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CONVOLUTION KERNEL DETERMINING PROBLEM FOR AN INTEGRO-DIFFERENTIAL HEAT EQUATION WITH NONLOCAL INITIAL-BOUNDARY AND OVERDETERMINATION CONDITIONS

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Abstract
In this paper, we consider an inverse problem of determining $u(x, t)$ and $k(t)$ functions in the one-dimensional integro-differential heat equation with the nonlocal initial-boundary and over-determination conditions. The unique solvability of the direct problem is rigorously proved using the Fourier method and Schauder principle. To investigate the solvability of the inverse problem, we first consider an auxiliary inverse boundary value problem, which is equivalent to the original one. Then using the Fourier method, the problem is reduced by an equivalent closed system of integral equations with respect to unknown functions. The existence and uniqueness theorem for this system of integral equations is proved by contraction mappings principle.

Keywords Integro-differential equation · Nonlocal initial-boundary problem · Inverse problem · Integral equation · Schauder principle

AMS subject classification: 35A01 · 35A02 · 35L02 · 35L03 · 35R03

Introduction

Nonlocal initial-boundary value problems are quite an interesting generalization of classical problems and at the same time they are naturally obtained when constructing mathematical models in physics and applied mathematics, engineering, sociology, ecology, and so on [1].

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Problems with nonlocal conditions for partial differential equations have been studied by many authors. In the articles [1–5], the unique solvability of nonlocal inverse boundary value problems for hyperbolic equation with overdetermination conditions was considered. In these problems, the existence and uniqueness theorems for the classical solution of the considered inverse coefficient problems are proved for the smaller value of time.
The inverse problem of determining the time-dependent thermal diffusivity and the temperature distribution in a parabolic equation in the case of nonlocal initial-boundary conditions containing a real parameter and integral overdetermination conditions are investigated in the works [6–11]. Sufficient conditions for the existence and uniqueness of the classical solution to inverse problems are obtained for small time.
The paper [11] studied the inverse problem of determination of the time-dependent coefficient of the higher-order derivative for a parabolic equation with degeneration by the Neumann boundary conditions and a nonlocal overdetermination condition. There were found sufficient conditions for the existence and uniqueness of the classical solution of the problem.
The problem of determining the kernel $k(t)$ of the integral term in an integro-differential heat equation was studied in many publications [12–21], in which both one- and multidimensional inverse problems with classical initial, initial-boundary conditions were investigated. The existence and uniqueness theorems of inverse problem solutions were proved.

Formulation of problem

We consider the nonlocal initial-boundary problem for the heat equation with a convolution-type integral term on the right-hand side

$$u_t - u_{xx} = \int_0^t k(t-\tau)u(x,\tau)d\tau, \quad (x,t) \in D_T, \tag{1}$$

$$u(x,0) + \lambda u(x,T) = \varphi(x), \tag{2}$$

$$u(0,t) = u(l,t) = 0, \;\; \varphi(0) = \varphi(l) = 0, \tag{3}$$

$\lambda, \;\; l, \;\; T$ are arbitrary positive numbers and $D_T := \{(x,t) : 0 < x < l, 0 < t \le T\})$.
The problem of determining a function $u(x,t), (x,t) \in D_T$, that satisfies (1)–(3) with known functions $k(t)$ and $\varphi(x)$ will be called the direct problem.
In the inverse problem, it is required to determine the kernel $k(t), t > 0$, of the integral in (1) using overdetermination condition about the solution of the direct problem (1)–(3):

$$u(x_0,t) = h(t), \quad h(0) + \lambda h(T) = \varphi(x_0), \;\; x_0 \in (0,l), \tag{4}$$

where $h(t)$ is a given function.
Let $C^m(0;l)$ be the class of m times continuously differentiable with all derivatives up to the $m-$th order (inclusive) in $(0;l)$ functions. In the case $m=0$, this space coincides with the class of continuous functions. $C^{m,k}(D_T)$ is the class of m times continuously differentiable with respect to t and k times continuously differentiable with respect to x with all derivatives in the domain D_T functions.

