Research Article

Durdimurod Kalandarovich Durdiev* and Jonibek Jamolovich Jumaev

One-dimensional inverse problems of determining the kernel of the integro-differential heat equation in a bounded domain

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Abstract: The integro-differential equation of heat conduction with the time-convolution integral on the right side is considered. The direct problem is the initial-boundary problem for this integro-differential equation. Two inverse problems are studied for this direct problem consisting in determining a kernel of the integral member on two given additional conditions with respect to the solution of the direct problems, respectively. The problems are replaced with the equivalent system of the integral equations with respect to unknown functions and on the basis of contractive mapping the unique solvability inverse problem.

Keywords: integro-differential equation, inverse problem, kernel, resolvent, contraction mapping principle

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1 Introduction: setting up the problems

Problems of determining coefficients, right sides, or other physical parameters in differential equations and partial differential equations (PDEs) equations, given additional "experimental" information about their solutions, arise quite often in various applications. These problems are inverse to the "direct" ones where a differential equation and initial and boundary data are given [1]. Inverse problems for parabolic and hyperbolic PDEs arise naturally in geophysics, oil prospecting, in the design of optical devices, and in many other areas where the interior of an object is to be imaged by measuring field in available domains. Problems of identification of memory kernels in such equations have been intensively studied starting at the end of the last century (see [2–6]). Nowadays, the study of inverse problems for parabolic integro-differential equations is the subject of many studies, of which we mention previous works [7–12] as being closest to the topic of this work. We consider the initial-boundary problem of determining a function $u(x, t)$, $x \in (0, l)$, $t \in (0, T)$ from the following equations:

$$u_t - a^2 u_{xx} = \int_0^t k(\tau) u(x, t - \tau) d\tau + h(x, t), \quad x \in (0, l), \; 0 < t < T; \tag{1}$$

$$u|_{t=0} = \varphi(x), \quad x \in [0, l]; \tag{2}$$

*** Corresponding author: Durdimurod Kalandarovich Durdiev,** Bukhara Branch of the Institute of Mathematics named after V.I. Romanovskiy Academy of Sciences of the Republic of Uzbekistan, Bukhara, Uzbekistan, e-mail: d.durdiev@mathinst.uz
Jonibek Jamolovich Jumaev: Department of Differential Equations, Bukhara State University, Bukhara 200100, Uzbekistan, e-mail: jonibekjj@mail.ru

$$u|_{x=0} = \mu_1(t), \quad u|_{x=l} = \mu_2(t), \quad 0 \le t \le T, \;\; \varphi(0) = \mu_1(0), \varphi(l) = \mu_2(0), \tag{3}$$

where a is a positive constant, and l and T are arbitrary positive numbers. When $k(t)$, $h(x, t)$, $\varphi(x)$, $\mu_1(t)$, and $\mu_2(t)$ are given functions, this problem is called as a direct problem.

In the inverse problem, it is assumed that the kernel $k(t)$, $t > 0$ of the integral term in (1) is unknown and it is required to determine it using additional information about the solution of the direct problem:

$$\int_0^l u(x, t)\mathrm{d}x = f(t), \quad t \in (0, T) \tag{4}$$

or

$$u(x_0, t) = f(t), \quad x_0 \in (0, l), \;\; t \in (0, T). \tag{5}$$

In this case, $\varphi(x)$, $x \in [0, l]$, $\mu_1(t)$, $\mu_2(t)$, $t \in (0, T)$ are assumed to be given functions. In the sequel, we will call the problem of determining functions $u(x, t)$, $k(t)$, $x \in (0, l)$, $t \in (0, T)$ from equations (1)–(4) as **Inverse problem 1** and the problem of determining functions $u(x, t)$, $k(t)$, $x \in (0, l)$, $t > 0$ from equations (1)–(3), (5) as **Inverse problem 2**.

For simplicity, we denote by ϑ the function u_t, i.e., $u_t = \vartheta$. We differentiate the equalities equation (1) with respect to t, and using condition (2), we obtain

$$\vartheta_t - a^2\vartheta_{xx} = k(t)\varphi(x) + \int_0^t k(\tau)\vartheta(x, t - \tau)\mathrm{d}\tau + h_t(x, t). \tag{6}$$

The initial condition for ϑ will be obtained by setting $t = 0$ in equality (1) and using equality (2):

$$\vartheta|_{t=0} = a^2\varphi''(x). \tag{7}$$

To obtain the boundary conditions for $\vartheta(\cdot,t)$ the equation differentiates equality (3) with respect to t:

$$\vartheta|_{x=0} = \mu_1'(t), \vartheta|_{x=l} = \mu_2'(t), 0 < t \le T, a^2\varphi''(0) = \mu_1'(0), a^2\varphi''(l) = \mu_2'(0). \tag{8}$$

By differentiating the additional conditions (4) and (5) with respect to t, we obtain these conditions with respect to the function ϑ for inverse problem 1:

$$\int_0^l \vartheta(x, t)\mathrm{d}x = f'(t), t \in (0, T) \tag{9}$$

and for inverse problem 2:

$$\vartheta(x_0, t) = f'(t), t \in (0, T). \tag{10}$$

We replace the initial-boundary problems (6)–(8) with the equivalent Volterra integral equation. To do this, from equations (6) to (8), we derive for $\vartheta(x, t)$ the equation [see [13, pp. 180–219]]:

$$\vartheta(x, t) = \psi(x, t) + \int_0^t \int_0^l G(x, \xi, t - \tau)\left[k(\tau)\varphi(\xi) + \int_0^\tau k(\alpha)\vartheta(\xi, \tau - \alpha)\mathrm{d}\alpha\right]\mathrm{d}\xi\mathrm{d}\tau, \tag{11}$$

where

$$\psi(x, t) = \int_0^t \int_0^l G(x, \xi, t - \tau)h_\tau(\xi, \tau)\mathrm{d}\xi\mathrm{d}\tau$$
$$+ \sum_{n=1}^{\infty}\left[\frac{2}{l}\int_0^l \varphi(x)\sin\frac{\pi n}{l}x\mathrm{d}x + \frac{2\pi a^2 n}{l^2}\int_0^t (\mu_1'(\tau) - (-1)^n\mu_2'(\tau))e^{\left(\frac{\pi a n}{l}\right)^2\tau}\mathrm{d}\tau\right]e^{\left(\frac{\pi a n}{l}\right)^2 t}\sin\frac{\pi n}{l}x$$

and

$$G(x,\xi,t-\tau)=\frac{2}{l}\sum_{n=1}^{\infty}e^{-\left(\frac{\pi an}{l}\right)^{2}(t-\tau)}\sin\frac{\pi an}{l}\xi\,\sin\frac{\pi an}{l}x$$

is the Green function of the initial-boundary problem for one-dimensional heat equation.

