AIP Conference Proceedings

RESEARCH ARTICLE | MARCH 11 2024

Inverse problem of determining the coefficient and kernel in an integro - differential equation of parabolic type

Jonibek Jumaev; Durdimurod Durdiev; Shakhnoza Ibragimova; Bekhzod Zaripov

Check for updates

AIP Conf. Proc. 3004, 040011 (2024)
https://doi.org/10.1063/5.0199968

View Online
Export Citation
CrossMark

13 March 2024 09:56:06

Boost Your Optics and Photonics Measurements
Lock-in Amplifier
Zurich Instruments
Find out more
Boxcar Averager

AIP Publishing

Inverse Problem of Determining the Coefficient and Kernel in an Integro - Differential Equation of Parabolic Type

Jonibek Jumaev,[1, 2, a)] Durdimurod Durdiev,[1, 2, b)] Shakhnoza Ibragimova,[2, c)] and Bekhzod Zaripov[2, d)]

1)*V.I. Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences, 4b, University street, Tashkent 100174, Uzbekistan.*

2)*Bukhara State University, 11, M.Ikbal street, Bukhara 200100, Uzbekistan.*

a)*Corresponding author: jonibekjj@mail.ru*
b)*d.durdiev@mathinst.ru*
c)*shaxnoza_i@mail.ru*
d)*begzodzaripov331@gmail.com*

Abstract. This article is concerned with the study of the unique solvability of inverse boundary value problem for integro-differential heat equation. To study the solvability of the inverse problem, we first reduce the considered problem to an auxiliary system with trivial data and prove its equivalence (in a certain sense) to the original problem. Then using the Banach fixed point principle, the existence and uniqueness of a solution to this system is shown.

INTRODUCTION AND FORMULATION OF PROBLEM

Mathematical physics usually studies well-posed problems, that is, the problems that have solutions which are unique and stable to small changes in the data in suitable functional spaces. Such problems, as a rule, are called direct problems. In this case, it is assumed that a differential equation is specified, as well as certain initial and boundary conditions. However, in applications there are interesting problems, where the differential (or integro-differential) equation is only partially specified, namely, some functions that are part of the differential equation (either in the right-hand side, integrant in integro-differential equations or the initial and boundary conditions) remain unknown. The problems, in which these unknowns are to be determined on some information about the solutions of direct problems for differential equations, are called inverse problems.

Inverse problems are widely used to solve practical problems in many branches of science and engineering. The study of inverse kernel determination problems for hyperbolic and parabolic integro-differential equations with an integral term of convolution type is very interesting from both the practical and theoretical viewpoint. Such equations in the case of a parabolic equation arise in problems of heat propagation in media whose state at a given time instant depends on their state at all previous time instants.

The inverse problem finding coefficient for parabolic and hyperbolic equations is studied in the works of many authors; a detailed bibliography can be found in [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. In last articles are proved that the problem is locally solvable in time or in the case where the norms of its data are sufficiently small.

In [12, 13, 14, 15, 16, 17, 18] papers are considered inverse problem for a fractional diffusion equation determining the reaction coefficient depending of the spatial variable and on time. Local existence and global uniqueness, a stability theorems are proved.

Inverse problems for integro-differential equation of multi and one-dimensional parabolic type are investigated by many author with initial, initial-boundary, overdetermination conditions (see [19, 20, 21, 22, 23, 24, 25, 26, 27]). As a result, is proved the global unique solvability of the inverse problem and obtained a stability estimate of a solution of the inverse problem.

Inverse problems for integro-differential equation of parabolic type can be found in papers [28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41] and the problems studied in them are close to the problem considered in this article. In the above papers, the existence and uniqueness theorems were proved for the solution of the problem of finding the kernel for various overdetermination conditions.

In the present paper, we investigate the inverse problem of the simultaneous determination of two unknowns: the coefficient $a(t)$ and the heat relaxation function $k(t)$ in the integro-differential heat equation. For this, two simple observations are given at two different points.

13 March 2024 09:56:06

International Scientific and Practical Conference on "Modern Problems of Applied Mathematics and Information Technology (MPAMIT2022)"
AIP Conf. Proc. 3004, 040011-1–040011-11; https://doi.org/10.1063/5.0199968
Published by AIP Publishing. 978-0-7354-4876-6/$30.00

Consider the problem of determining of functions $u(x,t)$, $a(t), k(t)$, from the following equations:

$$u_t - u_{xx} + a(t)u = \int_0^t k(t-\tau)u(x,\tau)d\tau, \quad (x,t) \in D_T, \tag{1}$$

$$u|_{t=0} = \varphi(x), \ \ x \in [0,1], \tag{2}$$

$$u|_{x=0} = u|_{x=1} = 0, \ \ \varphi(0) = \varphi(1) = 0, \ \ t \in [0,T], \tag{3}$$

$$u(x_0,t) = h_0(t), \ \ \int_0^1 u(x,t)dx = h_1(t), \ \ x_0 \in (0,1), \ \ \varphi(x_0) = h_0(0), \ \ \int_0^1 \varphi(x)dx = h_1(0), \tag{4}$$

where $D_T = \{(x,t)|x \in (0,1), \ 0 < t \leq T\}$, $T > 0$ is arbitrary fixed number.

In equation (1) on the left side there is a heat conduction operator acting on the function $u(x,t)$, and on the right side there is a convolution type integral. In fact, if a and the kernel k of the integral are known in equation (1), then the problem of finding the function u from equation (1), based on conditions (2), (3) is called the direct problem. Note that direct problem in this case is initial-boundary problem for the equation (1).

Since after finding functions a and k the solution of the direct problem becomes known, it is convenient to call the solution of the inverse problem (1)-(4) the problem of finding functions $u, \ k, \ a$.

In the inverse problem, it is required to find a and the kernel k of the integral in equation (1) if the additional conditions (4) are known with respect to the solution of the direct problem. The functions φ in condition (2) and the function h_i in (4) are called the data of the direct and inverse problems, respectively. The last conditions in (4) are matching conditions for given functions.

INVESTIGATION OF DIRECT PROBLEM

Before we proceed to studying the inverse problem, let us show that the direct problem has a unique solution.

Let $a(t), \ k(t) \in C[0,T], \ T > 0$ being some number. Then the direct problem (1)-(3) is equivalent to the problem of determining the function $u(x,t)$ from the equation

$$U_t - U_{xx} = \int_0^t k_0(t,\tau)U(x,\tau)d\tau, \tag{5}$$

$$U|_{t=0} = \varphi(x), \tag{6}$$

$$U|_{x=0} = 0, U|_{x=1} = 0, \tag{7}$$

where $U(x,t) = e^{-\int_0^t a(\tau)d\tau}u(x,t), \ k_0(t,\tau) = k(t-\tau)e^{-\int_\tau^t a(\alpha)d\alpha}$.

