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Investigation of Initial-boundary Value Problem for Integro-differential Fractional Diffusion Equation

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Abstract. In this paper, we consider the time-fractional integro-differential diffusion equation with initial-boundary conditions. At the beginning, an auxiliary problem, which is equivalent to the original problem, is introduced. Then the auxiliary problem is reduced to an equivalent of Volterra-type integral equation with respect to unknown function. Applying the method of successive approximation, we prove the main result of the article, which is a global existence and uniqueness theorem of problem solution.

INTRODUCTION AND SETTING UP THE PROBLEM

Nowadays, the fractional differential equations have many successful applications in various modern science and engineering technology areas [1, 2, 3, 4, 5]. Fractional differential equations have been adapted as mathematical models [6]. For example, in models of viscoelastic behavior [7, 8], anomalous diffusion [9], compartment models , economics [10], epidemiology [11], dynamics of particles , biology [12] and etc.

Researchers have proposed several new methods to find an analytical solution of fractional differential equations. Such as, In the paper [13], a maximum principle for the generalized time-fractional diffusion equation over an open bounded domain is formulated and proved. In the article [14] is considered initial value/boundary value problems for fractional diffusion-wave equation and established the unique existence of the weak solution. In this paper [15], is investigated the functional variable method and the modified Riemann-Liouville derivative for the exact solitary wave solutions and periodic wave solutions of the time-fractional Klein-Gordon equation. This method is effective for handling nonlinear time-fractional differential equations.

Direct and inverse problems for integro-differential equations of parabolic and hyperbolic type have been studied by many researcher(see [16, 17, 18, 19, 20, 21, 22, 23, 24]). In last articles are proved for posed problem existence and global uniqueness, a stability theorems.The inverse problem of determining the convolution kernels of integral terms from a system of first-order integro-differential equations of general form with two independent variables was studied in [25]. The theorem of local existence and global uniqueness is obtained. In the work of [26] the method for studying the work of [25] was applied to the investigating of the inverse problem of determining the diagonal relaxation matrix from the system of Maxwell's integro-differential equations.

In this paper, we investigate the existence and uniqueness for integro-differential time–fractional diffusion equation with initial-boundary conditions.

Consider the problem of determining of functions $u(x,t)$, from the following equations with fractional derivative in time t :

$$\partial_t^{\alpha} u - u_{xx} + a(x)u = \int_0^t k(t-\tau)u(x,\tau)d\tau, \quad (x,t) \in D_T, \alpha \in (0,1), \tag{1}$$

$$u|_{t=0} = \varphi(x), \ \ x \in [0,l], \tag{2}$$

$$u|_{x=0} = \mu_1(t), \ \ u|_{x=l} = \mu_2(t), \ \varphi(0) = \mu_1(0), \ \ \varphi(l) = \mu_2(0), \ t \in [0,T], \tag{3}$$

with the Caputo time fractional derivative ∂_t^{α} of order $0 < \alpha < 1$, defined by

$$\partial_t^{\alpha} u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} u'(\tau)d\tau, \tag{4}$$

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where Γ is the Euler's Gamma function, $D_T = \{(x,t)|x \in (0,l),\ t \in (0,T]\}$, $l > 0, T > 0$ are arbitrary fixed numbers, $\varphi(x), \mu_1(t), \mu_2(t)$ are given functions of $x \in [0,l]$ and $t \in [0,T]$.

Problem (1)-(3) is equivalent to the auxiliary problem of determining the function $\vartheta(x,t)$, from the equations

$$\partial_t^\alpha \vartheta - \vartheta_{xx} + a(x)\vartheta(x,t) = \int_0^t k(\tau)\vartheta(x,t-\tau)d\tau - a(x)b(x,t) + \int_0^t k(t-\tau)b(x,\tau)d\tau - \partial_t^\alpha b(x,t), \tag{5}$$

$$\vartheta|_{t=0} = \varphi(x) - \mu_1(0), \tag{6}$$

$$\vartheta|_{x=0} = \vartheta|_{x=l} = 0, \tag{7}$$

where $\vartheta(x,t) = u(x,t) - \mu_1(t) - \frac{x}{l}(\mu_2(t) - \mu_1(t)) = u(x,t) - b(x,t)$.