Direct problem

According to the Fourier method, problem (1)–(3) are equivalent to the integral equation

$$u(x,t) = \Phi(x,t) - \lambda \int_0^T \int_0^l G_0(x,\xi,T+t-\tau) \int_0^\tau k(\alpha)u(\xi,\tau-\alpha)d\alpha d\xi d\tau +$$

$$+\int_0^t \int_0^l G(x,\xi,t-\tau)\int_0^\tau k(\alpha)u(\xi,\tau-\alpha)d\alpha d\xi d\tau, \tag{5}$$

where

$$\Phi(x,t)=\int_0^l G_0(x,\xi,t)\varphi(\xi)d\xi,$$

$$G_0(x,\xi,t)=\frac{2}{l}\sum_{n=1}^{\infty}\frac{1}{1+\lambda e^{-(\frac{\pi n}{l})^2 T}}\cdot e^{-(\frac{\pi n}{l})^2 t}\sin\left(\frac{\pi n}{l}\xi\right)\sin\left(\frac{\pi n}{l}x\right),$$

$$G(x,\xi,t)=\frac{2}{l}\sum_{n=1}^{\infty}e^{-(\frac{\pi n}{l})^2 t}\sin\left(\frac{\pi n}{l}\xi\right)\sin\left(\frac{\pi n}{l}x\right)$$

is the Green function of the initial-boundary problem for one-dimensional heat equation.

Denote the operator acting the function $u(x, t)$ in the right-hand side of (5) by L. Then, (5) can be written as the operator equation

$$u = Lu. \tag{6}$$

Denote also

$$\phi_0=\max_{(x,t)\in D_T}|\Phi(x,t)|,\quad k_0=\max_{t\in[0,T]}|k(t)|,$$

and note $|\Phi(x,t)|\le\phi_0$ for $(x,t)\in\overline{D}_T$.

Let $S_d(0)=\{u : \|u\|\le d\}$ be the ball in the functional space $C\left(\overline{D}_T\right)$ with the center at origin and radius d, and d is some positive number.

We use the Schauder principle (see [22], p. 411) to the existence of solution of the operator Eq. (5).

Definition 1 An operator L is said to be equicontinuous if for each $\varepsilon>0$ there exists a $\delta_0=\delta_0(\varepsilon)>0$ such that the inequality

$$\|Lu_1-Lu_2\|\le\varepsilon \tag{7}$$

holds for all $(x,t)\in D_T, u_1(x,t), u_2(x,t)\in S_d(0)$ with $\|u_1(x,t)-u_2(x,t)\|\le\delta_0$.

Lemma 1 *Suppose that the following conditions are satisfied*: $\varphi(x)\in C[0,l], k(t)\in C[0,T], \varphi(0)=\varphi(l)=0.$ *Then for all* T *and* $d>\phi_0$ *satisfying the estimate*

$$0<T\le\sqrt{\frac{2(d-\phi_0)}{dk_0(1+\lambda)}}, \tag{8}$$

the operator L is uniformly bounded and equicontinuous. Then according to the Schauder principle, there exists at least one classical solution of problems (1)–(3) *in the space* $C^{2,1}(D_T)$.

Proof First, we establish the uniform boundedness of the operator L. To this end, we show that there exists a $\sigma\in(0,d]$ such that $\|Lu\|\le\sigma$, where $\|Lu\|=\max_{(x,t)\in\overline{D}_T}|Lu|$. For $u\in S_d(0)$ and $(x,t)\in\overline{D}_T$, we find estimate $\|Lu\|\le\phi_0+k_0d(1+\lambda)\frac{T^2}{2}\equiv\sigma$. For T that satisfies the estimate (7), the operator L is uniformly bounded.

