Uzbek Mathematical Journal
2024, Volume 68, Issue 1, pp.110-117
DOI: 10.29229/uzmj.2024-1-??

SOLVABILITY OF INVERSE PROBLEM FOR INTEGRO-DIFFERENTIAL HEAT EQUATION WITH PERIODIC AND INTEGRAL CONDITIONS

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Abstract. In this paper, we consider inverse problem of determining $u(x,t)$ and $k(t)$ functions in the one-dimensional integro-differential heat equation with the initial- periodic boundary and overdetermination conditions. The unique solvability of the direct problem are proved. To investigate the solvability of the inverse problem, we first consider an auxiliary inverse boundary value problem, which is equivalent to the original one. Existence and uniqueness of the solution of the equivalent problem is proved using a contraction mapping. Finally, using the equivalency, the existence and uniqueness of classical solution is obtained.

Keywords: integro-differential equation, initial-periodic boundary problem, inverse problem, integral equation.

MSC (2010): 35A01; 35A02; 35L02; 35L03; 35R03.

1. Introduction

Periodic boundary conditions (PBCs) are a set of boundary conditions which are often chosen for approximating a large (infinite) system by using a small part called a unit cell. PBCs are often used in computer simulations and mathematical models. The topology of two-dimensional PBC is equal to that of a world map of some video games; the geometry of the unit cell satisfies perfect two-dimensional tiling, and when an object passes through one side of the unit cell, it re-appears on the opposite side with the same velocity (see [21]).

The periodic boundary conditions arise from many important applications in heat transfer, life sciences [20, 1, 15, 16].

In the papers [20, 1, 15, 16], it was prove the existence, the uniqueness and the continuous dependence on the data of the solution and the numerical solution of diffusion problem with periodic boundary conditions.

The paper [17] investigated the inverse problem of finding a time-dependent diffusion coefficient in a parabolic equation with the periodic boundary and integral overdetermination conditions. Under some assumption on the data, the existence, uniqueness, and continuous dependence on the data of the solution were shown by using the generalized Fourier method.

The problem of determining the kernel $k(t)$ of the integral term in an integro-differential heat equation were studied in many publications [3]-[10], in which both one- and multidimensional inverse problems with classical initial, initial-boundary conditions were investigated. The existence and uniqueness theorems of inverse problem solutions were proved.

In the present work, one-dimensional integro-differential heat equation is used with periodic boundary condition for the determination of kernel. The existence and uniqueness of the classical solution of the problem (2.1)-(2.4) is reduced to fixed point principles by applying the Fourier method.

2. Formulation of problem

We consider the initial-periodic boundary problem for the heat equation with a convolution-type integral term on the right-hand side

$$u_t - u_{xx} = \int_0^t k(t-\tau)u(x,\tau)d\tau, \quad (x,t) \in D_T, \tag{2.1}$$

$$u(x,0) = \varphi(x), \tag{2.2}$$

$$u(0,t) = u(1,t), \quad u_x(0,t) = u_x(1,t), \varphi(0) = \varphi(1), \quad \varphi'(0) = \varphi'(1), \tag{2.3}$$

T is arbitrary positive number and $D_T := \{(x,t) : 0 < x < 1, 0 < t \leq T\})$.

The problem of determining a function $u(x,t),(x,t)\in D_T$, that satisfies (2.1)-(2.3) with known functions $k(t)$ and $\varphi(x)$ will be called the direct problem.

In the inverse problem, it is required to determine the kernel $k(t), t>0$, of the integral in (2.1) using overdetermination condition about the solution of the direct problem (2.1)-(2.3):

$$\int_0^1 \omega(x)u(x,t)dx = h(t), \qquad x\in(0,1), \tag{2.4}$$

where $\omega(x), h(t)$ are given functions. In heat propagation in a thin rod in which the law of variation $h(t)$ of the total quantity of heat in the rod is given in [11]. This integral condition in parabolic problems is also called heat moments which are analyzed in [12].

Definition 2.1. The pair $\{u(x,t),k(t)\}$ from the class $C^{2,1}(D_T)\cap C^{1,0}(\overline{D}_T)\times C[0,T]$ is said to be a classical solution of problem (2.1)-(2.4), if the functions $u(x,t)$ and $k(t)$ satisfy the following conditions:
(1) The function $u(x,t)$ and its derivatives $u_t(x,t), u_{xx}(x,t)$ are continuous in the domain D_T ;
(2) the function $k(t)$ is continuous on the interval $[0,T]$;
(3) equation (2.1) and conditions (2.2)-(2.4) are satisfied in the classical (usual) sense.