PRELIMINARIES

In this section, we recall basic definitions and notations from fractional calculus (see [4], pp. 69-79), which will be use in the future.

Two parameter Mittag-Leffler function. The two parameter M-L function $E_{\alpha,\beta}(z)$ is defined by the following series:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

where $\alpha, \beta, z \in \mathbb{C}$ with $\Re(\alpha) > 0$, $\Re(\alpha)$ is denote the real part of the complex number α. The Mittag-Leffler function has been studied by many authors who have proposed and studied various generalizations and applications.

Proposition 1. *Let $0 < \alpha < 2$ and $\beta \in \mathbb{R}$ be arbitrary. We suppose that κ is such that $\pi\alpha/2 < \kappa < \min\{\pi, \pi\alpha\}$. Then there exists a constant $C = C(\alpha, \beta, \kappa) > 0$ such that*

$$\left|E_{\alpha,\beta}(z)\right| \leq \frac{C}{1+|z|}, \quad \kappa \leq |arg(z)| \leq \pi.$$

For the proof, we refer to ([4], pp. 40-45) for example.

Proposition 2. (see [4], pp. 40-45) *For $0 < \alpha \leq \beta \leq 1$, the following hold:*

1. *For $\lambda > 0, t^{\beta-1}E_{\alpha,\beta}(-\lambda t^\alpha)$ is completely monotonic function.*
2. *For $t \in [0,T]$, we have*
   $E_{\alpha,\beta}(-\lambda t^\alpha) < \infty$ and $\int_0^t (t-s)^{\alpha-1}E_{\alpha,\beta}(-\lambda(t-s)^\alpha)ds < \infty$.
3. *Futhermore, for $\lambda \in \mathbb{R}^+, t \in (0,T]$*

$$\lambda t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha) \leq \frac{1}{t}\frac{\lambda t^\alpha}{1+\lambda t^\alpha} < \infty.$$

Proposition 3. (see [4], pp. 42-45) *Let $0 < \alpha < 1$ and $\lambda > 0$, then we have*

$$\frac{d}{dt}E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha), \quad t > 0.$$

Proposition 4. (see [4], pp. 42-45) *For $0 < \alpha < 1$, $\eta > 0$, we have $0 \leq E_{\alpha,\alpha}(-\eta) \leq \frac{1}{\Gamma(\alpha)}$. Moreover, $E_{\alpha,\alpha}(-\eta)$ is a monotonic decreasing function with $\eta > 0$.*

We will use these facts everywhere in this article.

THE MAIN RESULT

In this section we show that the problem has a unique solution.

Theorem 1. *If problem (5)-(7) has a solution such that*

$$\lim_{x\to l-0} u_x \sin(\lambda_n x) = 0, \quad 0 \le t \le T, \tag{8}$$

then this solution is unique.

Applying the method of separation of variables, we seek a solution of (5)-(7) with the form

$$\vartheta(x,t) = T(t)X(x). \tag{9}$$

Substituting (9) into (5) with $F(x,t;u,a,k,\mu_1,\mu_2) \equiv 0$, where

$$F(x,t;u,a,k,\mu_1,\mu_2) := a(x)\vartheta(x,t) + \int_0^t k(\tau)\vartheta(x,t-\tau)d\tau - a(x)b(x,t) + \int_0^t k(\tau)b(x,\tau)d\tau - \partial_t^\alpha b(x,t). \tag{10}$$

Carrying out a separation of the variables, we obtain the following one-dimensional eigenvalue problem:

$$\begin{cases} X'' + \lambda^2 X = 0, \\ X(0) = 0, \quad X(l) = 0, \end{cases} \tag{11}$$

where λ is a constant of the separation of variable. The boundary condition for $X(x)$ follows from the corresponding conditions for function ϑ. For example from

$$\vartheta(0,t) = X(0)T(t) = 0 \tag{12}$$

it follows $X(0) = 0$, since $T(t) \neq 0$ (we have only non-trivial solution).