We consider the estimates

$$\|Lu_1 - Lu_2\| \le$$

$$\le \lambda \max_{(x,t)\in\overline{D}_T} \left| \int_0^T \int_0^l G_0(x,\xi,T+t-\tau) \int_0^\tau k(\alpha)(u_1(\xi,\tau-\alpha) - u_2(\xi,\tau-\alpha))d\alpha d\xi d\tau \right| +$$

$$+ \max_{(x,t)\in\overline{D}_T} \left| \int_0^t \int_0^l G(x,\xi,t-\tau) \int_0^\tau k(\alpha)(u_1(\xi,\tau-\alpha) - u_2(\xi,\tau-\alpha))d\alpha d\xi d\tau \right| \le$$

$$\le (1+\lambda)k_0 \frac{T^2}{2} \|u_1 - u_2\| \le (1+\lambda)k_0 \frac{T^2}{2} \delta.$$

Consequently, if we take $\delta_0 = \frac{2\varepsilon}{(1+\lambda)k_0T^2}$, then inequality (7) will hold for $\delta \in (0, \delta_0]$, the operator L is equicontinuous. Then, the operator L is completely continuous on S_d, and it has at least one fixed point on S_d by the Schauder principle. The proof of the lemma is complete.
Consequently, under condition (7), Lemma 1 implies the existence of a solution of the operator Eq. (6), thus the existence of a solution to direct problem (1)–(3).
Let us prove the uniqueness of this solution.

Theorem 1 *For all $u \in S_d(0)$ and*

$$T < \sqrt{\frac{2}{k_0(1+\lambda)}}, \tag{9}$$

the operator Eq. (6) has a unique solution in the class $C^{2,1}(D_T)$.

Proof Let problems (1)–(3) have two solutions $u_1, u_2, u_1 \neq u_2$. We denote their differences by $\omega = u_1 - u_2$. For the difference ω, we obtain the problem

$$\omega_t - \omega_{xx} = \int_0^t k(t-\tau)\omega(x,\tau)d\tau, \quad (x,t) \in D_T, \tag{10}$$

$$\omega(x,0) + \lambda\omega(x,T) = 0, \ \ x \in [0,l], \tag{11}$$

$$\omega|_{x=0} = \omega|_{x=l} = 0, \ t \in [0,T], \tag{12}$$

The solution of this problem can be written as

$$\omega(x,t) = -\lambda \int_0^T \int_0^l G_0(x,\xi,T+t-\tau) \int_0^\tau k(\alpha)\omega(\xi,\tau-\alpha)d\alpha d\xi d\tau +$$

$$+ \int_0^t \int_0^l G(x,\xi,t-\tau) \int_0^\tau k(\alpha)\omega(\xi,\tau-\alpha)d\alpha d\xi d\tau. \tag{13}$$

Estimating we have

$$\|\omega\|_{C(\overline{D}_T)} \le \frac{1}{2}k_0T^2(1+\lambda)\|\omega\|_{C(\overline{D}_T)}.$$

Hence, for T that satisfies estimate (9), we obtain $u_1 = u_2$. The proof of the theorem is complete. □

Remark 1 It follows from (7) and (9) that under condition (7) there exists a unique solution to direct problems (1)–(3).

Solvability of the inverse problem

In this section, we consider the problem of simultaneously determining the functions $u(x, t)$, $k(t)$ from the integro-differential Eq. (1) with nonlocal initial-boundary condition (2), (3), and additional condition (4).
We introduce the notation

$$\vartheta(x, t) = u_{xx}(x, t), \tag{14}$$

and obtain the following equivalent problem with respect to function $\vartheta(x, t)$:

$$\vartheta_t - \vartheta_{xx} = \int_0^t k(t-\tau)\vartheta(x, \tau)d\tau, \tag{15}$$

$$\vartheta(x, 0) + \lambda\vartheta(x, T) = \varphi''(x), \quad \varphi''(0) = \varphi''(l) = 0, \tag{16}$$