We introduce the notation

$$\vartheta(x,t) = u_t(x,t)$$

and obtain the following equivalent problem with respect to function $\vartheta(x,t)$:

$$\vartheta_t - \vartheta_{xx} = k(t)\varphi(x) + \int_0^t k(\tau)\vartheta(x,t-\tau)d\tau, \tag{2.5}$$

$$\vartheta(x,0) = \varphi''(x), \tag{2.6}$$

$$\vartheta(0,t) = \vartheta(1,t), \quad \vartheta_x(0,t) = \vartheta_x(1,t), \tag{2.7}$$

$$\int_0^1 \omega(x)\vartheta(x,t)dx = h'(t). \tag{2.8}$$

Let $C^m\Big(0;l\Big)$ be the class of m times continuously differentiable with all derivatives up to the $m-$th order (inclusive) in $(0;l)$ functions. In the case $m=0$ this space coincides with the class of continuous functions. $C^{m,k}(D_T)$ is the class of m times continuously differentiable with respect to t and k times continuously differentiable with respect to x with all derivatives in the domain D_T functions.

The functions φ, ω and h satisfy the following assumptions:
$(A1)$ $\varphi(x)\in C^4[0,1]$; $\varphi^{(5)}(x)\in L_2[0,1]$; $\varphi(0)=\varphi(1); \varphi'(0)=\varphi'(1); \varphi''(0)=\varphi''(1)$; $\varphi^{(3)}(0) = \varphi^{(3)}(1); \varphi^{(4)}(0)=\varphi^{(4)}(1)$;
$(A2)$ $h(t)\in C^2[0,T]$; $h(0)\neq 0$;
$(A3)$ $\omega(x)\in C^2[0,1]$; $\int_0^1\omega(x)\varphi'(x)dx = h'(0)$; $\int_0^1 \omega(x)\varphi(x)dx = h(0)\neq 0$.

3. Direct problem

As we shall use separation of variables methods, let us denote by λ_n its eigenvalues and eigenfunctions by $X_n(x)$, i.e

$$X_n''(x) + \lambda^2 X_n(x) = 0, \ x\in(0,1),$$
$$X_n(0) = X_n(1), \quad X_n'(0) = X_n'(1), n=0,1,2,....$$

In [2] , it is known that the system

$$1, \cos\lambda_1 x, \sin\lambda_1 x, \cos\lambda_2 x, \sin\lambda_2 x, ..., \cos\lambda_n x, \sin\lambda_n x, ... \tag{3.1}$$

where $\lambda_n = 2\pi n(n=0,1,...)$, is a basis for $L_2(0,1)$.

Since the system (3.1) form a basis in $L_2(0,1)$, we shall seek the $\vartheta(x,t)$ of classical solution of the problem (2.5)-(2.7) in the form

$$\vartheta(x,t) = \sum_{n=0}^{\infty}\vartheta_{1n}(t)\cos\lambda_n x + \sum_{n=1}^{\infty}\vartheta_{2n}(t)\sin\lambda_n x, \lambda = 2\pi n, \tag{3.2}$$

where

$$\vartheta_{10}(t) = \int_0^1 \vartheta(x,t)dx,$$

$$\vartheta_{1n}(t) = 2\int_0^1 \vartheta(x,t)\cos\lambda_n x dx,$$

$$\vartheta_{2n}(t) = 2\int_0^1 \vartheta(x,t)\sin\lambda_n x dx.$$

Then applying the formal scheme of the Fourier method, for determining of unknown coefficients $\vartheta_{10}(t)$ and $\vartheta_{in}(t)(i := 1,2; n = 1,2,...)$ of function $\vartheta(x,t)$ from (2.5) and (2.6) we have

$$\vartheta_{10}'(t) = k(t)\varphi_{10} + \int_0^t k(\tau)\vartheta_{10}(x,t-\tau)d\tau, \tag{3.3}$$

$$\vartheta_{10}(t)|_{t=0} = 0, \tag{3.4}$$

$$\vartheta_{in}'(t) + \lambda_n^2\vartheta_{in}(t) = k(t)\varphi_{in} + \int_0^t k(\tau)\vartheta_{in}(x,t-\tau)d\tau, \tag{3.5}$$

$$\vartheta_{in}(t)|_{t=0} = -\lambda_n^2\varphi_{in}, \quad i = 1,2, \quad n = 1,2,..., \tag{3.6}$$