The solution of equation (11) have the form

$$X_n(x) = \sin(\lambda_n x). \tag{13}$$

The eigenvalues

$$\lambda_n^2 = \left(\frac{\pi n}{l}\right)^2, \quad n \in \mathbb{N}, \tag{14}$$

have corresponding eigenfunctions

$$X_n(x) = A_n \sin(\lambda_n x), \tag{15}$$

where A_n is some constant factor. We choose this so that the norm in $L^2[0,l]$ of the function X_n with weight 1 equals unity

$$\int_0^l X_n(x)^2 dx = A_n^2 \int_0^l \sin^2(\lambda_n x) dx = 1. \tag{16}$$

Hence $A_n = \frac{2}{l}$. So,

$$X_n(x) = \frac{2}{l}\sin(\lambda_n x) \tag{17}$$

is the orthonormal eigenfunction corresponding to $-\lambda_n$. Then the sequence $\{X_n\}_{n\in\mathbb{N}}$ is orthonormal basis in $L^2[0,l]$.

Let a function $\vartheta(x,t)$ stand for a solution to problem (5)-(7) which satisfies condition (8).

We consider the integral

$$T_n(t) = \int_0^l \vartheta(x,t) X_n(x) dx. \tag{18}$$

Introduce an auxiliary integral, namely,

$$T_n^\varepsilon(t) = \int\limits_{\varepsilon}^{l-\varepsilon} \vartheta(x,t) X_n(x) dx, \tag{19}$$

here ε is given sufficiently small positive value. Fractional differentiating equality (18) and make use of Eq. (5), we get the correlation

$$\left(\partial_t^\alpha T_n^\varepsilon\right)(t) = \int\limits_{\varepsilon}^{l-\varepsilon} \partial_t^\alpha \vartheta(x,t) X_n(x) dx = \int\limits_{\varepsilon}^{l-\varepsilon} \left(a^2\vartheta_{xx} + F(x,t;\vartheta,a,k,\mu_1,\mu_2)\right) X_n(x) dx$$

$$= a^2 \int\limits_{\varepsilon}^{l-\varepsilon} \vartheta_{xx} X_n(x) dx + \int\limits_{\varepsilon}^{l-\varepsilon} F(x,t;\vartheta,a,k,\mu_1,\mu_2) X_n(x) dx = a^2 I_1 + F_n^\varepsilon(t;\vartheta,a,k,\mu_1,\mu_2), \tag{20}$$

where

$$F_n^\varepsilon(t;\vartheta,a,k,\mu_1,\mu_2) = \int\limits_{\varepsilon}^{l-\varepsilon} F(x,t;\vartheta,a,k,\mu_1,\mu_2) X_n(x) dx. \tag{21}$$

Calculating integral I_1 by part, we conclude that

$$I_1 = \int_\varepsilon^{l-\varepsilon} \vartheta_{xx} X_n(x) dx = \left(\vartheta_x X_n(x)|_\varepsilon^{l-\varepsilon} - \int_\varepsilon^{l-\varepsilon} \vartheta_x X_{nx}(x) dx \right)$$

$$= \left(\vartheta_x X_n(x)|_\varepsilon^{l-\varepsilon} - \vartheta X_{nx}(x)|_\varepsilon^{l-\varepsilon} + \int_\varepsilon^{l-\varepsilon} \vartheta X_{nxx}(x) dx \right). \tag{22}$$