$$\vartheta|_{x=0} = \vartheta|_{x=l} = 0, \tag{17}$$

$$\vartheta|_{x=x_0} = h'(t) - \int_0^t k(t-\tau)h(\tau)d\tau. \tag{18}$$

Problems (15)–(17) are equivalent to the problem of finding the function $\vartheta(x, t)$ from the following integral equation:

$$\vartheta(x, t) = F(x, t) - \lambda \int_t^{T+t} \int_0^l G_0(x, \xi, \tau) \int_0^{T+t-\tau} k(\alpha)\vartheta(\xi, T + t - \tau - \alpha)d\alpha d\xi d\tau +$$

$$+ \int_0^t \int_0^l G(x, \xi, \tau) \int_0^{t-\tau} k(\alpha)\vartheta(\xi, t - \tau - \alpha)d\alpha d\xi d\tau, \tag{19}$$

$$F(x, t) = \int_0^l G_0(x, \xi, t)\varphi''(\xi)d\xi.$$

Differentiate the integral Eq. (19) once with respect to the variable t:

$$\vartheta_t(x, t) = F_t(x, t) + \lambda \int_0^l G_0(x, \xi, t) \int_0^T k(\alpha)\vartheta(\xi, T - \alpha)d\alpha d\xi -$$

$$-\lambda \int_t^{T+t} \int_0^l G(x, \xi, \tau) \int_0^{T+t-\tau} k(\alpha)\vartheta_t(\xi, T + t - \tau - \alpha)d\alpha d\xi d\tau -$$

$$-\lambda \int_t^{T+t} \int_0^l G_0(x, \xi, \tau)k(T + t - \tau)\vartheta(x, 0)d\xi d\tau + \int_0^t \int_0^l G(x, \xi, \tau)k(t - \tau)\vartheta(x, 0)d\xi d\tau +$$

$$+ \int_0^t \int_0^l G(x, \xi, \tau) \int_0^{t-\tau} k(\alpha)\vartheta_t(\xi, t - \tau - \alpha)d\alpha d\xi d\tau. \tag{20}$$

We use the following relation to calculate the free term $F_t(x, t)$ of the integral Eq. (20) :

$$G_{0t}(x,\xi,t) = G_{0\xi\xi}(x,\xi,t).$$

In view of relations $\varphi''(0) = \varphi''(l) = 0$ and integrating by part, we have

$$F_t(x,t) = \frac{\partial}{\partial t}\left(\int_0^l G_0(x,\xi,t)\varphi''(\xi)d\xi\right) = \int_0^l G_{0\xi\xi}(x,\xi,t)\varphi''(\xi)d\xi =$$

$$= \int_0^l G_0(x,\xi,t)\varphi^{(4)}(\xi)d\xi.$$

Taking into account the last equalities and condition (18), from (20), we obtain the integral equation for the unknown function $k(t)$ as follows:

$$k(t) = \frac{h''(t)}{h(0)} - \frac{1}{h(0)}\int_0^t k(\tau)h'(t-\tau)d\tau - \frac{1}{h(0)}\int_0^l G_{0t}(x_0,\xi,t)\varphi''(\xi)d\xi -$$

$$-\frac{\lambda}{h(0)}\int_0^l G_0(x_0,\xi,t)\int_0^T k(\alpha)\vartheta(\xi,T-\alpha)d\alpha d\xi +$$

$$+\frac{\lambda}{h(0)}\int_t^{T+t}\int_0^l G(x_0,\xi,\tau)\int_0^{T+t-\tau} k(\alpha)\vartheta_t(\xi,T+t-\tau-\alpha)d\alpha d\xi d\tau +$$

$$+\frac{\lambda}{h(0)}\int_t^{T+t}\int_0^l G_0(x_0,\xi,\tau)k(T+t-\tau)\vartheta(x,0)d\xi d\tau -$$

$$-\frac{1}{h(0)}\int_0^t\int_0^l G(x_0,\xi,\tau)k(t-\tau)\vartheta(x,0)d\xi d\tau -$$

$$-\frac{1}{h(0)}\int_0^t\int_0^l G(x_0,\xi,\tau)\int_0^{t-\tau} k(\alpha)\vartheta_t(\xi,t-\tau-\alpha)d\alpha d\xi d\tau. \tag{21}$$