where

$$\varphi_{10} = \int_0^1 \varphi(x)dx,$$

$$\varphi_{1n} = 2\int_0^1 \varphi(x)\cos\lambda_n x dx,$$

$$\varphi_{2n} = 2\int_0^1 \varphi(x)\sin\lambda_n x dx.$$

The solutions of problems (3.3)-(3.4) and (3.5)-(3.6) satisfy the following integral equations

$$\vartheta_{10}(t) = \varphi_{10}\int_0^t k(\tau)d\tau + \int_0^t\int_0^\tau k(\alpha)\vartheta_{10}(\tau-\alpha)d\alpha d\tau \tag{3.7}$$

and

$$\vartheta_{in}(t) = -\lambda_k^2\varphi_{in}e^{-\lambda_n^2 t} + \varphi_{in}\int_0^t e^{-\lambda_n^2(t-\tau)}k(\tau)d\tau +$$

$$+\int_0^t e^{-\lambda_n^2(t-\tau)}\int_0^\tau k(\alpha)\vartheta_{in}(\tau-\alpha)d\alpha d\tau, \quad (i = 1,2, \quad n = 1,2,...) \tag{3.8}$$

Estimating the functions $\vartheta_{10}(t), \vartheta_{in}(t)$ we obtain the following integral inequality

$$|\vartheta_{10}(t)| \le t|\varphi_{10}|\|k\| + t\|k\|\int_0^t |\vartheta_{10}(t-\tau)|d\tau, \tag{3.9}$$

$$|\vartheta_{in}(t)| \le \lambda_n^2|\varphi_{in}| + \frac{|\varphi_{in}|}{\lambda_n^2}\|k\| + \|k\|t\int_0^t |\vartheta_{in}(t-\tau)|d\tau, \quad (i = 1,2, \quad n = 1,2,...) \tag{3.10}$$

where $\|k\| = \max_{t\in[0,T]}|k(t)|$. Applying Gronwall's lemma, we obtain the following estimate

$$|\vartheta_{10}(t)| \le t|\varphi_{10}|\|k\|e^{t^2\|k\|},$$

$$|\vartheta_{in}(t)| \le \left(\lambda_k^2|\varphi_{in}| + \frac{|\varphi_{in}|}{\lambda_n^2}\|k\|\right)e^{\|k\|t^2}.$$

Using equalities (3.3) and (3.5), we obtain estimates for $\vartheta_{10}'(t), \vartheta_{in}'(t)$:

$$|\vartheta_{10}'(t)| \le |\varphi_{10}|\|k\| + t^2|\varphi_{10}|\|k\|^2e^{t^2\|k\|},$$

$$|\vartheta'_{in}(t)| \le (\lambda_n^2 + t\|k\|)\left(\lambda_n^2|\varphi_{in}| + \frac{|\varphi_{in}|}{\lambda_n^2}\|k\|\right)e^{\|k\|t^2} + |\varphi_{in}|\|k\|.$$

Thus we have proved the following lemma:

Lemma 3.1. *For any $t \in [0;T]$ and for sufficiently large n, the estimates are valid*

$$|\vartheta_{10}(t)| \le C_1|\varphi_{10}|, \quad |\vartheta'_{10}(t)| \le C_2|\varphi_{10}|$$

$$|\vartheta_{in}(t)| \le C_3\lambda_n^2|\varphi_{ik}|, \quad |\vartheta'_{in}(t)| \le C_4\lambda_n^4|\varphi_{in}|.$$

here C_i are positive constants.

Formally, from (3.2) by term-by-term differentiation we compose the series

$$\vartheta_t(x,t) = \sum_{n=0}^{\infty}\vartheta'_{1n}(t)\cos\lambda_n x + \sum_{n=1}^{\infty}\vartheta'_{2n}(t)\sin\lambda_n x, \tag{3.11}$$

$$\vartheta_{xx}(x,t) = \sum_{n=0}^{\infty}\lambda_n^2\vartheta_{1n}(t)\cos\lambda_n x + \sum_{n=1}^{\infty}\lambda_n^2\vartheta_{2n}(t)\sin\lambda_n x, \tag{3.12}$$

In view of Lemma (3.1), the series (3.2), (3.11), and (3.12) for any $(x,t) \in D_T$

$$C_4\sum_{n=1}^{\infty}\lambda_n^4|\varphi_{in}|.$$

We hold the following auxiliary lemma.