Lemma 1. *If* $\varphi(x) \in C(0,1), \ k(t) \in C([0,T]), \ \varphi(0) = \varphi(1) = 0$ *then there is the unique classical solution* $U(x,t)$ *to the problem (5)-(7) of the class* $C^{2,1}(D_T)$.

$C^{2,1}(D_T)$ is the class of twice continuously differentiable with respect to x and once continuously differentiable with respect to t in the domain D_T of functions. In what follows we also use the usual class $C(D_T)$ of continuous in D_T functions.

Proof. Problem (5)-(7) is equivalent to the integral equation

$$U(x,t) = \int_0^1 G(x,\xi,t-\tau)\varphi(\xi)d\xi + \int_0^t \int_0^1 G(x,\xi,t-\tau) \int_0^\tau k_0(t,\tau)U(\xi,\alpha)d\alpha d\xi d\tau, \tag{8}$$

where

$$G(x,\xi,t-\tau) = 2\sum_{n=1}^{\infty} e^{-(\pi n)^2(t-\tau)} \sin(\pi n\xi)\sin(\pi nx)$$

is the Green function of the initial-boundary problem for one-dimensional heat equation.

Let us denote the sum of the first one summand on the right-hand side of (8) by $\Phi(x,t)$, for this equation, we consider in the domain D_T the sequence of functions

$$U_n(x,t) = \Phi(x,t) + \int_0^t \int_0^1 G(x,\xi,t-\tau) \int_0^\tau k_0(t,\tau)U_{n-1}(\xi,\alpha)d\alpha d\xi d\tau, \ n = 1,2,..., \tag{9}$$

13 March 2024 09:56:06

where $U_0(x,t)=0$ for $(x,t)\in D_T$. If the conditions of Lemma 1 are fulfilled, we have that $\Phi(x,t)\in C^{2,1}(D_T)$ [42]. Then, it follows from (8) that all $U_n(x,t)$ in D_T possess the same property.

Denote $Z_n(x,t)=U_n(x,t)-U_{n-1}(x,t)$ and $\Phi_0=\|\Phi\|_{C(D_T)}$. According to the formula (8), we estimate $Z_n(x,t)$ in the domain D_T:

$$
\begin{aligned}
&|Z_1(x,t)|\le \Phi_0,\\
&|Z_2(x,t)|\le \int\limits_0^t d\tau \int\limits_{\mathbb{R}^n} G(x-\xi,t-\tau)\int\limits_0^\tau |k_0(\tau,\alpha)|\,|Z_1(\xi,\alpha)|\,d\alpha d\xi \le\\
&\le \Phi_0 k_0 k_{00}\frac{t^2}{2!},\ k_0=\max_{t\in[0,T]}|k(t)|,\ k_{00}=e^{a_0T}, a_0=\max_{t\in[0,T]}|a(t)|,\\
&|Z_3(x,t)|\le \int_0^t\int_0^l G(x,\xi,t-\tau)\\
&\times\int\limits_0^\tau |k_0(\tau,\alpha)|\,|Z_2(\xi,\tau-\alpha)|\,d\alpha d\xi d\tau \le \Phi_0\,(k_0k_{00})^2\,\frac{t^4}{4!}.
\end{aligned}
$$

Thus, for arbitrary $n=j$, we have

$$\left|Z_j(x,t)\right|\le \Phi_0\,(k_0k_{00})^{j-1}\,\frac{t^{2(j-1)}}{2(j-1)!}.$$

It follows from the above estimates that the series

$$\sum_{n=1}^{\infty}\left[U_n(x,t)-U_{n-1}(x,t)\right]$$

converges in D_T, and its sum $U(x,t)$ belongs to the functional space $C^{2,1}(D_T)$. Since the sequence $U_n(x,t)$, determined by equality (9) converges to $U(x,t)$ uniformly in D_T, then $U(x,t)$ is a solution of equation (8).

Now show that this solution is the only one. Suppose that there are two solutions $U^1(x,t)$ and $U^2(x,t)$. Then their difference $Z(x,t)=U^2(x,t)-U^1(x,t)$ is a solution to the equation

$$Z(x,t)=\int\limits_0^t\int\limits_0^l G(x,\xi,t-\tau)\int\limits_0^\tau k_0(\tau,\alpha)Z(\xi,\alpha)d\alpha d\xi d\tau.$$

Let $\tilde{Z}(t)$ denote the supremum of the module of the function $Z(x,t)$ for $x\in(0,1)$ at each fixed $t\in(0,T)$. Then we have the inequality

$$\tilde{Z}(t)\le k_0k_{00}T\int\limits_0^t \tilde{Z}(\tau)d\tau,\ t\in[0,T].$$

Applying the Gronwall lemma here, we obtain that $\tilde{Z}(t)=0$ for $t\in[0,T]$, which means that $Z(x,t)=0$ in D_T, i.e. $U^1(x,t)=U^2(x,t)$ in D_T. Therefore, equation (8) has a unique solution in D_T. The lemma is proved.

AUXILIARY PROBLEM

Suppose that the data of problem (1)-(4) are sufficiently smooth functions. The degree of smoothness for each function will be determined later.

The following assertation is true:

Lemma 2. *Problem (1)–(4) is equivalent to the following auxiliary problem of determining functions $u(x,t)$, $k(t)$, $a(t)$:*

$$\omega_t-\omega_{xx}+a'(t)\varphi''(x)+a'(t)\int_0^t\omega(x,\tau)d\tau+a(t)\omega(x,t)=k(t)\varphi''(x)+\int_0^t k(\tau)\omega(x,t-\tau)d\tau,\ (x,t)\in D_T, \tag{10}$$

13 March 2024 09:56:06

$$\omega|_{t=0} = \varphi^{(IV)}(x) - a(0)\varphi'(x),\ x \in [0,1], \tag{11}$$

$$\omega|_{x=0} = 0, \omega|_{x=1} = 0,\ \ t \in [0,T], \tag{12}$$

$$\omega|_{x=x_0} = h_0''(t) + a'(t)h_0(t) + a(t)h_0'(t) - k(t)\varphi(x_0) - \int_0^t k(\tau)h_0'(t-\tau)d\tau,\ \ t \in [0,T], \tag{13}$$

$$\int_0^1 \omega(x,t)dx = h_1''(t) + a'(t)h_1(t) + a(t)h_1'(t) - k(t)h_1(0) - \int_0^t k(\tau)h_1'(t-\tau)d\tau,\ \ t \in [0,T], \tag{14}$$

where $\omega(x,t) = u_{txx}(x,t)$,

$$a(0) = \frac{\varphi''(x_0) - h_0'(0)}{\varphi(x_0)}, \tag{15}$$

$\varphi^{(IV)}$ *is the fourth derivative of the function* $\varphi(x)$.