Now we write two properties of Green function (see [13, pp. 200–221]), which will be needed in the future.

Remark 1. The integral of the Green function does not exceed 1:

$$\int_0^l G(x,\xi,t)\mathrm{d}\xi\le 1,\ x\in(0,l),\ t\in(0,T].$$

Remark 2. The function $G(x,\xi,t)$ is infinitely continuously differentiable with respect to x, ξ, t and $G_t(x,\xi,t)$ is bounded for $0<x<l$, $0<\xi<l$, $0<t\le T$, i.e.,

$$|G_t(x,\xi,t-\tau)|\le\frac{2}{l}.$$

2 Direct problem

Lemma 1. *Let*

$$(\varphi(x),\varphi'(x),\varphi''(x))\in C(0,l),\ (h(x,t),h_t(x,t))\in C(D_{lT}),$$
$$(\mu_1(t),\mu_1'(t),\mu_2(t),\mu_2'(t))\in C(0,T),\ k(t)\in C(0,T),$$

and the matching conditions in (3) *and* (8) *are met. Then there is the unique classical solution* $\vartheta(x,t)$ *to problems* (6)–(8) *of the class* $C^{2,1}(D_{lT})$*, which is twice continuously differentiable with respect to x and once continuously differentiable with respect to t in the domain* D_{lT} *functions,* $D_{lT}=\{0<x<l,0<t<T\}$.

We also use the usual class $C(D_{lT})$ of continuous in D_{lT} functions.

To **prove** Lemma 1, we rewrite equation (11) in the form

$$\begin{aligned}\vartheta(x,t)=&\int_0^t\int_0^l G(x,\xi,t-\tau)h_\tau(\xi,\tau)\mathrm{d}\xi\mathrm{d}\tau+\sum_{n=1}^{\infty}\left[\frac{2}{l}\int_0^l\varphi(x)\sin\frac{\pi n}{l}x\mathrm{d}x\right.\\&\left.+\frac{2\pi a^2n}{l^2}\int_0^t(\mu_1'(\tau)-(-1)^n\mu_2'(\tau))e^{\left(\frac{\pi an}{l}\right)^2\tau}\mathrm{d}\tau\right]e^{\left(\frac{\pi an}{l}\right)^2t}\sin\frac{\pi n}{l}x\\&+\int_0^t\int_0^l G(x,\xi,t-\tau)k(\tau)\varphi(\xi)\mathrm{d}\xi\mathrm{d}\tau+\int_0^t\int_0^l G(x,\xi,t-\tau)\int_0^\tau k(\alpha)\vartheta(\xi,\tau-\alpha)\mathrm{d}\alpha\mathrm{d}\xi\mathrm{d}\tau,\end{aligned}\tag{12}$$

and denoting the sum of the first three summands on the right-hand side of (12) by $\Phi(x,t)$, for this equation, we consider in the domain D_{lT} the sequence of functions

$$\vartheta_n(x,t)=\Phi(x,t)+\int_0^t\int_0^l G(x,\xi,t-\tau)\int_0^\tau k(\alpha)\vartheta_{n-1}(\xi,\tau-\alpha)\mathrm{d}\alpha\mathrm{d}\xi\mathrm{d}\tau,\quad n=1,2,\dots,\tag{13}$$

where $\vartheta_0(x,t)=0$ for $(x,t)\in D_{lT}$. If the conditions of Lemma 11 are fulfilled, we have that $\Phi(x,t)\in C^{2,1}(D_T)$. (see [14, pp. 39–44]) Then, it follows from (13) that all $\vartheta_n(x,t)$ in D_{lT} possess the same property.

Denote $Z_n(x,t) = \vartheta_n(x,t) - \vartheta_{n-1}(x,t)$ and $\Phi_0 = \|\Phi\|_{C(D_{lT})}$. According to formula (13), we estimate $Z_n(x,t)$ in the domain D_{lT}:

$$
\begin{aligned}
&|Z_1(x,t)| \le \Phi_0,\\
&|Z_2(x,t)| \le \int_0^t \int_0^l G(x,\xi,t-\tau) \int_0^\tau |k(\alpha)||Z_1(\xi,\tau-\alpha)|\mathrm{d}\alpha\mathrm{d}\xi\mathrm{d}\tau \le \Phi_0 k_0 \frac{t^2}{2!},\\
&\qquad k_0 = \max_{t\in[0,T]} |k(t)|,\\
&|Z_3(x,t)| \le \int_0^t \int_0^l G(x,\xi,t-\tau) \int_0^\tau |k(\alpha)||Z_2(\xi,\tau-\alpha)|\mathrm{d}\alpha\mathrm{d}\xi\mathrm{d}\tau \le \Phi_0 k_0^2 \frac{t^4}{4!}.
\end{aligned}
$$

Thus, for arbitrary $n = k$, we have

$$|Z_k(x,t)| \le \Phi_0 k_0^{k-1} \frac{t^{2(k-1)}}{2(k-1)!}.$$

It follows from the aforementioned estimates that the series

$$\sum_{n=1}^{\infty} [\vartheta_n(x,t) - \vartheta_{n-1}(x,t)]$$

converges in D_{lT}, and its sum $u(x,t)$ belongs to the functional space $C^{2,1}(D_T)$. Since the sequence $\vartheta_n(x,t)$ determined by equality (13) converges to $\vartheta(x,t)$ uniformly in D_{lT}, then $\vartheta(x,t)$ is a solution of equation (11).