Lemma 3.2. *If the conditions (A1) then there is equality*

$$\varphi_{in} = \frac{1}{\lambda_n^5}\varphi_{in}^{(5)}, \quad (i = 1,2) \tag{3.13}$$

where

$$\varphi_{1n}^{(5)} = 2\int_0^1 \varphi^{(5)}(x)\cos\lambda_n x dx, \quad \varphi_{2n}^{(5)} = 2\int_0^1 \varphi^{(5)}(x)\sin\lambda_n x dx$$

with the following estimate:

$$\sum_{n=1}^{\infty}|\varphi_{in}^{(5)}|^2 \le \|\varphi^{(5)}\|_{L_2(0,1)}, \quad (i = 1,2). \tag{3.14}$$

If the functions $\varphi(x)$ satisfy the conditions of Lemma 3.2, then due to representations (3.13) and (3.14) series (3.2), (3.9) and (3.10) converge uniformly in the rectangle D_T, therefore, function $\vartheta(x,t)$ satisfies relations (2.5)-(2.7).

Using the above results, we obtain the following assertion.

Lemma 3.3. *Let $k(t) \in C[0,T], (A1)$ are satisfied, then there exists a unique solution of the direct problem* (2.5)-(2.7) $\vartheta(x,t) \in C^{2,1}(D_T) \cap C^{1,0}(\overline{D}_T)$.

4. Solvability of inverse problem

In this section it is studied the inverse problem as the problem of determining of functions $\vartheta(x,t), k(t)$ from relations (2.5)-(2.8).

Let us multiply (2.5) by $\omega(x)$ and integrate over x from 0 to l. Taking into account conditions (2.6)-(2.7) and using (3.2), we obtain the relation

$$k(t) = \frac{1}{\Theta}\Big(h''(t) - \int_0^t k(\tau)h'(t-\tau)d\tau - \sum_{n=0}^{\infty}\vartheta_{1n}(t)\int_0^1\omega''(x)\cos\lambda_n x dx +$$

$$+\sum_{n=1}^{\infty}\vartheta_{2n}(t)\int_0^1\omega''(x)\sin\lambda_n x dx\Big), \tag{4.1}$$

where $\Theta = \int_0^1 \omega(x)\varphi(x)dx$.

The following lemma plays an important role in studying the uniqueness of the solution to problem (2.5)-(2.8):

Lemma 4.1. *If $\vartheta(x,t), k(t)$ is a solution of (2.5)-(2.8), then the functions*

$$\vartheta_{10}(t) = \int_0^1 \vartheta(x,t)dx,$$

$$\vartheta_{1n}(t) = 2\int_0^1 \vartheta(x,t)\cos\lambda_n x dx,$$

$$\vartheta_{2n}(t) = 2\int_0^1 \vartheta(x,t)\sin\lambda_n x dx$$

satisfy system (3.8), (3.9) on the interval $[0,T]$.

Now, consider the space $B^2_{2,T}$ consisting of functions of the form $\vartheta(x,t)$ in domain D_T, where the functions $\vartheta_{1n}(t)(n=0,1,2,..), \vartheta_{2n}(t)(n=1,2,...)$ are continuous on $[0,T]$ and satisfy the condition

$$\|\vartheta_{10}(t)\|_{C[0,T]} + \Big(\sum_{n=1}^{\infty}(\lambda_n^2\|\vartheta_{1n}(t)\|_{C[0,T]})^2\Big)^{1/2} + \Big(\sum_{n=1}^{\infty}(\lambda_n^2\|\vartheta_{2n}(t)\|_{C[0,T]})^2\Big)^{1/2} < +\infty.$$

The norm in the space $B^2_{2,T}$ is

$$\|\vartheta(x,t)\|_{B^2_{2,T}} = \|\vartheta_{10}(t)\|_{C[0,T]} + \Big(\sum_{n=1}^{\infty}(\lambda_n^2\|\vartheta_{1n}(t)\|_{C[0,T]})^2\Big)^{1/2} + \Big(\sum_{n=1}^{\infty}(\lambda_n^2\|\vartheta_{2n}(t)\|_{C[0,T]})^2\Big)^{1/2}.$$