Proof. By setting $\vartheta(x,t) = u_t(x,t)$ and differentiating in t, we reduce (1)-(4) to the problem

$$\vartheta_t - \vartheta_{xx} + a'(t)\int_0^t \vartheta(x,\tau)d\tau + a'(t)\varphi(x) + a(t)\vartheta(x,t) = k(t)\varphi(x) + \int_0^t k(\tau)\vartheta(x,t-\tau)d\tau,\ (x,t) \in D_T, \tag{16}$$

$$\vartheta|_{t=0} = \varphi''(x) - a(0)\varphi(x),\ x \in [0,1], \tag{17}$$

$$\vartheta|_{x=0} = 0, \vartheta|_{x=1} = 0,\ \ t \in [0,T], \tag{18}$$

$$\vartheta(x_0,t) = h_0'(t),\ \ \int_0^1 \vartheta(x,t) = h_1'(t),\ \ t \in [0,T]. \tag{19}$$

From condition (17) and (19), requiring the matching condition at the points $(0,x_0)$ we obtain

$$a(0) = \frac{\varphi''(x_0) - h_0'(0)}{\varphi(x_0)} = \frac{\varphi'(1) - \varphi'(0) - h_1'(0)}{\int_0^1 \varphi(x)dx}.$$

Hence it follows that if (u,k,a) is solution of problem (1)-(4), then (16)-(19) has a solution (ϑ,k,a) with the same k,a as well. Let us prove the converse. Let (ϑ,k,a) satisfy relations(16)-(19), then

$$u(x,t) = \int_0^t \vartheta(x,\tau)d\tau + \varphi(x).$$

Let us show that relation (1) holds. It follows from (16)-(19) that

$$\begin{aligned}
&u_t - u_{xx} + a(t)u - \int_0^t k(t-\tau)u(x,\tau)d\tau = \\
&= \vartheta(x,t) - \int_0^t \vartheta_{xx}(x,\tau)d\tau - \varphi''(x) + a(t)\int_0^t \vartheta(x,\tau)d\tau + a(t)\varphi(x) - \\
&- \int_0^t k(\tau)\int_0^{t-\tau} \vartheta(x,\alpha)d\alpha d\tau - \int_0^t k(\tau)\varphi(x)d\tau = \\
&= \int_0^t \Big[\vartheta_\tau(x,\tau) - \vartheta_{xx}(x,\tau) + a'(\tau)\int_0^\tau \vartheta(x,\alpha)d\alpha + a(\tau)\vartheta(x,\tau) + a'(\tau)\varphi(x) - k(\tau)\varphi(x) - \\
&- \int_0^\tau k(x,\alpha)\vartheta(x,\tau-\alpha)d\alpha\Big]d\tau = 0.
\end{aligned}$$

This completes the proof of the equivalence of problems (1)-(4) and (16)-(19).

Now consider the second auxiliary problem. It is obtained from problem (16)-(19) for the function $p(x,t) = \vartheta_x(x,t)$,

13 March 2024 09:56:06

$$p_t - p_{xx} + a'(t)\int_0^t p(x,\tau)d\tau + a'(t)\varphi'(x) + a(t)p(x,t) = k(t)\varphi'(x) + \int_0^t k(\tau)p(x,t-\tau)d\tau, \tag{20}$$

$$p|_{t=0} = \varphi'''(x) - a(0)\varphi'(x); \tag{21}$$

$$p_x|_{x=0} = 0, \;\; p_x|_{x=1} = 0; \tag{22}$$

$$p_x|_{x=x_0} = h_0''(t) + a'(t)h_0(t) + a(t)h_0'(t) - k(t)\varphi(x_0) - \int_0^t k(\tau)h_0'(t-\tau)d\tau, \tag{23}$$

$$\int_0^1 p_x(x,t)dx = h_1''(t) + a'(t)h_1(t) + a(t)h_1'(t) - k(t)h_1(0) - \int_0^t k(\tau)h_1'(t-\tau)d\tau. \tag{24}$$

The derivation of the equations in the opposite way under the matching conditions

$$\begin{aligned}
&\varphi^{(IV)}(0) - a(0)\varphi''(0) = \varphi^{(IV)}(1) - a(0)\varphi''(1) = 0,\\
&h_0''(0) + a'(0)h_0(0) + a(0)h_0'(0) - k(0)\varphi(x_0) = \varphi^{(IV)}(x_0) - a(0)\varphi''(x_0),\\
&\varphi^{(III)}(1) - \varphi^{(III)}(0) + a(0)(\varphi'(0) - \varphi'(1)) = h_1''(0) + a'(0)h_1(0) + a(0)h_1'(0) - k(0)h_1(0),
\end{aligned}$$

which follows (21)-(23), can be proved by complete analogy with the previous case.

Therefore, if the problem (16)-(19) has a solution (ϑ, k, a), then the problem (20)-(23) has a solution (p,k,a) with the same k,a, moreover, $p(x,t) = \vartheta_x(x,t)$. Conversely, let (p,k,a) satisfy relations (20)-(23).

Hence it follows that

$$\vartheta(x,0) = \int_0^x p(y,0)dy = \int_0^t (\varphi'''(y) - a(0)\varphi'(y))dy = \varphi''(x) - a(0)\varphi(x);$$

i.e., condition (17) is satisfied. It remains to show that equation (16) holds. It follows from relations (20)-(23) that

$$\begin{aligned}
&\vartheta_t - \vartheta_{xx} + a'(t)\int_0^t \vartheta(x,\tau)d\tau + a'(t)\varphi(x) + a(t)\vartheta(x,t) - k(t)\varphi(x) - \int_0^t k(\tau)\vartheta(x,t-\tau)d\tau =\\
&= \int_0^x \Big[p_t(y,t) - p_{yy}(y,t) + a'(t)\int_0^t p(y,\tau)d\tau + a'(t)\varphi'(y) + a(t)p(y,t) -\\
&-k(t)\varphi'(y) - \int_0^t k(\tau)p(y,t-\tau)d\tau\Big]dy + a'(t)\varphi(0) - k(t)\varphi(0) = 0.
\end{aligned}$$

We have thereby proved the equivalence of problems (16)-(19) and (20)-(23). In similar way, one can show that problem (20)-(23) is equivalent to problem (10)-(13) for the function $\omega = p_x$. This implies the equivalence of problems (1)-(4) and (10)-(13). The lemma is proved.