Now show that this solution is the only one. Suppose that there are two solutions $\vartheta^1(x,t)$ and $\vartheta^2(x,t)$. Then their difference $Z(x,t) = \vartheta^2(x,t) - \vartheta^1(x,t)$ is a solution to the equation

$$Z(x,t) = \int_0^t \int_0^l G(x,\xi,t-\tau) \int_0^\tau k(\alpha) Z(\xi,\alpha)\mathrm{d}\alpha\mathrm{d}\xi\mathrm{d}\tau.$$

Let $\tilde{Z}(t)$ denote the supremum of the module of the function $Z(x,t)$ for $x \in (0,l)$ at each fixed $t \in (0,T)$. Then we have the inequality

$$\tilde{Z}(t) \le k_0 T \int_0^t \tilde{Z}(\tau)\mathrm{d}\tau, \quad t \in [0,T].$$

Applying the Gronwall lemma here, we obtain that $\tilde{Z}(t) = 0$ for $t \in [0,T]$, which means that $Z(x,t) = 0$ in D_T, i.e., $\vartheta^1(x,t) = \vartheta^2(x,t)$ in D_{lT}. Therefore, equation (11) has a unique solution in D_{lT}. The lemma is proved.

3 Inverse problem 1

Using the additional condition for inverse problem 1, from (11) we have

$$f'(t) = \int_0^l \psi(x,t)\mathrm{d}x + \int_0^l \int_0^t \int_0^l G(x,\xi,t-\tau)k(\tau)\varphi(\xi)\mathrm{d}\xi\mathrm{d}\tau\mathrm{d}x + \int_0^l \int_0^t \int_0^l G(x,\xi,t-\tau) \int_0^\tau k(\alpha)\vartheta(\xi,\tau-\alpha)\mathrm{d}\alpha\mathrm{d}\xi\mathrm{d}\tau\mathrm{d}x.$$

Differentiating this equality with respect to t, we arrive at equation:

$$f''(t) = \int_0^l \psi_t(x,t)\mathrm{d}x \int_0^l \int_0^l G(x,\xi,0)k(t)\varphi(\xi)\mathrm{d}\xi\mathrm{d}x + \int_0^l \int_0^t k(\tau)\int_0^l G_t(x,\xi,t-\tau)\varphi(\xi)\mathrm{d}\xi\mathrm{d}\tau\mathrm{d}x$$
$$+ \int_0^l \int_0^t \int_0^l G_t(x,\xi,t-\tau)\int_0^\tau k(\alpha)\vartheta(\xi,\tau-\alpha)\mathrm{d}\alpha\mathrm{d}\xi\mathrm{d}\tau\mathrm{d}x + \int_0^l \int_0^l G(x,\xi,0)\int_0^t k(\alpha)\vartheta(\xi,t-\alpha)\mathrm{d}\alpha\mathrm{d}\xi\mathrm{d}x.$$

Since $G(x,\xi,0) = \delta(x-\xi)$, where $\delta(\cdot)$ is the Dirac's delta function, and taking into account the following relations:

$$\int_0^l g(\xi)\delta(x-\xi)\mathrm{d}\xi = g(x), \quad \int_0^l G(x,\xi,0)\int_0^t k(\alpha)\vartheta(\xi,t-\alpha)\mathrm{d}\alpha\mathrm{d}\xi = \int_0^t k(\alpha)\vartheta(x,t-\alpha)\mathrm{d}\alpha,$$

we rewrite the last equation in the form

$$\begin{aligned} f''(t) &= \int_0^l \psi_t(x,t)\mathrm{d}x + k(t)\int_0^l \varphi(x)\mathrm{d}x + \int_0^l \int_0^t k(\tau)\int_0^l G_t(x,\xi,t-\tau)\varphi(\xi)\mathrm{d}\xi\mathrm{d}\tau\mathrm{d}x \\ &+ \int_0^l \int_0^t \int_0^l G_t(x,\xi,t-\tau)\int_0^\tau k(\alpha)\vartheta(\xi,\tau-\alpha)\mathrm{d}\alpha\mathrm{d}\xi\mathrm{d}\tau\mathrm{d}x + \int_0^l \int_0^t k(\alpha)\vartheta(x,t-\alpha)\mathrm{d}\alpha\mathrm{d}x. \end{aligned} \tag{14}$$

In what follows, we denote

$$\varphi_0 = \int_0^l \varphi(x)\mathrm{d}x.$$

Next we write equality (14) as the integral equation of the second order with respect to unknown function $k(t)$

$$\begin{aligned} k(t) = \frac{1}{\varphi_0}\Bigg[& f''(t) - \int_0^l \psi_t(x,t)\mathrm{d}x - \int_0^l \int_0^t k(\tau)\int_0^l G_t(x,\xi,t-\tau)\varphi(\xi)\mathrm{d}\xi\mathrm{d}\tau\mathrm{d}x \\ & - \int_0^l \int_0^t k(\alpha)\vartheta(x,t-\alpha)\mathrm{d}\alpha\mathrm{d}x - \int_0^l \int_0^t \int_0^l G_t(x,\xi,t-\tau)\int_0^\tau k(\alpha)\vartheta(\xi,\tau-\alpha)\mathrm{d}\alpha\mathrm{d}\xi\mathrm{d}\tau\mathrm{d}x \Bigg]. \end{aligned} \tag{15}$$

We represent the system of equations (11) and (15) in the form

$$Ag = g, \tag{16}$$

where $g = (g_1, g_2) = (\vartheta(x,t), k(t))$ is the vector function and $A = (A_1, A_2)$ is defined by the right sides of equations (11) and (15):

$$A_1 g = g_{01}(x,t) + \int_0^t \int_0^l G(x,\xi,t-\tau)\left[g_2(\tau)\varphi(\xi) + \int_0^\tau g_2(\alpha)g_1(\xi,\tau-\alpha)\mathrm{d}\alpha \right]\mathrm{d}\xi\mathrm{d}\tau, \tag{17}$$

$$\begin{aligned} A_2 g = g_{02}(t) - \frac{1}{\varphi_0}\Bigg[& \int_0^l \int_0^t g_2(\tau)\int_0^l G_t(x,\xi,t-\tau)\varphi(\xi)\mathrm{d}\xi\mathrm{d}\tau\mathrm{d}x - \int_0^l \int_0^t g_2(\alpha)g_1(x,t-\alpha)\mathrm{d}\alpha\mathrm{d}x \\ & - \int_0^l \int_0^t \int_0^l G_t(x,\xi,t-\tau)\int_0^\tau g_2(\alpha)g_1(\xi,\tau-\alpha)\mathrm{d}\alpha\mathrm{d}\xi\mathrm{d}\tau\mathrm{d}x \Bigg]. \end{aligned} \tag{18}$$