A system of closed integral Eqs. (19)–(21) was obtained for the unknown functions $\vartheta(x,t)$, $\vartheta_t(x,t)$, $k(t)$. The following theorem on the existence and uniqueness of the solution of this system of integral equations is valid.

Theorem2 *Suppose* $\varphi(x) \in C^4[0,l]$; $h(t) \in C^2[0,T]$; $h(0) \neq 0$; $\lambda \geq 0$ *and* $\varphi(0) = \varphi(l) = \varphi''(0) = \varphi''(l) = 0$, $h(0) + \lambda h(T) = \varphi(x_0)$ *are met. Then, there exists sufficiently small number* T^*, *such that for* $T \in (0, T^*]$ *the solution to the integral Eqs.* (19)–(21) *in the class of functions* $\vartheta(x,t) \in C^{2,1}(D_T)$, $k(t) \in C[0,T]$ *exists and it is unique.*

Proof We write the system of Eqs. (19)–(21) in the form of a nonlinear operator equation:

$$Aq = q,$$

where

$$q = (q_1, q_2, q_3) = \big(\vartheta(x,t), \vartheta_t(x,t), k(t)\big).$$

$A = (A_1,\ A_2,\ A_3)$ is defined by the right sides of Eqs. (19)–(21):

$$(Aq)_1(x,t) = q_{01}(x,t) - \lambda \int_0^T \int_0^l G_0(x,\xi,T+t-\tau) \int_0^\tau q_3(\tau) q_1(\xi,\tau-\alpha) d\alpha d\xi d\tau +$$

$$+ \int_0^t \int_0^l G(x,\xi,t-\tau) \int_0^\tau q_3(\alpha) q_1(\xi,\tau-\alpha) d\alpha d\tau d\xi,$$

$$(Aq)_2(x,t) = q_{02}(x,t) + \lambda \int_0^l G_0(x,\xi,t) \int_0^T q_3(\alpha) q_1(\xi,T-\alpha) d\alpha d\xi -$$

$$-\lambda \int_t^{T+t} \int_0^l G_0(x_0,\xi,\tau) \int_0^{T+t-\tau} q_3(\alpha) q_2(\xi,T+t-\tau-\alpha) d\alpha d\xi d\tau -$$

$$-\lambda \int_t^{T+t} \int_0^l G_0(x_0,\xi,\tau) q_3(T+t-\tau) q_4(\xi) d\xi d\tau +$$

$$+ \int_0^t \int_0^l G(x,\xi,\tau) q_3(t-\tau) q_4(\xi) d\xi d\tau +$$

$$+ \int_0^t \int_0^l G(x,\xi,\tau) \int_0^{t-\tau} q_3(\alpha) q_2(\xi,t-\tau-\alpha) d\alpha d\xi d\tau,$$

$$(Aq)_3(t) = q_{03}(t) - \frac{1}{h(0)} \int_0^t q_3(\tau) h'(t-\tau) d\tau -$$

$$-\frac{\lambda}{h(0)} \int_0^l G_0(x_0,\xi,t) \int_0^T q_3(\alpha) q_1(\xi,T-\alpha) d\alpha d\xi +$$

$$+\frac{\lambda}{h(0)} \int_0^{T+t} \int_0^l G_0(x_0,\xi,\tau) \int_0^{T+t-\tau} q_3(\alpha) q_4(\xi,T+t-\tau-\alpha) d\alpha d\xi d\tau +$$

$$+\frac{\lambda}{h(0)} \int_t^{T+t} \int_0^l G_0(x_0,\xi,\tau) q_4(\xi) q_3(T+t-\tau) d\xi d\tau -$$

$$-\frac{1}{h(0)} \int_0^t \int_0^l G(x_0,\xi,\tau) q_3(t-\tau) q_4(\xi) d\xi d\tau -$$

$$-\frac{1}{h(0)} \int_0^t \int_0^l G(x_0,\xi,\tau) \int_0^{t-\tau} q_3(\alpha) q_2(\xi,t-\tau-\alpha) d\alpha d\xi d\tau.$$