FORMULATION OF MAIN RESULT AND ITS PROOF

In this section existence and uniqueness for the problem (10)-(13) is proved using the contraction mapping principle. The idea is to write the integral equations for unknown functions $\omega(x,t), k(t), a(t)$ as a system with a nonlinear operator, and prove that this operator is a contraction mapping operator for sufficiently small T. The existence and uniqueness then follow immediately.

From problem (10)-(13), we obtain

$$\begin{aligned}
&\omega(x,t) = \omega_0(x,t) + \int_0^t\int_0^1 G(x,\xi,t-\tau)k(\tau)\varphi(\xi)d\xi d\tau - \int_0^t\int_0^1 G(x,\xi,t-\tau)a(\tau)\omega(\xi,\tau)d\xi d\tau -\\
&-\int_0^t\int_0^1 G(x,\xi,t-\tau)a'(\tau)\varphi(\xi)d\xi d\tau - \int_0^t\int_0^1 G(x,\xi,t-\tau)a'(\tau)\int_0^\tau \omega(\xi,\alpha)d\alpha d\xi d\tau +\\
&+\int_0^t\int_0^1 G(x,\xi,t-\tau)\int_0^\tau k(\alpha)\omega(\xi,\tau-\alpha)d\alpha d\xi d\tau,
\end{aligned} \tag{25}$$

13 March 2024 09:56:06

where

$$\omega_0(x,t)=\int_0^1 G(x,\xi,t)(\varphi^{(IV)}(\xi)-a(0)\varphi'(\xi))d\xi.$$

By setting $x=x_0$ in integral equation (25) and and integrated over x from 0 to 1 in equation (25) by taking into account condition (13), for the functions $k(t),a'(t))$, we obtain the integral equations

$$\begin{aligned}
k(t)&=\frac{1}{\Delta}\Big[h_1(t)\Big(\omega_0(x_0,t)-h_0''(t)\Big)-h_0(t)(\int_0^1\omega_0(x,t)dx-h_1''(t))\Big]+\\
&+\frac{1}{\Delta}\int_0^t\int_0^1 h_1(t)G(x_0,\xi,t-\tau)\Big[k(\tau)\varphi(\xi)-a(\tau)\omega(\xi,\tau)-a'(\tau)\varphi(\xi)-a'(\tau)\int_0^\tau\omega(\xi,\tau-\alpha)d\alpha\Big]d\xi d\tau-\\
&-\frac{1}{\Delta}\int_0^1\int_0^t\int_0^1 h_0(t)G(x,\xi,t-\tau)\Big[k(\tau)\varphi(\xi)-a(\tau)\omega(\xi,\tau)-a'(\tau)\varphi(\xi)-a'(\tau)\int_0^\tau\omega(\xi,\tau-\alpha)d\alpha\Big]d\xi d\tau dx-\\
&-\frac{a(t)}{\Delta}\Big(h_1(t)h_0'(t)-h_0(t)h_1'(t)\Big)+\frac{1}{\Delta}\int_0^t k(\tau)\Big(h_1(t)h_0'(t-\tau)-h_0(t)h_1'(t-\tau)\Big)d\tau,
\end{aligned}\tag{26}$$

where

$$\Delta=h_1(0)h_0(t)-\varphi(x_0)h_1(t).$$

$$\begin{aligned}
a'(t)&=\frac{1}{\Delta}\Big[h_1(0)(\omega_0(x_0,t)-h_0''(t))-\varphi(x_0)(\int_0^1\omega_0(x,t)dx-h_1''(t))\Big]+\\
&+\frac{1}{\Delta}\int_0^t\int_0^1 h_2(0)G(x_0,\xi,t-\tau)\Big[k(\tau)\varphi(\xi)-a(\tau)\omega(\xi,\tau)-a'(\tau)\varphi(\xi)-a'(\tau)\int_0^\tau\omega(\xi,\tau-\alpha)d\alpha\Big]d\xi d\tau-\\
&-\frac{1}{\Delta}\int_0^1\int_0^t\int_0^1\varphi(x_0)G(x,\xi,t-\tau)\Big[k(\tau)\varphi(\xi)-a(\tau)\omega(\xi,\tau)-a'(\tau)\varphi(\xi)-a'(\tau)\int_0^\tau\omega(\xi,\tau-\alpha)d\alpha\Big]d\xi d\tau dx-\\
&-\frac{a(t)}{\Delta}\Big(h_1(0)h_0'(t)-\varphi(x_0)h_1'(t)\Big)+\frac{1}{\Delta}\int_0^t k(\tau)\Big(h_1(0)h_0'(t-\tau)-\varphi(x_0)h_1'(t-\tau)\Big)d\tau.
\end{aligned}\tag{27}$$

$$a(t)=a(0)+\int_0^t a'(\tau)d\tau.\tag{28}$$

Equations (25)-(28) form a complete system of integral equations for the unknown functions $\omega(x,t),k(t),a'(t),a(t)$. We represent this system in the form of the operator equation

$$\psi=A\psi,\tag{29}$$

where $\psi=(\psi_1,\psi_2,\psi_3,\psi_4):=(\omega(x,t),k(t)+a(t)\beta(t),a'(t)+a(t)\gamma(t),a(t))$, $\Big(\beta(t)=\frac{1}{\Delta}\Big(h_1(t)h_0'(t)-h_0(t)h_1'(t)\Big),\gamma(t)=\frac{1}{\Delta}\Big(h_1(0)h_0'(t)-\varphi(x_0)h_1'(t)\Big)\Big)$ is the vector-function and unknown functions are represented by functions $\psi_1,\psi_2,\psi_3,\psi_4$ as follows:

$$\omega(x,t)=\psi_1(x,t),\ \ k(t)=\psi_2(t)-\psi_4(t)\beta(t),\ \ a'(t)=\psi_3(t)-\psi_4(t)\gamma(t),\quad a(t)=\psi_4(t).$$