The following notations were introduced in equalities (17) and (18):

$$g_0(x,t) = (g_{01}(x,t), g_{02}(t)) = \left(\psi(x,t), \frac{1}{\varphi_0}\left[f''(t) - \int_0^l \psi_t(x,t)\mathrm{d}x\right]\right).$$

Theorem 1. *Suppose $f(t) \in C^2[0,T]$, $\varphi_0 \neq 0$ and the conditions of Lemma 1 are fulfilled. Then the operator equation (16) has a unique solution in domain D_{lT} for arbitrary fixed $l > 0$ and $T > 0$.*

To **prove** Theorem 1, we define for the unknown vector function $g(x,t) \in C(D_{lT})$ the following weight norm:

$$\|g\|_\sigma = \max\left\{\sup_{(x,t)\in\overline{D_T}} |g_1(x,t)e^{-\sigma t}|, \sup_{t\in[0,T]} |g_2(x,t)e^{-\sigma t}|\right\} = \max\{\|g_1\|_\sigma, \|g_2\|_\sigma\}, \quad \sigma \geq 0.$$

At $\sigma = 0$, this norm coincides with the usual norm

$$\|g\| = \max\left\{\sup_{(x,t)\in\overline{D_T}} |g_1(x,t)|, \quad \sup_{t\in[0,T]} |g_2(t)|\right\}.$$

The number $\sigma \geq 0$ will be chosen later. Denote by $B(g_0, \rho)$ the ball of vector functions g with center at the point g_0 and radius $\rho > 0$, i.e., $B(g_0,\rho) = \{g : \|g - g_0\|_\sigma \leq \rho\}$. The number $\rho > 0$ will also be chosen later.

Obviously, $\|g\| \leq \rho + \|g_0\|$ for $g(x,t) \in B(g_0,\rho)$. We prove that the operator A is contracting in the Banach space $B(g_0,\rho)$ if the numbers σ and ρ will be chosen in suitable way. Remind that operator A is contractive if the following two conditions are met (see [15, pp. 87–97]):

(1) If $g(x,t) \in B(g_0,\rho)$, then $Ag \in B(g_0,\rho)$;
(2) If g^1, g^2 are arbitrary two elements of $B(g_0,\rho)$, then the inequality $\|Ag^1 - Ag^2\|_\sigma \leq \mu\|g^1 - g^2\|_\sigma$ is valid with $\mu \in (0,1)$.

Note that the weight norm $\|\cdot\|_\sigma$ is equivalent to the usual norm $\|\cdot\|$:

$$\|\cdot\|_\sigma \leq \|\cdot\| \leq e^{\sigma T}\|\cdot\|_\sigma, \quad \sigma \geq 0. \tag{19}$$

The convolution operator is commutative and invariant with respect to multiplication by $e^{-\sigma t}$:

$$(h_1 * h_2)(t) = \int_0^t h_1(t-s)h_2(s)\mathrm{d}s = \int_0^t h_1(s)h_2(t-s)\mathrm{d}s = (h_2 * h_1)(t), \tag{20}$$

$$e^{-\sigma t}(h_1 * h_2)(t) = (e^{-\sigma t}h_1(t)) * (e^{-\sigma t}h_2(t)). \tag{21}$$

The last formula implies the estimation

$$\|h_1 * h_2\|_\sigma \leq \|h_1\|_\sigma \|h_2\|_\sigma T. \tag{22}$$

Moreover, since

$$\int_0^t e^{-\sigma s}\mathrm{d}s = \int_0^t e^{-\sigma(t-s)}\mathrm{d}s \leq \frac{1}{\sigma}, \quad \sigma \geq 0, \tag{23}$$

we have

$$\|h_1 * h_2\|_\sigma \leq \frac{1}{\sigma}\|h_1\|\|h_2\|_\sigma \leq \frac{1}{\sigma}\|h_1\|\|h_2\|, \quad \sigma \geq 0 \tag{24}$$

using (19) and the results of [12].

First, we check the first condition of contractive mapping. For simplicity, we denote $\varphi_1 = \max_{x\in(0,l)}|\varphi(x)|$. Let $g(x,t)$ be an element of $B(g_0,\rho)$, i.e., $g \in B(g_0,\rho)$. Then for $(x,t) \in D_{lT}$ we have

$$\begin{aligned}
\|A_1g - g_{01}\|_\sigma &= \sup_{(x,t)\in D_{lT}} |(A_1g - g_{01})e^{-\sigma t}| \\
&= \sup_{(x,t)\in D_{lT}} e^{-\sigma t}\left|\int_0^t\int_0^l G(x,\xi,t-\tau)g_2(\tau)\varphi(\xi)\mathrm{d}\xi\mathrm{d}\tau + \int_0^t\int_0^l G(x,\xi,t-\tau)\int_0^\tau g_2(\alpha)g_1(\xi,\tau-\alpha)\mathrm{d}\alpha\mathrm{d}\xi\mathrm{d}\tau\right| \\
&\le \sup_{(x,t)\in D_{lT}}\left|\int_0^t\int_0^l G(x,\xi,t-\tau)g_2(\tau)e^{-\sigma\tau}\varphi(\xi)e^{-\sigma(t-\tau)}\mathrm{d}\xi\mathrm{d}\tau\right| \\
&\quad + \sup_{(x,t)\in D_{lT}}\left|\int_0^t e^{-\sigma(t-\tau)}\int_0^l G(x,\xi,t-\tau)\int_0^\tau g_2(\alpha)e^{-\sigma\alpha}g_1(\xi,\tau-\alpha)e^{-\sigma(\tau-\alpha)}\mathrm{d}\alpha\mathrm{d}\xi\mathrm{d}\tau\right| \\
&\le \frac{(\rho + \|g_0\|)}{\sigma}(\varphi_1 + (\rho + \|g_0\|)T).
\end{aligned}$$

If we choose σ as

$$\sigma \ge \sigma_1 = \frac{\rho}{(\rho + g_0)(\varphi_1 + (\rho + \|g_0\|)T)},$$

then $\|A_1g - g_{01}\|_\sigma \le \rho$, i.e., the first condition of contractive mapping for A_1 is satisfied.