We denote by E^2_T, the Banach space $B^2_{2,T}\times C[0,T]$ of vector functions $z(x,t) = \{\vartheta(x,t), k(t)\}$ with norm

$$\|z(x,t)\|_{B^2_{2,T}} = \|\vartheta(x,t)\|_{B^2_{2,T}} + \|k(t)\|_{C[0,T]}.$$

Now we consider the operator

$$\Lambda(\vartheta,k) = \{\Lambda_1(\vartheta,k), \Lambda_2(\vartheta,k)\}$$

in the space E^2_T, where

$$\Lambda_1(\vartheta,k) = \widetilde{\vartheta}(x,t) \equiv \sum_{n=0}^{\infty}\widetilde{\vartheta}_{1n}(t)\cos\lambda_n x + \sum_{n=1}^{\infty}\widetilde{\vartheta}_{2n}(t)\sin\lambda_n x, \quad \Lambda_2(\vartheta,k) = \widetilde{k}(t)$$

and the functions $\widetilde{\vartheta}_{10}(t), \widetilde{\vartheta}_{in}(t)(i=1,2; n=1,2,...), \widetilde{k}(t)$ are equal to the right-hand sides of (3.7), (3.8) and (4.1) respectively.

Using simple transformations from (3.7), (3.8) and (4.1) we obtain following estimates

$$\begin{aligned}\|\widetilde{\vartheta}_{10}(t)\|_{C[0,T]} &\le |\varphi_{10}|\|k\|_{C[0,T]}T + \|k\|_{C[0,T]}\|\widetilde{\vartheta}_{10}(t)\|_{C[0,T]}T^2 \le \\ &\le \|\varphi\|_{L_2[0,1]}\|k\|_{C[0,T]}T + \|k\|_{C[0,T]}\|\widetilde{\vartheta}(x,t)\|_{B^2_{2,T}}T^2,\end{aligned} \tag{4.2}$$

$$\Big(\sum_{n=1}^{\infty}(\lambda_n^2\|\vartheta_{in}(t)\|_{C[0,T]})^2\Big)^{1/2} \le \Big(\sum_{n=1}^{\infty}(\lambda_n^4|\varphi_{in}| + \lambda_n^2\|k\|_{C[0,T]}|\vartheta_{in}|T + \lambda_n^2\|k\|_{C[0,T]}|\varphi_{in}|T^2)^2\Big)^{1/2} \le$$

$$\le \sqrt{3}\Big(\sum_{n=1}^{\infty}(\lambda_n^4|\varphi_{in}|)^2\Big)^{1/2} + \sqrt{3}\Big(\sum_{n=1}^{\infty}(\lambda_n^2\|k\|_{C[0,T]}|\vartheta_{in}|T)^2\Big)^{1/2} + \sqrt{3}\Big(\sum_{n=1}^{\infty}(\lambda_n^2\|k\|_{C[0,T]}|\varphi_{in}|T^2)^2\Big)^{1/2} \le$$

$$\le \sqrt{3}\|\varphi^{(4)}\|_{L_2[0,1]} + \sqrt{3}\|k\|_{C[0,T]}\|\varphi^{(2)}\|_{L_2[0,1]}T + \sqrt{3}\|k\|_{C[0,T]}\|\vartheta\|_{B^2_{2,T}}T^2 \tag{4.3}$$

$$\|k\|_{C[0,T]} \le \frac{h_1}{\Theta} + \frac{h_1}{\Theta} T\|k\|_{C[0,T]} + \frac{3\omega_0}{\Theta}\|\vartheta\|_{B^2_{2,T}}, \tag{4.4}$$

where $h_1 := \max_{t\in C^2[0,T]} |h(t)|$, $\omega_0 := \max_{x\in C^2[0,1]} |\omega(x)|$.

Then from (4.2)-(4.4) we find that

$$\|\vartheta\|_{B^2_{2,T}} \le A_1(T) + B_1(T)\|k\|_{C[0,T]} + C_1(T)\|\vartheta\|_{B^2_{2,T}} + D_1(T)\|k\|_{C[0,T]}\|\vartheta\|_{B^2_{2,T}}, \tag{4.5}$$

$$\|k\|_{C[0,T]} \le A_2(T) + B_2(T)\|k\|_{C[0,T]} + C_2(T)\|\vartheta\|_{B^2_{2,T}} + D_2(T)\|k\|_{C[0,T]}\|\vartheta\|_{B^2_{2,T}}, \tag{4.6}$$

where

$$A_1(T) = \sqrt{3}\|\varphi^{(4)}\|_{L_2[0,1]}, \quad B_1(T) = \|\varphi\|_{L_2[0,1]}T + \sqrt{3}\|\varphi^{(2)}\|_{L_2[0,1]}T,$$
$$C_1(T) = 0, \quad D_1(T) = (1+\sqrt{3})T^2,$$
$$A_2(T) = \frac{h_1}{\Theta}, \quad B_2(T) = \frac{h_1}{\Theta}T, \quad C_2(T) = \frac{3\omega_0}{\Theta}, \quad D_2(T) = 0.$$