$A=(A_1,\ A_2,\ A_3,\ A_4)$ is defined by the right sides of equations (25)-(28):

$$\begin{aligned}
(A\psi)_1(x,t)&=\omega_0(x,t)+\int_0^t\int_0^1 G(x,\xi,t-\tau)\Big(\psi_2(\tau)-\psi_4(\tau)\beta(\tau)\Big)\varphi(\xi)d\xi d\tau-\\
&-\int_0^t\int_0^1 G(x,\xi,t-\tau)\psi_4(\tau)\psi_1(\xi,\tau)d\xi d\tau-\int_0^t\int_0^1 G(x,\xi,t-\tau)\Big(\psi_3(\tau)-\psi_4(\tau)\gamma(\tau)\Big)\varphi(\xi)d\xi d\tau-\\
&-\int_0^t\int_0^1 G(x,\xi,t-\tau)\Big(\psi_3(\tau)-\psi_4(\tau)\gamma(\tau)\Big)\int_0^\tau\psi_1(\xi,\alpha)d\alpha d\xi d\tau+\\
&+\int_0^t\int_0^1 G(x,\xi,t-\tau)\int_0^\tau\Big(\psi_2(\alpha)-\psi_4(\alpha)\beta(\alpha)\Big)\psi_1(\xi,\tau-\alpha)d\alpha d\xi d\tau,
\end{aligned}$$

13 March 2024 09:56:06

$$(A\psi)_2(t)=\frac{1}{\Delta}\left[h_1(t)\Big(\omega_0(x_0,t)-h_0''(t)\Big)-h_0(t)(\int_0^1\omega_0(x,t)dx-h_1''(t))\right]+$$
$$+\frac{1}{\Delta}\int_0^t\int_0^1 h_1(t)G(x_0,\xi,t-\tau)\Big[\Big(\psi_2(\tau)-\psi_4(\tau)\beta(\tau)\Big)\varphi(\xi)-\psi_4(\tau)\psi_1(\xi,\tau)-$$
$$-\Big(\psi_3(\tau)-\psi_4(\tau)\gamma(\tau)\Big)\varphi(\xi)-\Big(\psi_3(\tau)-\psi_4(\tau)\gamma(\tau)\Big)\int_0^\tau\psi_1(\xi,\tau-\alpha)d\alpha\Big]d\xi d\tau-$$
$$-\frac{1}{\Delta}\int_0^1\int_0^t\int_0^1 h_0(t)G(x,\xi,t-\tau)\Big[\Big(\psi_2(\tau)-\psi_4(\tau)\beta(\tau)\Big)\varphi(\xi)-\psi_4(\tau)\psi_1(\xi,\tau)-$$
$$-\Big(\psi_3(\tau)-\psi_4(\tau)\gamma(\tau)\Big)\varphi(\xi)-\Big(\psi_3(\tau)-\psi_4(\tau)\gamma(\tau)\Big)\int_0^\tau\psi(\xi,\tau-\alpha)d\alpha\Big]d\xi d\tau dx+$$
$$+\frac{1}{\Delta}\int_0^t\Big(\psi_2(\tau)-\psi_4(\tau)\beta(\tau)\Big)\Big(h_1(t)h_0'(t-\tau)-h_0(t)h_1'(t-\tau)\Big)d\tau,$$

$$(A\psi)_3(t)=\frac{1}{\Delta}\Big[h_1(0)(\omega_0(x_0,t)-h_0''(t))-\varphi(x_0)(\int_0^1\omega_0(x,t)dx-h_1''(t))\Big]+$$
$$+\frac{1}{\Delta}\int_0^t\int_0^1 h_2(0)G(x_0,\xi,t-\tau)\Big[\Big(\psi_2(\tau)-\psi_4(\tau)\beta(\tau)\Big)\varphi(\xi)-\psi_4(\tau)\psi_1(\xi,\tau)-$$
$$-\Big(\psi_3(\tau)-\psi_4(\tau)\gamma(\tau)\Big)\varphi(\xi)-\Big(\psi_3(\tau)-\psi_4(\tau)\gamma(\tau)\Big)\int_0^\tau\psi_1(\xi,\tau-\alpha)d\alpha\Big]d\xi d\tau-$$
$$-\frac{1}{\Delta}\int_0^1\int_0^t\int_0^1\varphi(x_0)G(x,\xi,t-\tau)\Big[\Big(\psi_2(\tau)-\psi_4(\tau)\beta(\tau)\Big)\varphi(\xi)-\psi_4(\tau)\psi_1(\xi,\tau)-$$
$$-\Big(\psi_3(\tau)-\psi_4(\tau)\gamma(\tau)\Big)\varphi(\xi)-\Big(\psi_3(\tau)-\psi_4(\tau)\gamma(\tau)\Big)\int_0^\tau\psi(\xi,\tau-\alpha)d\alpha\Big]d\xi d\tau dx-$$
$$+\frac{1}{\Delta}\int_0^t\Big(\psi_2(\tau)-\psi_4(\tau)\beta(\tau)\Big)\Big(h_1(0)h_0'(t-\tau)-\varphi(x_0)h_1'(t-\tau)\Big)d\tau,$$

13 March 2024 09:56:06

$$(A\psi)_4(t)=a(0)+\int_0^t\Big(\psi_3(\tau)-\psi_4(\tau)\gamma(\tau)\Big)d\tau.$$

Let $\psi_0:=(\psi_{01},\psi_{02},\psi_{03},\psi_{04})$, where

$$\psi_{01}=\omega_0(x,t),\psi_{02}=\frac{1}{\Delta}\left[h_1(t)\Big(\omega_0(x_0,t)-h_0''(t)\Big)-h_0(t)(\int_0^1\omega_0(x,t)dx-h_1''(t))\right],$$
$$\psi_{03}=\frac{1}{\Delta}\Big[h_1(0)(\omega_0(x_0,t)-h_0''(t))-\varphi(x_0)(\int_0^1\omega_0(x,t)dx-h_1''(t))\Big],\psi_{04}=a(0).$$

Theorem 1. *If the conditions $\varphi(x)\in C^4(0,1)$, $h_i(t)\in C^2[0,T], i=0,1$, $\varphi(0)=\varphi(1)=0$, $\varphi(x_0)=h_0(0)$, $\int_0^1\varphi(x)dx=h_1(0)$ are met, then there exists sufficiently small number $T^*\in(0,T)$ that the solution to the system of integral equations (25)–(28) in the class of functions $\omega(x,t)\in C^{2,1}(D_T)$, $k(t)\in C(0,T]$, $a(t)\in C^1(0,T]$ exists is unique. Thus, there is the unique classical solution to the problem (1)-(4).*

Proof. Consider the functional space of vector functions $\psi\in C(D_T)$ with the norm given by the relation

$$\|\psi\|=\max\left\{\max_{(x,t)\in\overline{D}_T}|\psi_1(x,t)|,\max_{t\in[0,T]}|\psi_2(t)|,\max_{t\in[0,T]}|\psi_3(t)|,\max_{t\in[0,T]}|\psi_4(t)|\right\}=$$
$$=\max\Big\{\|\psi_1\|,\|\psi_2\|,\|\psi_3\|,\|\psi_4\|\Big\}.$$

In this space, by $B(\psi_0,\|\psi_0\|)$ we denote the ball with center ψ_0 and radius $\|\psi_0\|$, i.e. $B(\psi_0,\|\psi_0\|)=\{\psi:\|\psi-\psi_0\|\leq\|\psi_0\|\}$. Obviously, $\|\psi\|\leq 2\|\psi_0\|$.