Proceeding to the limit in integral I_1 as $\varepsilon \to 0$, taking into account conditions (6), (8), from formula (20) arrives

$$\left(\partial_t^\alpha T_n\right)(t) + \lambda_n^2 T_n(t) = F_n(t;\vartheta,a,k,\mu_1,\mu_2), \tag{23}$$

where

$$F_n(t;\vartheta,a,k,\mu_1,\mu_2) = \int\limits_0^l F(x,t;\vartheta,a,k,\mu_1,\mu_2) X_n(x) dx. \tag{24}$$

The initial conditions (6) give:

$$T_n(0) = \int\limits_0^l \vartheta(x,0) X_n(x) dx = \int\limits_0^l (\varphi(x) - \mu_0(0)) X_n(x) dx = \varphi_n. \tag{25}$$

The initial-value problem (23), (25) is equivalent in the space $C[0,T]$ to the Volterra integral equation of the second kind

$$T_n(t) = \varphi_n E_\alpha(-\lambda_n^2 t^\alpha) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_n^2 (t-\tau)^\alpha\right) F_n(\tau;\vartheta,a,k,\mu_1,\mu_2) d\tau. \tag{26}$$

Now, we investigate the integral equation (26) on $[0,T]$. The following assertion is true.

Lemma 1. *If $(\varphi(x), a(x)) \in C[0,l], (\mu_1(t),\mu_2(t)) \in C^1([0,T])$, $k(t) \in C([0,T])$, then there is a unique classical solution $u(x,t)$ to problem (1)-(3) of the class $C^{2,\alpha}([0,l]\times[0,T])$ ($C^{2,\alpha}(D_T) = \{u(x,t) \in C^2[0,l]; t \in (0,T]; u(x,t) \in AC[0,T]; x \in [0,l]\}$.)* In what follows we also use the usual class $C(D_T)$ of continuous in D_T functions.

Proof. Problem (5)-(7) is equivalent to the integral equation

$$\vartheta(x,t) = \int_0^l G_{\alpha,\alpha}(x,\xi,t)\left(\varphi(\xi) - \mu_1(0)\right) d\xi + \int_0^t \int_0^l G_{\alpha,1}(x,\xi,t-\tau) a(\xi) \vartheta(\xi,\tau) d\xi d\tau$$

$$+ \int_0^t \int_0^l G_{\alpha,1}(x,\xi,t-\tau) \int_0^\tau k(\alpha) \vartheta(\xi,\tau-\alpha) d\alpha d\xi d\tau - \int_0^t \int_0^l G_{\alpha,1}(x,\xi,t-\tau) a(\xi) b(\xi,\tau) d\xi d\tau$$

$$+ \int_0^t \int_0^l G_{\alpha,1}(x,\xi,t-\tau) \int_0^\tau k(\tau-\alpha) b(\xi,\tau) d\alpha d\xi d\tau - \int_0^t \int_0^l G_{\alpha,1}(x,\xi,t-\tau) \partial_\tau^\alpha b(\xi,\tau) d\xi d\tau, \tag{27}$$

where

$$G_{\alpha,\beta}(x,\xi,t)=\frac{2}{l}\sum_{n=1}^{\infty}t^{\alpha-\beta}E_{\alpha,\alpha-\beta+1}(-\lambda_n^2t^{\alpha})\sin\left(\frac{\pi n}{l}\xi\right)\sin\left(\frac{\pi n}{l}x\right)$$

is the Green function of the initial-boundary problem for one-dimensional fractional diffusion equation.