Now we carry out the estimations for A_2:

$$\begin{aligned}
\|A_2g - g_{02}\|_\sigma &= \sup_{t\in(0,T)} |(A_2g - g_{02})e^{-\sigma t}| \\
&= \sup_{t\in(0,T)} \frac{1}{\varphi_0}e^{-\sigma t}\left|\int_0^l\int_0^t g_2(\tau)\int_0^l G_t(x,\xi,t-\tau)\varphi(\xi)\mathrm{d}\xi\mathrm{d}\tau\mathrm{d}x + \int_0^l\int_0^t g_2(\alpha)g_1(x,t-\alpha)\mathrm{d}\alpha\mathrm{d}x\right. \\
&\quad \left. + \int_0^l\int_0^t\int_0^l G_t(x,\xi,t-\tau)\int_0^\tau g_2(\alpha)g_1(\xi,\tau-\alpha)\mathrm{d}\alpha\mathrm{d}\xi\mathrm{d}\tau\mathrm{d}x\right| \\
&\le \sup_{t\in(0,T)} \frac{1}{\varphi_0}\left|\int_0^l\int_0^t g_2(\tau)e^{-\sigma\tau}e^{-\sigma(t-\tau)}\int_0^l G_t(x,\xi,t-\tau)\varphi(\xi)\mathrm{d}\xi\mathrm{d}\tau\mathrm{d}x\right| \\
&\quad + \sup_{t\in(0,T)} \frac{1}{\varphi_0}\left|\int_0^l\int_0^t g_2(\alpha)e^{-\sigma\alpha}g_1(x,t-\alpha)e^{-\sigma(t-\alpha)}\mathrm{d}\alpha\mathrm{d}x\right| \\
&\quad + \sup_{t\in(0,T)} \frac{1}{\varphi_0}\left|\int_0^l\int_0^t e^{-\sigma(t-\tau)}\int_0^l G_t(x,\xi,t-\tau)\int_0^\tau g_2(\alpha)e^{-\sigma\alpha}g_1(\xi,\tau-\alpha)e^{-\sigma(\tau-\alpha)}\mathrm{d}\alpha\mathrm{d}\xi\mathrm{d}\tau\mathrm{d}x\right|.
\end{aligned}$$

Denoting each summand in last formula by I_i, $i = 1, 2, 3$, we estimate them. For the expression I_1 we obtain

$$\begin{aligned}
I_1 &= \sup_{t\in(0,T)} \frac{1}{\varphi_0}\left|\int_0^l\int_0^t g_2(\tau)e^{-\sigma\tau}e^{-\sigma(t-\tau)}\int_0^l G_t(x,\xi,t-\tau)\varphi(\xi)\mathrm{d}\xi\mathrm{d}\tau\mathrm{d}x\right| \\
&\le \frac{\varphi_1}{\varphi_0}\|g_2\|_\sigma \sup_{t\in(0,T)}\left|\int_0^t e^{-\sigma(t-\tau)}\int_0^l\int_0^l G_t(x,\xi,t-\tau)\mathrm{d}\xi\mathrm{d}\tau\mathrm{d}x\right| \\
&\le \frac{2l\varphi_1(\rho + \|g_0\|)}{\varphi_0}\frac{1}{\sigma}.
\end{aligned}$$

Using relations (19)–(24), we estimate I_2 as follows:

$$
\begin{aligned}
I_2 &= \sup_{t\in(0,T)} \frac{1}{\varphi_0} \left| \int_0^l \int_0^t g_2(\alpha) g_1(x, t-\alpha) e^{-\sigma t} \mathrm{d}\alpha \mathrm{d}x \right| \\
&= \sup_{t\in(0,T)} \frac{1}{\varphi_0} \left| \int_0^l (g_2 * g_1)(t) e^{-\sigma t} \mathrm{d}x \right| \\
&= \frac{1}{\varphi_0} \sup_{t\in(0,T)} \left| \int_0^l \{[(g_2 - g_{02}) * (g_1 - g_{01})](t) + (g_2 * g_{01})(t) + (g_1 * g_{02})(t) - (g_{02} * g_{01})(t)\} e^{-\sigma t} \mathrm{d}x \right| \\
&\le \frac{1}{\varphi_0} \int_0^l \left(\|g_2 - g_{02}\|_\sigma \|g_1 - g_{01}\|_\sigma T + \frac{1}{\sigma}\|g_2\|_\sigma \|g_{01}\| + \frac{1}{\sigma}\|g_1\|_\sigma \|g_{01}\| + \frac{1}{\sigma}\|g_{01}\|_\sigma \|g_{02}\| \right) \mathrm{d}x \\
&\le \frac{l}{\varphi_0} \left(\rho^2 T + \frac{2}{\sigma}(\rho + \|g_0\|)\|g_0\| + \frac{1}{\sigma}\|g_0\|^2 \right).
\end{aligned}
$$

Conducting the similar estimates alike as for the case I_1 we have for I_3

$$
\begin{aligned}
I_3 &= \sup_{t\in(0,T)} \frac{1}{\varphi_0} \left| \int_0^l \int_0^t e^{-\sigma(t-\tau)} \int_0^l G_t(x, \xi, t-\tau) \int_0^\tau g_2(\alpha) e^{-\sigma\alpha} g_1(\xi, \tau-\alpha) e^{-\sigma(\tau-\alpha)} \mathrm{d}\alpha \mathrm{d}\xi \mathrm{d}\tau \mathrm{d}x \right| \\
&\le \frac{1}{\varphi_0} \|g_1\|_\sigma \|g_2\|_\sigma \sup_{t\in(0,T)} \left| \int_0^l \int_0^t e^{-\sigma(t-\tau)} \int_0^l G_t(x, \xi, t-\tau) \mathrm{d}\xi \mathrm{d}\tau \mathrm{d}x \right| \le \frac{2lT(\rho + \|g_0\|)^2}{\varphi_0} \frac{1}{\sigma}.
\end{aligned}
$$

Accordingly, we obtain

$$
\begin{aligned}
\|A_2 g - g_{02}\|_\sigma &\le I_1 + I_2 + I_3 \\
&\le \frac{2l\varphi_1(\rho + \|g_0\|)}{\varphi_0} \frac{1}{\sigma} + \frac{\rho^2 lT}{\varphi_0} + \frac{2l(\rho + \|g_0\|)\|g_0\|}{\varphi_0} \frac{1}{\sigma} + \frac{l\|g_0\|^2}{\varphi_0} \frac{1}{\sigma} + \frac{2lT(\rho + \|g_0\|)^2}{\varphi_0} \frac{1}{\sigma}. \qquad (25)
\end{aligned}
$$

