^1 - g^2\right\|_\sigma \end{aligned}$$

Now we choose numbers σ, ρ so that the expression at $\|g^1 - g^2\|_\sigma$ becomes less than 1, that is, the inequality

$$(2+2\theta_0)\frac{\rho T}{\theta_0(x_0)} + (2\|g_0\| + 2\theta_0\|g_0\| + 2\theta_0)\frac{1}{\theta_0(x_0)\sigma} + (4\theta_0 T + 4T)(\rho + \|g_0\|)\frac{1}{\theta_0(x_0)\sigma} < 1$$

is fulfilled. This inequality is true if numbers σ, ρ will be chosen from conditions

$$\begin{cases} (2+2\theta_0)\frac{\rho T}{\theta_0(x_0)} < \frac{1}{3}, \\ (2\|g_0\| + 2\theta_0\|g_0\| + 2\theta_0)\frac{1}{\theta_0(x_0)\sigma} < \frac{1}{3}, \\ (4\theta_0 T + 4T)\,(\rho + \|g_0\|)\,\frac{1}{\theta_0(x_0)\sigma} < \frac{1}{3}. \end{cases}$$

Solving these inequalities with respect to σ, ρ we obtain

$$\begin{cases} \rho < \frac{\theta_0(x_0)}{3(2+2\theta_0)} = \rho_4, \\ \beta_9 = 3(2\|g_0\| + 2\theta_0\|g_0\| + 2\theta_0)\frac{1}{\theta_0(x_0)}\|g_0\| < \sigma, \\ \beta_{10} = 3(4\theta_0 T + 4T)\,(\rho_4 + \|g_0\|)\,\frac{1}{\theta_0(x_0)} < \sigma. \end{cases}$$

From these estimates, it follows that if σ and ρ are chosen from conditions $\sigma > \sigma_6 = \max(\beta_9, \beta_{10})$ and $\rho < (0, \rho_4)$, then the operator A_3 satisfies the second condition of contracting mapping.

As result, we conclude that if σ and ρ are taken from conditions $\sigma > \max(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6)$ and $\rho \in (0, \min(\rho_1, \rho_2, \rho_3, \rho_4))$, then the operator A carries out contracting mapping the ball $S(g_0, \rho)$ into itself, and according to Banach theorem in this ball, it has a unique fixed point; that is, there exists a unique solution of operator Equation (18). The proof of the theorem is complete.

Having found the functions $\vartheta(t,x)$ and $y(t)$, we determine the functions $\theta(t,x), a(t)$ by integral Equation (11):

$$a(t) = b(t) - y(t) - \int_0^t b(t-\tau)y(\tau)d\tau.$$

$$\theta(t,x) = \theta_0(x) + \int_0^t \vartheta(\tau,x)d\tau.$$

With the known function $a(t)$, solving the differential equation $a(t) = \alpha'(t)/\alpha(0)$, we find the function

$$\alpha(t) = \alpha(0) + \alpha(0)\int_0^t a(\tau)d\tau,$$

the solution of the inverse problem 1 (5–8).

5 | INVERSE PROBLEM 2

This section deals with the problem of finding $\theta(t,x)$ and $k(t)$ from equalities (5–8). According to Lemma 1, the Equation (5) is equivalent to Equation (9). The solution of the direct problems (9), (6), and (7) is expressed in the form of integral Equation (15). We rewrite this equation as follows:

$$\begin{aligned} \theta(t,x) = \Phi(t,x) + \int_0^t \int_0^l G(t-\tau,x,\xi)F(\tau,\xi)d\xi d\tau + \int_0^t \int_0^l G(t-\tau,x,\xi)\theta_0(\xi)y(\tau)d\xi d\tau + \\ + \int_0^t \int_0^l G(t-\tau,x,\xi)\int_0^\tau y(\alpha)\theta_\alpha(\tau-\alpha,\xi)d\alpha d\xi d\tau, \end{aligned} \tag{27}$$

where

$$\begin{aligned} \Phi(t,x) = \int_0^l G(t,x,\xi)\theta_0(\xi)d\xi \\ + \sum_{n=1}^{\infty}\int_0^t \frac{2\pi n}{l^2}\left[\mu_1(\tau) - (-1)^n\mu_2(\tau)\right]e^{-\left(\frac{\pi n}{l}\right)^2(t-\tau)} sin\left(\frac{\pi n}{l}x\right)d\tau. \end{aligned}$$