Let us show that A is a contraction operator in the ball $B(\psi_0, \|\psi_0\|)$ provided that T is sufficiently small number. For simplicity, we denote

$$h_0 := \max\left\{ \sup_{t\in(0;T)} |h_i(t)|, \sup_{t\in(0,T)} \left|h_i'(t)\right|, \sup_{t\in(0,T)} \left|h_i''(t)\right| \right\}, i = 0,1, \;\; \beta = \max_{t\in[0,T]} |\beta(t)|, \gamma = \max_{(t)\in[0,T]} |\gamma(t)|$$

$$\varphi_0 := \max\left\{ \sup_{x\in(0,1)} |\varphi(x)|, \sup_{x\in(0,1)} \left|\varphi'(x)\right|, \sup_{x\in(0,1)} \left|\varphi''(x)\right| \right\}.$$

Let us verify the first condition of a fixed point argument [43]. Let $\psi \in B$; then $\|\psi\| \le 2\|\psi_0\|$. In addition, for $(x,t) \in D_T$, we have estimates

$$\begin{aligned}
&\|(A\psi)_1 - \psi_{01}\| = \sup_{(x,t)\in D_T} |(A\psi)_1 - \psi_{01}| \le \\
&\le \sup_{(x,t)\in D_T} \left| \int_0^t \int_0^1 G(x,\xi,t-\tau)\Big(\psi_2(\tau) - \psi_4(\tau)\beta(\tau)\Big)\varphi(\xi)d\xi d\tau \right| + \\
&+ \sup_{(x,t)\in D_T} \left| \int_0^t \int_0^1 G(x,\xi,t-\tau)\psi_4(\tau)\psi_1(\xi,\tau)d\xi d\tau \right| + \\
&+ \sup_{(x,t)\in D_T} \left| \int_0^t \int_0^1 G(x,\xi,t-\tau)\Big(\psi_3(\tau) - \psi_4(\tau)\gamma(\tau)\Big)\varphi(\xi)d\xi d\tau \right| + \\
&\sup_{(x,t)\in D_T} \left| \int_0^t \int_0^1 G(x,\xi,t-\tau)\Big(\psi_3(\tau) - \psi_4(\tau)\gamma(\tau)\Big)\int_0^\tau \psi_1(\xi,\alpha)d\alpha d\xi d\tau \right| + \\
&+ \sup_{(x,t)\in D_T} \left| \int_0^t \int_0^1 G(x,\xi,t-\tau)\int_0^\tau \Big(\psi_2(\alpha) - \psi_4(\alpha)\beta(\alpha)\Big)\psi_1(\xi,\tau-\alpha)d\alpha d\xi d\tau \right| \le \\
&\le 2T\|\psi_0\| \left[(2+\beta+\gamma)\varphi_0 + \|\psi_0\|(1+2T+T\beta+T\gamma)\right].
\end{aligned}$$

$$\begin{aligned}
&\|(A\psi)_2 - \psi_{02}\| = \sup_{t\in(0,T)} |((A\psi)_2 - \psi_{02})| \le \\
&\le \frac{4Th_0\|\psi_0\|}{\Delta}\left(\varphi_0(2+\beta+\gamma) + \|\psi_0\|(1+2T+T\gamma+T\beta) + h_0(1+\beta)\right);
\end{aligned}$$

$$\begin{aligned}
&\|(A\psi)_3 - \psi_{03}\| = \sup_{t\in(0,T)} |((A\psi)_3 - \psi_{03})| \le \\
&\le \frac{4T\varphi_0\|\psi_0\|}{\Delta}\left(\varphi_0(2+\beta+\gamma) + \|\psi_0\|(1+2T+T\gamma+T\beta) + h_0(1+\beta)\right);
\end{aligned}$$

$$\|(A\psi)_4 - \psi_{04}\| = \sup_{t\in(0,T)} |((A\psi)_4 - \psi_{04})| \le 2\|\psi_0\|T(1+\gamma).$$

Denote $T_1 = \min\{T_{11}, T_{12}, T_{13}, T_{14}\}$, where $T_{1i}, i = \overline{1,4}$ are the positive roots of the following equations, respectively

$$\begin{aligned}
&2T\Big[(2+\beta+\gamma)\varphi_0 + \|\psi_0\|(1+2T+T\beta+T\gamma)\Big] = 1, \\
&\frac{4Th_0}{\Delta}\Big(\varphi_0(2+\beta+\gamma) + \|\psi_0\|(1+2T+T\gamma+T\beta) + h_0(1+\beta)\Big) = 1, \\
&\frac{4T\varphi_0}{\Delta}\Big(\varphi_0(2+\beta+\gamma) + \|\psi_0\|(1+2T+T\gamma+T\beta) + h_0(1+\beta)\Big) = 1, \\
&2T(1+\gamma) = 1.
\end{aligned}$$

If we choose T so that $T < T_1$, then $A\psi \in B(\psi_0, \|\psi_0\|)$.

13 March 2024 09:56:06

We now check the second condition of a fixed point argument

$$\|(A\psi^1-A\psi^2)_1\| \leq \sup_{(x,t)\in D_T}\left|\int_0^t\int_0^1 G(x,\xi,t-\tau)\Big([\psi_2^1(\tau)-\psi_2^2(\tau)]-[\psi_4^1(\tau)-\psi_4^2(\tau)]\beta(\tau)\Big)\varphi(\xi)d\xi d\tau\right|+$$

$$+\sup_{(x,t)\in D_T}\left|\int_0^t\int_0^1 G(x,\xi,t-\tau)[\psi_4^1(\tau)\psi_1^1(\xi,\tau)-\psi_4^2(\tau)\psi_1^2(\xi,\tau)]d\xi d\tau\right|+$$

$$+\sup_{(x,t)\in D_T}\left|\int_0^t\int_0^1 G(x,\xi,t-\tau)\Big([\psi_3^1(\tau)-\psi_3^2(\tau)]-[\psi_4^1(\tau)-\psi_4^2(\tau)]\gamma(\tau)\Big)\varphi(\xi)d\xi d\tau\right|+$$