From (4.5)-(4.6) we conclude that

$$\|\vartheta\|_{B^2_{2,T}} + \|k\|_{C[0,T]} \le A(T) + B(T)\|k\|_{C[0,T]} + C(T)\|\vartheta\|_{B^2_{2,T}} + D(T)\|k\|_{C[0,T]}\|\vartheta\|_{B^2_{2,T}}, \tag{4.7}$$

where

$$A(T) = A_1(T) + A_2(T), \quad B(T) = B_1(T) + B_2(T),$$
$$C(T) = C_1(T) + C_2(T), D(T) = D_1(T) + D_2(T).$$

Theorem 4.2. *If conditions (A1)-(A3) and condition*

$$(B(T) + C(T) + D(T)(A(T)+2))\,(A(T)+2) < 1 \tag{4.8}$$

hold, then problem (2.5)-(2.8) *has a unique solution in the ball* $S = S_R(\|z\|_{E^3_T} \le R \le (A(T)+2))$ *of the space* E^2_T.

Proof. In the space E^2_T consider the equation

$$z = \Phi z, \tag{4.9}$$

where $z = \{\vartheta, k\}$, the components $\Phi_i(\vartheta, k)(i = 1, 2)$, of operator $\Phi(\vartheta, k)$, defined by right side of equations (3.2) and (4.1).

Consider the operator $\Phi_i(\vartheta, k)(i = 1, 2)$, in the ball $S = S_R$ from E^2_T. Analogically to (4.7), we get that for any $z_1, z_2 \in S_R$ the following estimates are valid:

$$\|\Phi z\|_{E^2_T} \le \|\vartheta\|_{B^2_{2,T}} + \|k\|_{C[0,T]} \le$$
$$A(T) + B(T)\|k\|_{C[0,T]} + C(T)\|\vartheta\|_{B^2_{2,T}} + D(T)\|k\|_{C[0,T]}\|\vartheta\|_{B^2_{2,T}} \le$$
$$\le A(T) + (B(T) + C(T) + D(T)(A(T)+2))\,(A(T)+2) < A(T) + 2. \tag{4.10}$$

$$\|\Phi z_1 - \Phi z_2\|_{E^2_T} \le$$
$$\le B(T)\|k_1 - k_2\|_{C[0,T]} + C(T)\|\vartheta_1 - \vartheta_2\|_{B^2_{2,T}} + D(T)R\left(\|k_1 - k_2\|_{C[0,T]} + \|\vartheta_1 - \vartheta_2\|_{B^2_{2,T}}\right). \tag{4.11}$$

Then taking into account (4.8) in (4.10) and (4.11), it follows that the operator Φ acts in the ball $S = S_R$ and satisfy the conditions of the contraction mapping principle. Therefore, in the ball $S = S_R$ the operator Φ has a unique fixed point $\{\vartheta, k\}$ that is a unique solution of equation (4.9).

In this way we conclude that the function $\vartheta(x,t)$ as an element of space $B^2_{2,T}$ is continuous and has continuous derivatives $\vartheta(x,t)$ and $\vartheta_{xx}(x,t)$ in D_T.

From (3.5) it is easy to see that

$$\left(\sum_{n=1}^{\infty}(\lambda_n\|\vartheta'_{in}(t)\|_{C[0,T]})^2\right)^{1/2} \le \sqrt{2}(1 + \|k\|_{C[0,T]}T)\sum_{n=1}^{\infty}(\lambda_n^3\|\vartheta\|_{C[0,T]}) = \sqrt{2}(1 + \|k\|_{C[0,T]}T)\|\vartheta\|_{B^2_T}.$$

Thus $\vartheta(x,t)$ is continuous in the region D_T.

Further, it is possible to verify that equation (2.5) and conditions (2.6)-(2.8) are satisfied in the usual sense. Consequently, $\{\vartheta(x,t), k(t)\}$ is a solution of (2.5)-(2.8) by Lemma 3.2 it is unique. $\square$

Remark 4.3. Let the assumptions (A1) - (A3) are satisfied. Then we will derive from (2.5)-(2.8) the equations (2.1)-(2.4). By denoting $u_t(x,t) = \vartheta(x,t)$, we obtain

$$u(x,t) = \varphi(x) + \int_0^t \vartheta(x,t)dt.$$

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