The following notations were introduced in the equalities (19)–(21):

$$q_0 = (q_{01}, q_{02}, q_{03}) = \left(F(x,t), F_t(x,t), \frac{1}{h(0)} (h''(t) - F_t(x_0,t)) \right),$$

$$\|q\| = \max\left\{\max_{(x,t)\in\overline{D}_T}|q_1|,\ \max_{(x,t)\in\overline{D}_T}|q_2|,\ \max_{t\in[0,T]}|q_3|\right\},$$

$$\|q_0\| = \max\left\{\max_{(x,t)\in\overline{D}_T}|q_{01}|,\ \max_{(x,t)\in\overline{D}_T}|q_{02}|,\ \max_{t\in[0,T]}|q_{03}|\right\},$$

$$\varphi_0 = \|\varphi\|_{C^4[0,l]}, h_0 = \|h\|_{C^2[0,T]}.$$

Denote by $B(q_0, \|q_0\|)$ the ball of vector-functions q with the center at the point q_0 and radius $\|q_0\|$, i.e. $B(q_0, \|q_0\|) = \left\{q : \|q - q_0\| \leq \|q_0\|\right\}$.
Obviously, $\|q\| \leq 2\|q_0\|$ for $q(x,t) \in B(q_0, \|q_0\|)$. We prove that the operator A is contracting on the set $B(q_0, \|q_0\|)$ if the number T is chosen in a suitable way.
Let us prove that for suitable T the operator A maps the ball $B(q_0, \|q_0\|)$ into itself; i.e., the condition $q \in B(q_0, \|q_0\|)$ implies that $Aq \in B(q_0, \|q_0\|)$.

$$\|(Aq)_1 - q_{01}\| = \max_{(x,t)\in\overline{D}_T}|(Aq)_1 - q_{01}| \leq$$

$$\leq \max_{(x,t)\in\overline{D}_T}\left|\lambda\int_0^T\int_0^l G_0(x,\xi,T+t-\tau)\int_0^\tau q_3(\tau)q_1(\xi,\tau-\alpha)d\alpha d\xi d\tau\right| +$$

$$+ \max_{(x,t)\in\overline{D}_T}\left|\int_0^t\int_0^l G(x,\xi,t-\tau)\int_0^T q_3(\alpha)q_1(\xi,\tau-\alpha)d\alpha d\xi d\tau\right| \leq$$

$$\leq 2\lambda\|q_0\|^2T^2 + 2\|q_0\|^2T^2 = 2\|q_0\|^2T^2(1+\lambda).$$

Similarly, for other components of the vector A, we get:

$$\|(Aq)_2 - q_{02}\| = \max_{(x,t)\in\overline{D}_T}|(Aq)_2 - q_{02}| \leq$$

$$\leq 4\|q_0\|^2T(1+2\lambda) + 2\|q_0\|^2T^2(1+\lambda),$$

$$\|(Aq)_3 - q_{03}\| = \max_{(x,t)\in\overline{D}_T}|(Aq)_3 - q_{03}| \leq$$

$$\leq \frac{2\|q_0\|}{h(0)}(h_0 + 4\lambda\|q_0\| + 2\|q_0\|)T + \frac{2\|q_0\|^2}{h(0)}(1+4\lambda)T^2.$$