$$+\sup_{(x,t)\in D_T}\left|\int_0^t\int_0^1 G(x,\xi,t-\tau)\int_0^\tau[\psi_1^1(\xi,\alpha)\psi_3^1(\tau)-\psi_1^2(\xi,\alpha)\psi_3^2(\tau)]d\alpha d\xi d\tau\right|+$$

$$+\sup_{(x,t)\in D_T}\left|\int_0^t\int_0^1 G(x,\xi,t-\tau)\int_0^\tau[\psi_1^1(\xi,\alpha)\psi_4^1(\tau)-\psi_1^2(\xi,\alpha)\psi_4^2(\tau)]\gamma(\tau)d\alpha d\xi d\tau\right|+$$

$$+\sup_{(x,t)\in D_T}\left|\int_0^t\int_0^1 G(x,\xi,t-\tau)\int_0^\tau[\psi_1^1(\xi,\tau-\alpha)\psi_2^1(\alpha)-\psi_1^2(\xi,\tau-\alpha)\psi_2^2(\alpha)]d\alpha d\xi d\tau\right|+$$

$$+\sup_{(x,t)\in D_T}\left|\int_0^t\int_0^1 G(x,\xi,t-\tau)\int_0^\tau[\psi_1^1(\xi,\tau-\alpha)\psi_4^1(\alpha)-\psi_1^2(\xi,\tau-\alpha)\psi_4^2(\alpha)]\beta(\alpha)\Big)d\alpha d\xi d\tau\right|.$$

The integrand in the second integral can be estimated as follows:

$$\begin{aligned}&\left\|\psi_4^1\psi_1^1-\psi_4^2\psi_1^2\right\| = \left\|(\psi_4^1-\psi_4^2)\psi_1^1+\psi_4^2(\psi_1^1-\psi_1^2\right\| \leq \\ &\leq 2\left\|\psi^1-\psi^2\right\|\max\left(\left\|\psi_1^1\right\|,\left\|\psi_4^2\right\|\right) \leq 4\left\|\psi_0\right\|\left\|\psi^1-\psi^2\right\|.\end{aligned}$$

Therefore,

$$\|(A\psi^1-A\psi^2)_1\| \leq \|\psi^1-\psi^2\|\left(2\|\psi_0\|(2+\beta+\gamma)T^2+((2+\beta+\gamma)\varphi_0+4\|\psi_0\|)T\right);$$

The next components can be estimated in a similar way,

$$\begin{aligned}&\|(A\psi^1-A\psi^2)_2\| \leq \frac{2h_0}{\Delta}\left(((2+\beta+\gamma)\varphi_0+h_0(1+\beta)+4\|\psi_0\|)T+2\|\psi_0\|(2+\beta+\gamma)T^2\right)\|\psi^1-\psi^2\|;\\ &\|(A\psi^1-A\psi^2)_3\| \leq \frac{2\varphi_0}{\Delta}\left(((2+\beta+\gamma)\varphi_0+h_0(1+\beta)+4\|\psi_0\|)T+2\|\psi_0\|(2+\beta+\gamma)T^2\right)\|\psi^1-\psi^2\|;\\ &\|(A\psi^1-A\psi^2)_4\| \leq (1+\gamma)T\|\psi^1-\psi^2\|.\end{aligned}$$

Denote $T_2=(T_{21},T_{22},T_{23},T_{24})$; where $T_{2i}, i=\overline{1,4}$ are the positive roots of the following equations, respectively

$$\begin{aligned}&2\|\psi_0\|\Big(2+\beta+\gamma\Big)T^2+\Big((2+\beta+\gamma)\varphi_0+4\|\psi_0\|\Big)T=1,\\ &\frac{2h_0}{\Delta}\Big(((2+\beta+\gamma)\varphi_0+h_0(1+\beta)+4\|\psi_0\|)T+2\|\psi_0\|(2+\beta+\gamma)T^2\Big)=1,\\ &\frac{2\varphi_0}{\Delta}\Big(\Big((2+\beta+\gamma)\varphi_0+h_0(1+\beta)+4\|\psi_0\|\Big)T+2\|\psi_0\|(2+\beta+\gamma)T^2\Big)=1,\\ &\Big(1+\gamma\Big)T=1.\end{aligned}$$

Therefore, if the number T^* is small enough to ensure that condition $T^*\in(0,\min(T_1,T_2))$ is satisfied, then A is contraction operator on B. Then, by the Banach principle, integral equations (25)-(28) has a unique solution in B. Theorem is proved.

If we put the coefficient $a(t)$ and the kernel $k(t)$ to the problem (1), get the direct problem (1) - (3). In this case, the function $u(x,t)$ is defined as uniquely.

CONCLUSION

In this work, inverse problem was considered for determining the coefficient $a(t)$ and the kernel $k(t)$ included in the equation (1) with by using overdetermination conditions (4) of the solution of problem with the initial and boundary

13 March 2024 09:56:06

conditions (2), (3). Sufficiently conditions are obtained for given functions, under which the inverse problem have unique solutions for small interval.

ACKNOWLEDGMENTS

We would like to thank our colleagues at Bukhara State University and the Institute of Mathematics named V.I.Romanovskiy for making convenient research facilities.