There exist some positive constants L_j, M_i, K such that

$$L_1:=\max_{0\le t\le T}E_{\alpha,1}(-\lambda_n t^{\alpha}),\ \ L_2:=\max_{0\le s<t\le T}E_{\alpha,\alpha}(-\lambda_n(t-s)^{\alpha}),$$

$$M_0:=\|a\|_{C[0,l]},\ \ M_1:=\max(\|\mu_i\|_{C^1[0,T]}),\ \ i:=1,2,$$

$$M_2:=\|\varphi\|_{C[0,l]},\ \ M_3:=\|\,k\|_{C[0,T]},$$

$$\max_{0<\tau<t\le T}\lambda_n(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^{\alpha})\le K.$$

Let us denote the sum of the first three summand on the right-hand side of (27) by $F(x,t)$, for this equation, we consider in the domain D_T the sequence of functions

$$\vartheta_n(x,t)=\int_0^t\int_0^l G_{\alpha,1}(x,\xi,\tau)a(\xi)\vartheta_{n-1}(\xi,t-\tau)d\xi d\tau+\int_0^t\int_0^l G_{\alpha,1}(x,\xi,\tau)\int_0^{t-\tau}k(\beta)\vartheta_{n-1}(\xi,t-\tau-\beta)d\beta d\xi d\tau, \tag{28}$$

where

$$\vartheta_0(x,t)=\int_0^l G_{\alpha,\alpha}(x,\xi,t)\left(\varphi(\xi)-\mu_1(0)\right)d\xi-\int_0^t\int_0^l G_{\alpha,1}(x,\xi,t-\tau)a(\xi)b(\xi,\tau)d\xi d\tau$$
$$+\int_0^t\int_0^l G_{\alpha,1}(x,\xi,t-\tau)\int_0^{\tau}k(\tau-\beta)b(\xi,\beta)d\beta d\xi d\tau-\int_0^t\int_0^l G_{\alpha,1}(x,\xi,t-\tau)\partial_{\tau}^{\alpha}b(\xi,\tau)d\xi d\tau,$$

for $(x,t)\in D_T$.

Firstly, used proposition 2 we estimate following integral term

$$\left|G_{\alpha,1}(x,\xi,t)\right|=\frac{2}{l}\sum_{n=1}^{\infty}\left|t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n t^{\alpha})\right|\left|\sin(\frac{\pi n}{l}x)\right|\left|\sin(\frac{\pi n}{l}\xi)\right|$$
$$\le\frac{2}{l}\sum_{n=1}^{\infty}\frac{T^{\alpha}}{T(1+\lambda_n T^{\alpha})}\left|\sin(\frac{\pi n}{l}x)\right|\sin(\frac{\pi n}{l}\xi)\right|:=M<\infty.$$

If the conditions of Lemma 1 are fulfilled, we have that $\vartheta_0(x,t)\in C^{2,1}(D_T)$. Then, it follows from (28) that all $\vartheta_n(x,t)$ in D_T possess the same property.

According to the formula (27), we estimate $\vartheta_n(x,t)$ in the domain D_T:

$$|\vartheta_0(x,t)|\le F_0,$$

$$|\vartheta_1(x,t)|\le\int_0^t\int_0^l G_{\alpha,1}(x,\xi,\tau)|a(\xi)||\vartheta_0(\xi,t-\tau)|d\xi d\tau$$

$$+\int_0^t\int_0^l G_{\alpha,1}(x,\xi,\tau)\int_0^{t-\tau}|k(\alpha)||\vartheta_0(\xi,t-\tau-\alpha)|d\alpha d\xi d\tau\le tF_0Ml\left(M_0+M_3\frac{T}{2}\right),$$

$$|\vartheta_2(x,t)|\le\int_0^t\int_0^l G_{\alpha,1}(x,\xi,\tau)|a(\xi)||\vartheta_1(\xi,t-\tau)|d\xi d\tau$$

$$+\int_0^t\int_0^l G_{\alpha,1}(x,\xi,\tau)\int_0^{t-\tau}|k(\alpha)||\vartheta_1(\xi,t-\tau-\alpha)|d\alpha d\xi d\tau \le \frac{t^2}{2!}F_0M^2l^2\left(M_0+M_3\frac{T}{2}\right)\left(M_0+M_3\frac{T}{3}\right),$$

$$|\vartheta_3(x,t)|\le\int_0^t\int_0^l G_{\alpha,1}(x,\xi,\tau)|a(\xi)||\vartheta_2(\xi,t-\tau)|d\xi d\tau$$

$$+\int_0^t\int_0^l G_{\alpha,1}(x,\xi,\tau)\int_0^{t-\tau}|k(\alpha)||\vartheta_2(\xi,t-\tau-\alpha)|d\alpha d\xi d\tau \le \frac{t^3}{3!}F_0M^3l^3\left(M_0+M_3\frac{T}{2}\right)\left(M_0+M_3\frac{T}{3}\right)\left(M_0+M_3\frac{T}{4}\right).$$