Now we can choose ρ, σ such that there hold the inequalities:

$$
\begin{cases}
\dfrac{\rho^2 lT}{\varphi_0} < \dfrac{1}{3}\rho, \\[2mm]
\dfrac{l\|g_0\|^2}{\varphi_0 \sigma} < \dfrac{1}{3}\rho, \\[2mm]
\dfrac{2l(\rho + \|g_0\|)(\varphi_1 + \|g_0\| + T(\rho + \|g_0\|))}{\varphi_0 \sigma} < \dfrac{1}{3}\rho.
\end{cases}
$$

It follows that if

$$
\begin{cases}
\rho < \dfrac{\varphi_0}{3Tl} = \rho_1, \\[2mm]
\beta_1 = \dfrac{9l^2 \|g_0\|^2 T}{\varphi_0^2} < \sigma, \\[2mm]
\beta_2 = \dfrac{18Tl^2}{\varphi_0^2} \left(\dfrac{\varphi_0}{3Tl} + \|g_0\| \right) \left(\varphi_1 + \|g_0\| + T\left(\dfrac{\varphi_0}{3Tl} + \|g_0\| \right) \right) < \sigma,
\end{cases}
$$

then $A_2 g \in B(g_0, \rho)$.

So, if the inequality

$$
\sigma > \sigma_2 = \max\{\beta_1, \beta_2\} \qquad (26)
$$

and $\rho \in (0, \rho_1)$ hold, then the operator A_2 maps $B(g_0, \rho)$ into itself, i.e., $A_2 g \in B(g_0, \rho)$.

As a result, we conclude that if σ, ρ satisfy the conditions $\sigma > \max\{\sigma_1, \sigma_2\}$, $\rho \in (0, \rho_1)$, then operator A maps $B(g_0, \rho)$ into itself, i.e., $Ag \in B(g_0, \rho)$.

Second, we check the second condition of contractive mapping. In accordance with (17) for the first component of operator A we obtain

$$\|(Ag^1 - Ag^2)_1\|_\sigma \le \sup_{(x,t)\in D_{lT}} \left| \int_0^t \int_0^l G(x, \xi, t-\tau)[g_2^1(\tau) - g_2^2(\tau)]\varphi(\xi)\mathrm{d}\xi\mathrm{d}\tau e^{-\sigma t} \right|$$
$$+ \sup_{(x,t)\in D_{lT}} \left| \int_0^t \int_0^l G(x, \xi, t-\tau) \times \int_0^\tau [g_2^1(\alpha)g_1^1(\xi, \tau-\alpha) - g_2^2(\alpha)g_1^2(\xi, \tau-\alpha)]\mathrm{d}\alpha\mathrm{d}\xi\mathrm{d}\tau e^{-\sigma t} \right|.$$

Here the integrand in the last integral can be estimated as follows:

$$\begin{aligned} \|g_2^1 g_1^1 - g_2^2 g_1^2\|_\sigma &= \|(g_2^1 - g_2^2)g_1^1 + g_2^2(g_1^1 - g_1^2)\|_\sigma \\ &\le 2\,\|g^1 - g^2\|_\sigma \max(\|g_1^1\|_\sigma, \|g_2^2\|_\sigma) \\ &\le 2(\|g_0\| + \rho)\|g^1 - g^2\|_\sigma. \end{aligned}$$

Therefore,

$$\|(Ag^1 - Ag^2)_1\|_\sigma \le \frac{1}{\sigma}(\varphi_1 + 2(\rho + \|g_0\|)T)\|g^1 - g^2\|_\sigma.$$

It is obvious that if we choose σ as $\sigma > \sigma_3 = \varphi_1 + 2(\rho + \|g_0\|)T$, then $\|(Ag^1 - Ag^2)_1\|_\sigma \le \frac{\sigma_3}{\sigma}\|g^1 - g^2\|_\sigma$, i.e., the second condition of contractive mapping for A_1 is satisfied.

The second component of A can be estimated in the following form:

$$\|(Ag^1 - Ag^2)_2\|_\sigma = \sup_{t\in(0,T)} \frac{1}{\varphi_0} \left| \int_0^l \int_0^t [g_2^1 - g_2^2](\tau) \int_0^l G_t(x, \xi, t-\tau)\varphi(\xi)\mathrm{d}\xi\mathrm{d}\tau\mathrm{d}x e^{-\sigma t} \right|$$
$$+ \sup_{t\in(0,T)} \frac{1}{\varphi_0} \left| \int_0^l \int_0^t [g_2^1 g_1^1 - g_2^2 g_1^2] e^{-\sigma t}\mathrm{d}\alpha\mathrm{d}x \right| + \sup_{t\in(0,T)} \frac{1}{\varphi_0} \left| \int_0^l \int_0^t \int_0^l G_t(x, \xi, t-\tau) \int_0^\tau [g_2^1 g_1^1 \right.$$
$$\left. - g_2^2 g_1^2] e^{-\sigma t}\mathrm{d}\alpha\mathrm{d}\xi\mathrm{d}\tau\mathrm{d}x \right|.$$

We denote the summands in this equality by J_1, J_2, J_3, respectively, and carry out the estimates for them separately. The estimate for J_1 has the form

$$J_1 = \sup_{t\in(0,T)} \frac{1}{\varphi_0} \left| \int_0^l \int_0^t [g_2^1 - g_2^2](\tau)e^{-\sigma\tau}e^{-\sigma(t-\tau)} \int_0^l G_t(x, \xi, t-\tau)\varphi(\xi)\mathrm{d}\xi\mathrm{d}\tau\mathrm{d}x \right| \le \frac{1}{\sigma}\frac{2l\varphi_1}{\varphi_0}\|g^1 - g^2\|_\sigma.$$