Therefore, $Aq \in B(q_0, \|q_0\|)$ if T satisfies the conditions

$$2\|q_0\|T^2(1+\lambda) \leq 1,$$

$$2\|q_0\|(1+\lambda)T^2 + 4\|q_0\|(1+2\lambda)T \leq 1, \tag{22}$$

$$\frac{2\|q_0\|}{h(0)}(1+4\lambda)T^2 + \frac{2}{h(0)}(h_0 + 4\lambda\|q_0\| + 2\|q_0\|)T \leq 1.$$

Now, consider two functions q^1 and q^2 belonging to the ball B and estimate the distance between their images Aq^1 and Aq^2 in the space C.

$$\|(Aq^1 - Aq^2)_1\| =$$

$$= \max_{(x,t)\in\overline{D}_T} \left| \lambda \int_0^T \int_0^l G_0(x,\xi,T+t-\tau) \int_0^\tau [q_3^1(\tau)q_1^1(\xi,\tau-\alpha) - q_3^2(\tau)q_1^2(\xi,\tau-\alpha)]d\alpha d\xi d\tau \right| +$$

$$+ \max_{(x,t)\in\overline{D}_T} \left| \int_0^t \int_0^l G(x,\xi,t-\tau) \int_0^T [q_3^1(\alpha)q_1^1(\xi,\tau-\alpha) - q_3^2(\alpha)q_1^2(\xi,\tau-\alpha)]d\alpha d\xi d\tau \right|.$$

The sub-integral expression in the second integral can be estimated as follows:

$$\left\| q_2^1 q_1^1 - q_2^2 q_1^2 \right\| = \left\| (q_2^1 - q_2^2) q_1^1 + q_2^2 (q_1^1 - q_1^2) \right\| \leq$$

$$\leq 2\left\| q^1 - q^2 \right\| \max\left(\left\| q_1^1 \right\|, \left\| q_2^2 \right\| \right) \leq 4\|q_0\| \left\| q^1 - q^2 \right\|.$$

Therefore,

$$\|(Aq^1 - Aq^2)_1\| \leq 2T^2(1+\lambda)\|q_0\|\|q^1 - q^2\|.$$

The next components can be estimated in a similar way,

$$\|(Aq^1 - Aq^2)_2\| \leq \left(2(\lambda+1)T^2 + 4(2\lambda+1)T\right)\|q_0\|\|q^1 - q^2\|,$$

$$\|(Aq^1 - Aq^2)_3\| \leq \left(\frac{2\|q_0\|}{h(0)}(4\lambda+1)T^2 + \frac{(h_0 + 8\lambda\|q_0\| + 4\|q_0\|)}{h(0)}T \right)\|q^1 - q^2\|.$$

Consequently, $\|(Aq^1 - Aq^2)\| \leq \rho\|q^1 - q^2\|$, where $\rho < 1$ provided that T satisfies the condition

$$2T^2(1+\lambda)\|q_0\| \leq 1,$$

$$2\|q_0\|(\lambda+1)T^2 + 4\|q_0\|(2\lambda+1)T \leq 1, \tag{23}$$

$$\frac{2\|q_0\|}{h(0)}(4\lambda+1)T^2 + \frac{(h_0 + 8\lambda\|q_0\| + 4\|q_0\|)}{h(0)}T \leq 1.$$

Therefore, if the number T is small enough to ensure that conditions (22) and (23) are satisfied, then A is a contraction operator on $B(q_0, \|q_0\|)$. Then, by the Banach principle, equation $q = Aq$ has a unique solution in $B(q_0, \|q_0\|)$.
The proof of the theorem is complete.
By the found function $k(t)$, the function $u(x, t)$ is determined as a solution to integral Eq. (5) (see the "Formulation of problem" section). Thus, the solution of the inverse problems (1)–(4) exists and is unique in the class of functions $u(x,t) \in C^{4,1}(D_T), k(t) \in C[0,T]$, where T satisfies inequalities (7), (22), and (23).

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Declarations

Conflict of interest The authors declare no competing interests.

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