Differentiating equation (27) in t, we use the equality (10) and notation $\vartheta(t,x) := \theta_t(t,x)$. Then we have

$$\begin{aligned}\vartheta(t,x) = \Phi_t(t,x) + f(t,x) - \int_0^t D(t-\tau)f(\tau,x)d\tau + b(t)\theta_0(x) - \theta_0(x)\int_0^t b(t-\tau)D(\tau)d\tau + \\ + \theta_0(x)y(t) + \int_0^t y(\tau)\vartheta(t-\tau,\xi)d\tau + \int_0^t\int_0^l G_t(t-\tau,x,\xi)F(\tau,\xi)d\xi d\tau + \\ + \int_0^t\int_0^l G_t(t-\tau,x,\xi)\theta_0(\xi)y(\tau)d\xi d\tau + \int_0^t\int_0^l G_t(t-\tau,x,\xi)\int_0^\tau y(\alpha)\vartheta(\tau-\alpha,\xi)d\alpha d\xi d\tau,\end{aligned} \tag{28}$$

and we obtained the following equation using the additional condition (8):

$$\begin{aligned}\psi'(t) = \Phi_t(t,x_0) + f(t,x_0) - \int_0^t D(t-\tau)f(\tau,x_0)d\tau + b(t)\theta_0(x_0) - \theta_0(x_0)\int_0^t b(t-\tau)D(\tau)d\tau + \\ + \theta_0(x_0)y(t) + \int_0^t y(\tau)\vartheta(t-\tau,\xi)d\tau + \int_0^t\int_0^l G_t(t-\tau,x_0,\xi)F(\tau,\xi)d\xi d\tau + \\ + \int_0^t\int_0^l G_t(t-\tau,x_0,\xi)\theta_0(\xi)y(\tau)d\xi d\tau + \int_0^t\int_0^l G_t(t-\tau,x_0,\xi)\int_0^\tau y(\alpha)\vartheta(\tau-\alpha,\xi)d\alpha d\xi d\tau.\end{aligned}$$

From the above equation, the unknown function $b(t)$ is found:

$$\begin{aligned}b(t) = -\frac{1}{\theta_0(x_0)}\Bigg[\Phi_t(t,x_0) + f(t,x_0) - \psi'(t) - \int_0^t D(t-\tau)f(\tau,x_0)d\tau - \\ - \theta_0(x_0)\int_0^t b(t-\tau)D(\tau)d\tau + \theta_0(x_0)y(t) + \int_0^t y(\tau)\vartheta(t-\tau,\xi)d\tau + \\ + \int_0^t\int_0^l G_t(t-\tau,x_0,\xi)F(\tau,\xi)d\xi d\tau + \int_0^t\int_0^l G_t(t-\tau,x_0,\xi)\theta_0(\xi)y(\tau)d\xi d\tau + \\ + \int_0^t\int_0^l G_t(t-\tau,x_0,\xi)\int_0^\tau y(\alpha)\vartheta(\tau-\alpha,\xi)d\alpha d\xi d\tau\Bigg].\end{aligned} \tag{29}$$

The existence and uniqueness of the solution of the system of closed integral Equations (28) and (29) are proved by applying the principle of contraction mapping as in Section 3. Therefore, it is true the following assertion:

Theorem 2 (existence and uniqueness). *Assume the conditions $\theta_0(x) \in C(0,l)$, $\psi(t)\in C[0;T]$, $r(t,x)\in C(D_T)$, $\mu_i(t) \in C[0,T]$, $i = 1,2$, $\alpha(t)\in C^2[0,T]$, $\theta_0(x_0) = \psi(0)$, $\theta_0(x_0)\neq 0$, $\theta_0(0) = \mu_1(0)$, $\theta_0(l) = \mu_2(0)$ are hold. Then there exists sufficiently small number $T^*\in(0,T)$ that the solution to the integral Equations (28) and (29) in the class of functions $\vartheta(t,x) \in C^{1,2}(D_{T^*})$, $b(t)\in C[0;T^*]$ exist and unique, where $D_{T^*} = \{(x,t)|x\in(0,l), t\in[0,T^*]\}$.*

From the found function $b(t)$, the unknown function $k(t)$ is determined as follows:

$$k(t) = k(0) + k(0)\int_0^t b(\tau)d\tau.$$

6 | CONCLUSION

In this work, two inverse problems were considered for determining the kernels $\alpha(t)$ and $k(t)$ included in the system of Equation (3) with a simple observation (8) at the point $x_0\in(0,l)$ of the solution of this system with the initial and boundary conditions (5) and (6). Conditions for given functions are obtained under which the inverse problems have unique solutions for a sufficiently small time interval. When determining one of the kernels, it was assumed that the other is known. In this case, it should be noted the question of the simultaneous determination of two kernels in the system of Equation (3) remains open using some additional conditions of the corresponding measurement.

ACKNOWLEDGEMENT

The research was supported by Ministry of Innovation Development of the Republic of Uzbekistan (project no. F-4-02).

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How to cite this article: Durdiev DK, Zhumaev ZZh. Memory kernel reconstruction problems in the integro–differential equation of rigid heat conductor. *Math Meth Appl Sci*. 2020;1–15. https://doi.org/10.1002/mma.7133