Taking into account the relation

$$g_2^1 * g_1^1 - g_2^2 * g_1^2 = (g_1^2 - g_2^2) * (g_1^1 - g_{01}) + (g_1^1 - g_1^2) * (g_2^2 - g_{02}) + g_{01} * (g_2^1 - g_2^2) + g_{02} * (g_1^1 - g_1^2),$$

estimate J_2 and J_3 as follows:

$$J_2 = \sup_{t\in(0,T)} \frac{1}{\varphi_0}\left| \int_0^l \int_0^t [g_2^1 g_1^1 - g_2^2 g_1^2] e^{-\sigma t} \mathrm{d}\alpha \mathrm{d}x \right|$$
$$= \sup_{t\in(0,T)} \frac{1}{\varphi_0}\left| \int_0^l [g_2^1 * g_1^1 - g_2^2 * g_1^2] e^{-\sigma t} \mathrm{d}x \right|$$
$$\le \frac{l}{\varphi_0}[\|g_1^2 - g_2^2\|_\sigma \|g_1^1 - g_{01}\|_\sigma T + \|g_1^1 - g_1^2\|_\sigma \|g_2^2 - g_{02}\|_\sigma T + \|g_{01}\|_\sigma \|g_2^1 - g_2^2\|_\sigma + \|g_{02}\|_\sigma \|g_1^1 - g_1^2\|_\sigma]$$
$$\le \frac{2l}{\varphi_0}\left(\rho T + \frac{1}{\sigma}\|g_0\|\right) \|g^1 - g^2\|_\sigma,$$
$$J_3 = \sup_{t\in(0,T)} \frac{1}{\varphi_0}\left| \int_0^l \int_0^t \int_0^l G_t(x,\xi,t-\tau) \int_0^\tau ((g_2^1 - g_2^2) g_1^1 + (g_1^1 - g_1^2) g_2^2) e^{-\rho t} \mathrm{d}\alpha \mathrm{d}\xi \mathrm{d}\tau \mathrm{d}x \right|$$
$$\le \sup_{t\in(0,T)} \frac{1}{\varphi_0}\left| \int_0^l \int_0^t e^{-\sigma(t-\tau)} \int_0^l G_t(x,\xi,t-\tau) \int_0^\tau (g_2^1 - g_2^2) e^{-\sigma\alpha} g_1^1 e^{-\sigma(\tau-\alpha)} \mathrm{d}\alpha \mathrm{d}\xi \mathrm{d}\tau \mathrm{d}x \right|$$
$$+ \sup_{t\in(0,T)} \frac{1}{\varphi_0}\left| \int_0^l \int_0^t e^{-\sigma(t-\tau)} \int_0^l G_t(x,\xi,t-\tau) \int_0^\tau (g_1^1 - g_1^2) e^{-\sigma\alpha} g_2^2 e^{-\sigma(\tau-\alpha)} \mathrm{d}\alpha \mathrm{d}\xi \mathrm{d}\tau \mathrm{d}x \right|$$
$$\le \frac{1}{\sigma} \frac{4(\rho + \|g_0\|) l T}{\varphi_0} \|g^1 - g^2\|_\sigma.$$

Summing the obtained estimates for J_i, $i = 1, 2, 3$, we have

$$\|(Ag^1 - Ag^2)_2\|_\sigma \le J_1 + J_2 + J_3 \le \frac{2l}{\varphi_0}\left(\rho T + \frac{\varphi_1}{\sigma} + \frac{\|g_0\|}{\sigma} + \frac{2(\rho + \|g_0\|)T}{\sigma}\right) \|g^1 - g^2\|_\sigma.$$

Now we choose numbers σ, ρ so that the expression at $\|g^1 - g^2\|_\sigma$ becomes less than 1, i.e., the inequality

$$\frac{2l}{\varphi_0}\left(\frac{\varphi_1}{\sigma} + \rho T + \frac{\|g_0\|}{\sigma} + \frac{2(\rho + \|g_0\|)T}{\sigma}\right) < 1$$

is fulfilled. This inequality is valid if numbers σ, ρ will be chosen from conditions

$$\begin{cases} \frac{2\rho T l}{\varphi_0} < \frac{1}{3}, \\ \frac{2l}{\varphi_0 \sigma}(\varphi_1 + \|g_0\|) < \frac{1}{3}, \\ \frac{4lT}{\varphi_0 \sigma}(\rho + \|g_0\|) < \frac{1}{3}. \end{cases}$$

Solving these inequalities with respect to σ, ρ we obtain

$$\begin{cases} \rho < \frac{\varphi_0}{6Tl} = \rho_2, \\ \sigma_4 = \frac{6l}{\varphi_0}(\varphi_1 + \|g_0\|) < \sigma, \\ \sigma_5 = \frac{2\varphi_0 + 12lT\|g_0\|}{\varphi_0} < \sigma. \end{cases}$$

From these estimates, it is clear that if σ and ρ are chosen from condition $\sigma > \sigma_4$ and $\rho < (0, \rho_2)$, then the operator A_2 satisfies the second condition of contracting mapping.

As a result, we conclude that if σ and ρ are taken from conditions

$$\sigma > \max(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5)$$

and $\rho \in (0, \min(\rho_1, \rho_2)) = (0, \rho_2)$, then the operator A carries out contracting mapping the ball $B(g_0, \rho)$ into itself and according to the Banach theorem in this ball it has a unique fixed point, i.e., there exists a unique solution of operator equation (16). Hence, Theorem 1 is proved.

4 Inverse problem 2

In Section 1, **Inverse problem 2** was reduced to the problem of determining the kernel $k(t), \quad t \in (0, T)$ from equations (6)–(8) and (10). To obtain the integral equation for $k(t)$ in this case we use equation (11) for solution direct problem and additional condition (10). As a result, we have

$$\begin{aligned} k(t) = {} & \frac{1}{\varphi(x_0)}(f''(t) - \psi_t'(x_0, t)) - \frac{1}{\varphi(x_0)}\int_0^l G(x_0, \xi, 0)\int_0^t k(\alpha)\vartheta(\xi, t-\alpha)\mathrm{d}\alpha\mathrm{d}\xi \\ & - \frac{1}{\varphi(x_0)}\int_0^t\int_0^l G_t(x_0, \xi, t-\tau)\int_0^\tau k(\alpha)\vartheta(\xi, \tau-\alpha)\mathrm{d}\alpha\mathrm{d}\xi\mathrm{d}\tau. \end{aligned} \tag{27}$$

We represent the system of equations (11) and (27) in the form of operator equation

$$Ag = g, \tag{28}$$

where $g = (g_1, g_2) = (\vartheta(x, t), k(t))$ is the vector function, and $A = (A_1, A_2)$ are defined by the right parts of equations (11) and (27).