Thus, for arbitrary $n=j$, we have

$$\left|\vartheta_j(x,t)\right|\le F_0\frac{t^j}{j!}(Ml)^j\prod_{i=1}^{j}\left(M_0+M_3\frac{T}{i+1}\right)\le F_0\frac{T^j}{j!}(Ml)^j\prod_{i=1}^{j}\left(M_0+M_3\frac{T}{i+1}\right),\ \ (j\ge 1).$$

Now based on the terms on the right side of the inequalities, we can construct the following positive series:

$$F_0+TF_0Ml\left(M_0+M_3\frac{T}{2}\right)+F_0\frac{T^2}{2!}(Ml)^2\prod_{i=1}^{2}\left(M_0+M_3\frac{T}{i+1}\right)+F_0\frac{T^3}{3!}(Ml)^3\prod_{i=1}^{3}\left(M_0+M_3\frac{T}{i+1}\right)$$

$$...+F_0(Ml)^j\frac{T^j}{j!}\prod_{i=1}^{j}\left(M_0+M_3\frac{T}{i+1}\right)+...$$

The convergence of this series can be indicated by the D'Alambert criterion

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{T}{n+1}\left(a_0+k_0\frac{T}{n+2}\right)=0.$$

It follows from the above estimates that the series

$$\vartheta(x,t)=\sum_{n=0}^{\infty}\vartheta_n(x,t)$$

converges in D_T, and its sum $\vartheta(x,t)$ belongs to the functional space $C^{2,1}(D_T)$. Since the sequence $u_n(x,t)$, determined by equality (28) converges to $u(x,t)$ uniformly in D_T, then $u(x,t)$ is a solution of equation (27).

Now show that this solution is the only one. Suppose that there are two solutions $\vartheta^1(x,t)$ and $\vartheta^2(x,t)$. Then their difference $Z(x,t)=\vartheta^2(x,t)-\vartheta^1(x,t)$ is a solution to the equation

$$Z(x,t)=\int_0^t\int_0^l G_{\alpha,1}(x,\xi,\tau)a(\xi)Z(\xi,t-\tau)d\xi d\tau$$
$$+\int_0^t\int_0^l G_{\alpha,1}(x,\xi,\tau)\int_0^{t-\tau}k(\alpha)Z(\xi,t-\tau-\alpha)d\alpha d\xi d\tau.$$

Let $\tilde{Z}(t)$ denote the supremum of the module of the function $Z(x,t)$ for $x\in(0,l)$ at each fixed $t\in(0,T)$. Then we have the inequality

$$\tilde{Z}(t)\le(M_0+M_3TMl)\int_0^t\tilde{Z}(\tau)d\tau,\ t\in[0,T].$$

Applying the Gronwall lemma here, we obtain that $\tilde{Z}(t)=0$ for $t\in[0,T]$, which means that $Z(x,t)=0$ in D_T, i.e. $\vartheta^1(x,t)=\vartheta^2(x,t)$ in D_T. Therefore, equation (27) has a unique solution in D_T. The lemma is proved.

CONCLUSION

In this work, the initial-boundary problem for integro-differential diffusion equation was considered. Used the Fourier method and approximation series methods, to investigate the solvability of the problem. Further, on the basis of the equivalency of integral equation's the existence and uniqueness theorem for the classical solution of the considered problem is proved.

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