REFERENCES

1. S. G. Pyatkov, "Certain inverse problems for parabolic equations," Journal of Mathematical Sciences **5**, 2422–2433 (2008).
2. Y. E. Anikonov and Y. Y. Belov, "Determining two unknown coefficients of parabolic-type equation," Journal of Mathematical Sciences **5**, 469–488 (2001).
3. S. N. Baranov and Y. Y. Belov, "On the identification problem of two unknown coefficients with inhomogeneous overdetermination conditions," Nonclassical Equations of Mathematical Physics **1**, 11–22 (2002).
4. Y. Y. Belov, "Inverse problems for parabolic equations," J. Inverse Ill-Posed Probl. **4**, 283–301 (1993).
5. Y. Y. Belov and T. N. Shipina, "The problem of determining a coefficient in the parabolic equation and some properties of its solution," J. Inverse Ill-Posed Probl. **9**, 31–48 (2001).
6. V. Isakov, *Inverse Problems for Partial Differential Equations* (Springer, Berlin, 1998).
7. M. V. Klibanov, "Carleman estimates and inverse problems in the last two decades," Surveys on Solutions Methods for Inverse Problems , 119–149 (2000).
8. A. I. Kozhanov, *Composite-Type Equations and Inverse Problems* (VSP, Utrecht, 1999).
9. M. M. Lavrentiev, V. G. Romanov, and V. G. Vasiliev, "Multidimensional inverse problems for differential equations," Lect. Notes Math. **167** (1972).
10. M. Yamamoto, "Conditional stability in the determination of densities of heat sources in a bounded domain," Estimation and Control of Distributed Parameter Systems , 359–370 (1994).
11. M. Huntul, D. Lesnic, and M. Hussein, "Reconstruction of time-dependent coefficients from heat moments," Applied Mathematics and Computation, **301**, 233–253 (2017).
12. U. D. Durdiev, "Problem of determining the reaction coefficient in a fractional diffusion equation," Differential Equations, **59**, 1195–1204 (2021).
13. L. Miller and M. Yamamoto, "Coefficient inverse problem for a fractional diffusion equation," Inverse Problems, **29** (2013).
14. X. Xiong, Q. Zhou, and Y. Hon, "An inverse problem for fractional diffusion equation in 2-dimensional case: Stability analysis and regularization," Journal of Mathematical Analysis and Applications **393**, 185–199 (2012).
15. X. Xiong, H. Guo, and X. Liu, "An inverse problem for a fractional diffusion equation," Journal of Computational and Applied Mathematics **236**, 4474–4484 (2012).
16. M. Kirane, S. Malik, and M. Al-Gwaiz, "An inverse source problem for a two dimensional time fractional diffusion equation with nonlocal boundary conditions," Mathematical Methods in the Applied Sciences **36**, 1056–1069 (2013).
17. A. Hazanee, D. Lesnic, M. Ismailov, and N. Kerimov, "Inverse time-dependent source problems for the heat equation with nonlocal boundary conditions," Applied Mathematics and Computation **346**, 800–815 (2019).
18. U. Durdiev., "Inverse problem of determining an unknown coefficient in the beam vibration equation," Differential Equations **58**, 36–43 (2022).
19. D.K.Durdiev and Zh.D.Totieva, "The problem of determining the one-dimensional kernel of viscoelasticity equation with a source of explosive type," J. Inverse Ill Posed Probl. **28**, 43–52 (2020).
20. D.K.Durdiev and Z. Safarov, "Inverse problem of determining the one-dimensional kernel of the viscoelasticity equation in a bounded domain," Math. Notes **97**, 867–877 (2015).
21. D. Durdiev and Z. Totieva, "Problem of determining one-dimensional kernel of viscoelasticity equation," Sib. Zh. Ind. Mat. **16**, 72–82 (2013).
22. D. Durdiev and Z. Totieva, "Inverse problem for a second-order hyperbolic integro-differential equation with variable coefficients for lower derivatives," Sibirskie Elektronnye Matematicheskie Izvestiya [Siberian Electronic Mathematical Reports] **17**, 1106–1127 (2020).
23. D. Durdiev and A. Rakhmonov, "Inverse problem for a system of integro-differential equations for sh waves in a visco-elastic porous medium: global solvability," Theor. Math. Phys. **195**, 923–937 (2018).
24. D. Durdiev and A. Rakhmonov, "The problem of determining the 2d-kernel in a system of integro-differential equations of a viscoelastic porous medium," Sib. Zh. Ind. Mat. **23**, 63–80 (2020).
25. D. Durdiev and A. Rakhmonov, "A 2d kernel determination problem in a visco-elastic porous medium with a weakly horizontally inhomogeneity," Math Meth Appl Sci. **43**, 8776–8796 (2020).
26. D. Durdiev and Z. Totieva, "Problem of determining the multidimensional kernel of viscoelasticity equation," Vladikavkaz. Mat. Zh. **17**, 18–43 (2015).
27. U.D.Durdiev and Z.D.Totieva, "A problem of determining a special spatial part of 3d memory kernel in an integro-differential hyperbolic equation," Math. Methods Appl. Scie. **42**, 7440–7451 (2019).
28. J. Janno and L. Wolfersdorf, "Inverse problems for identification of memory kernels in heat flow," Inverse and Ill-Posed Problems **4**, 39–66 (1996).
29. E. Pais and J. Janno, "Inverse problem to determine degenerate memory kernel in heat flux third kind boundary conditions," Math. Modelling and Analysis **11**, 427–450 (2006).
30. F. Colombo, "A inverse problem for a parabolic integro-differential model in the theory of combustion," Phys. **236**, 81–89 (2007).

13 March 2024 09:56:06

31. D.K.Durdiev, "On the uniqueness of kernel determination in the integro-differential equation of parabolic type," J. Samara State Tech. Univ. **19**, 658–666 (2015).
32. D. K. Durdiev and Z. Z. Zhumaev, "One-dimensional inverse problems of finding the kernel of the integro-differential heat equation in a bounded domain," Ukrainskyi Matematychnyi Zhurnal **73**, 1492 –1506 (2021).
33. D. Durdiev and E. S. S. Sitnik, "The explicit formula for solution of anomalous diffusion equation in the multi-dimensional space," Lobachevskii Journal of Mathematics **42**, 1264–1273 (2021).
34. D. Durdiev and J.J.Jumaev, "Memory kernel reconstruction problems in the integro-differential equation of rigid heat conductor," Mathematical Methods in the Applied Sciences **45**, 8374–8388 (2022).
35. D. Durdiev and A. Rashidov, "Inverse problem of determining the kernel in an integro-differential equation of parabolic type," Differential Equations **50**, 110–116 (2014).
36. D. Durdiev and Z. Zhumaev, "Problem of determining a multidimensional thermal memory in a heat conductivity equation," Methods of Functional Analysis and Topology **25**, 219–226 (2019).
37. D. Durdiev and Z. Zhumaev, "Problem of determining the thermal memory of a conducting medium," Differential Equations **56**, 785–796 (2020).
38. D. Durdiev and Z.Z.Nuriddinov, "Determination of a multidimensional kernel in some parabolic integro-differential equation," Journal of Siberian Federal University - Mathematics and Physics **114**, 117–127 (2021).
39. S. Avdonin, S. Ivanov, and J. Wang, "Inverse problems for the heat equation with memory," Inverse problems and imaging **13**, 31–38 (2019).
40. D. Guidetti, "Reconstruction of a convolution kernel in a parabolic problem with a memory term in the boundary conditions," Bruno Pini Mathematical Analysis Seminar **4**, 47–55 (2013).
41. D. Guidetti, "Some inverse problems of identification for integro differential parabolic systems with a boundary memory term," Discrete & Continuous Dynamical Systems **8**, 749–756 (2015).
42. A. A. Kilbas, *Integral Equations: Course of Lectures* (Minsk, Belarusian State University, 2005).
43. A. N. Kolmogorov and S. V. Fomin, *Elements of the Theory of Functions and Functional Analysis* (Moscow:Nauka, 1989).

13 March 2024 09:56:06