Theorem 2. *Suppose $f(t) \in C^2[0, T]$, $\varphi(x_0) \neq 0$ and the conditions of Lemma 1 are fulfilled. Then the operator equation (28) has a unique solution in domain D_{lT} for arbitrary fixed $l > 0$ and $T > 0$.*

Proof. We introduce the vector function by formula

$$g_0(x, t) = (g_{01}, g_{02})(x, t) = \left(\psi(x, t), \frac{1}{\varphi(x_0)}(f''(t) - \psi_t'(x_0, t))\right).$$

Then, in accordance with equalities (11) and (27), the components of operator A will have the form:

$$Ag_1 = \psi(x, t) + \int_0^t\int_0^l G(x, \xi, t-\tau)\left[g_2(\tau)\varphi(\xi) + \int_0^\tau g_2(\alpha)g_1(\xi, \tau-\alpha)\mathrm{d}\alpha\right]\mathrm{d}\xi\mathrm{d}\tau,$$

$$\begin{aligned} Ag_2 = {} & \frac{1}{\varphi(x_0)}(f''(t) - \psi_t'(x_0, t)) - \frac{1}{\varphi(x_0)}\int_0^l G(x_0, \xi, 0)\int_0^t g_2(\alpha)g_1(\xi, t-\alpha)\mathrm{d}\alpha\mathrm{d}\xi \\ & - \frac{1}{\varphi(x_0)}\int_0^t\int_0^l G_t(x_0, \xi, t-\tau)\int_0^\tau g_2(\alpha)g_1(\xi, \tau-\alpha)\mathrm{d}\alpha\mathrm{d}\xi\mathrm{d}\tau. \end{aligned} \tag{29}$$

The conditions of contractive mapping for operator A_1 were received in the previous section. Here, it is shown that A_2 has the property of a contraction mapping operator. Let $g(x, t) \in B(D_T)$. Then, it is easy to see that

$$\begin{aligned} \|A_2 g - g_{02}\|_\sigma \leq {} & \sup_{t\in(0,T)} \frac{1}{\varphi_{x_0}}\left|\int_0^l G(x_0, \xi, 0)\int_0^t g_2(\alpha)g_1(\xi, t-\alpha)e^{-\sigma t}\mathrm{d}\alpha\mathrm{d}\xi\right| \\ & + \frac{1}{\varphi(x_0)}\left|\int_0^t\int_0^l G_t(x_0, \xi, t-\tau)\int_0^\tau g_2(\alpha)g_1(\xi, \tau-\alpha)e^{-\sigma t}\mathrm{d}\alpha\mathrm{d}\xi\mathrm{d}\tau\right| = P_1 + P_2; \end{aligned}$$

$$P_1 = \sup_{t\in(0,T)} \frac{1}{\varphi(x_0)} \left| \int_0^l G(x_0, \xi, 0)(g_2 * g_1)(t)e^{-\sigma t} d\xi \right| \le \frac{1}{\varphi_0}\left(\rho^2 T + \frac{2}{\sigma}(\rho + \|g_0\|)\|g_0\| + \frac{1}{\sigma}\|g_0\|^2\right);$$

$$P_2 = \frac{1}{\varphi(x_0)} \left| \int_0^t \int_0^l G_t(x_0, \xi, t-\tau) \int_0^\tau g_2(\alpha) g_1(\xi, \tau - \alpha) e^{-\sigma t} d\alpha d\xi d\tau \right| \le \frac{2T(\rho + \|g_0\|)^2}{\sigma\varphi(x_0)};$$

$$\|A_2 g - g_{02}\|_\sigma \le \frac{1}{\varphi(x_0)}(\rho^2 T + \frac{2}{\sigma}(\rho + \|g_0\|)\|g_0\| + \frac{1}{\sigma}\|g_0\|^2) + \frac{2T(\rho + \|g_0\|)^2}{\sigma\varphi(x_0)};$$

$$\begin{cases} \rho < \dfrac{\varphi(x_0)}{3T} = \kappa_3, \\ \theta_1 = \dfrac{6T}{\varphi^2(x_0)}\left(\dfrac{\varphi(x_0)}{3T} + \|g_0\|\right)\left(\|g_0\| + T\left(\dfrac{\varphi(x_0)}{3T} + \|g_0\|\right)\right) < \sigma, \\ \theta_2 = \dfrac{9T\|g_0\|^2}{\varphi^2(x_0)} < \sigma. \end{cases} \tag{30}$$

It follows that if $\sigma > \max(\theta_1, \theta_2) = \beta_5$, $\rho < \kappa_3$, then $A_2 g \in B(g_0, \rho)$.
So, if the inequality $\sigma > \sigma_1 = \max(\beta_0, \beta_5)$ holds, then the operator A maps $A_2 g \in B(g_0, \rho)$ into itself.
Consider next the property of contraction mapping operator for A. Then we have

$$\|(Ag^1 - Ag^2)_1\|_\sigma \le \frac{1}{\sigma}(\varphi_1 + 2(\rho + \|g_0\|)T)\|g^1 - g^2\|_\sigma.$$

The second component Ag can be estimated in the analogous way:

$$\|(Ag^1 - Ag^2)_2\|_\sigma \le \left(\frac{2\rho T}{\varphi(x_0)} + \frac{2\|g_0\|}{\sigma\varphi(x_0)} + \frac{4T(\rho + \|g_0\|)}{\varphi(x_0)\sigma}\right)\|g^1 - g^2\|_\sigma.$$

If $T > 0$ satisfies condition (26),

$$\begin{cases} \rho < \dfrac{\varphi(x_0)}{6T} = \kappa_4, \\ \theta_3 = \dfrac{6\|g_0\|}{\varphi(x_0)} < \sigma, \\ \theta_4 = \dfrac{2(\varphi(x_0) + 6T\|g_0\|)}{\varphi(x_0)} < \sigma, \\ \theta_5 = 2\varphi_1 < \sigma, \\ \theta_6 = \dfrac{2}{3}\varphi(x_0) + 4\|g_0\|T < \sigma. \end{cases} \tag{31}$$

From these estimates it is clear that if σ, ρ are chosen from condition $\sigma > \sigma_2 = \max(\theta_3, \theta_4, \theta_5, \theta_6)$, $\rho < \kappa_4$, then the operator A in the set $B(g_0, \rho)$ is a contraction map. According to the principle of contracting operators, therefore, if σ, ρ are taken from condition $\sigma > \max(\sigma_1, \sigma_2)$, $\rho < \min(\kappa_3, \kappa_4)$, then the operator A in the set $B(g_0, \rho)$ has a unique fixed point. Hence, the theorem is proved. □

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