O‘ZBEKISTON RESPUBLIKASI

OLIY TA’LIM, FAN VA INNOVATSIYALAR VAZIRLIGI

BUXORO DAVLAT UNIVERSITETI

J.J. Jumayev

KASR TARTIBLI DIFFERENSIAL TENGLAMALAR

Uslubiy qo‘llanma

“Durdona” nashriyoti

Buxoro – 2024

Ushbu uslubiy qo‘llanma Oliy ta’lim muassasalarining 70540101 - Matematika(yo’nalishlar bo’yicha) mutaxassisligida tahsil olayotgan magistrantlar uchun mo‘ljallab yozilgan. Qo‘llanmada asosan maxsus funksiyalar, integral almashtirishlar, Grunwald-Letnikov, Riman-Liuvill, Caputo kasr hosilasi kabi tushunchalar batafsil yoritilgan. Barcha mavzularda nazariy ma’lumotlar, namunaviy misol va masalalar yechimlari va talabalar mustaqil bajarishlari uchun mo‘ljallangan topshiriqlar keltirilgan.

Taqrizchilar:

Teshayev Muxsin Xudoyberdiyevich, Buxoro muhandislik- texnologiya instituti Oliy matematika kafedrasi professori, (DSc).

Rasulov Haydar Raupovich, Buxoro davlat universiteti Matematik analiz kafedrasi dotsenti, (PhD).

KIRISH

Kasr tartibli differensial tenglamalar nazariyasi matematikaning rivojlanayotgan va dolzarb sohalaridan biridir. Butun tartibli differensial tenglamalardan farqli o'laroq, kasr tartibli differensial tenglamalar (KTDT) hosilalarining tartibi butun son bo'lmagan, ya'ni kasr yoki irratsional son bo'lishi mumkin. Bu esa KTDTni qo'llash imkoniyatlarini sezilarli darajada kengaytiradi.

Kasr tartibli differensial tenglamalar dastlab nazariy tahlil uchun yaratilgan bo'lsa-da, hozirgi kunda ular fizikadan tortib biologiyaga, iqtisodiyotdan tortib muhandislikka qadar ko'plab sohalarda qo'llanilmoqda. Quyida KTD tenglamalarining ba'zi amaliy qo'llanilishlariga misollar keltiriladi:

1. Materiallarning Reologiyasi. Kasr tartibli differensial tenglamalar materiallarning elastiklik va yopishqoqlik xususiyatlarini aniqroq modellash uchun qo'llaniladi. Masalan, polimerlarning reologik xatti-harakatlarini o'rganishda KTD tenglamalari yordamida ularning vaqtga bog'liq deformatsiya xususiyatlarini ifodalash mumkin.

2. Difuziya Jarayonlari. KTD tenglamalari diffuziya jarayonlarini, ayniqsa anomal diffuziya hodisalarini, model qilishda ishlatiladi. Masalan, geofizikada neft va gaz rezervuarlarida moddaning anomal diffuziyasini ifodalash uchun KTD tenglamalari qo'llanilishi mumkin.

3. Moliyaviy Bozorlar. Moliyaviy bozorlar o'zgaruvchanligini model qilishda KTD tenglamalari qo'llaniladi. Ular bozorlardagi uzoq muddatli xotira va noaniqliklarni aniqlashda yordam beradi. Masalan, moliyaviy instrumentlarning baholari va ularning o'zgarish tezligini tahlil qilish uchun KTD tenglamalari qo'llanilishi mumkin.

4. Biologik Tizimlar. KTD tenglamalari biologik tizimlarning dinamikasini model qilishda qo'llaniladi. Masalan, hujayra ichidagi signalizatsiya yo'llarini va ularning vaqtga bog'liq o'zgarishlarini ifodalash uchun ishlatilishi mumkin.

5. Elektromagnit Tizimlar. Elektronika va elektrotexnika sohalarida KTD tenglamalari elektr zanjirlarining xatti-harakatlarini modellashda ishlatiladi. Masalan, zanjirning o'tkazuvchanligini va induktiv xususiyatlarini aniqroq ifodalash uchun KTD tenglamalari ishlatiladi.

6. Neytral Tarmoqlar. Sun'iy neyron tarmoqlarda va chuqur o'rganishda KTD tenglamalari neyronlarning faollashuv funksiyalarini va o'rganish jarayonlarini aniqroq ifodalashda qo'llaniladi. Bu usullar, ayniqsa, uzun muddatli xotiraga ega bo'lgan modellarda qo'llaniladi.

Ushbu misollar kasr tartibli differensial tenglamalarning turli sohalarda qanday qo'llanilishi haqida tasavvur beradi. Ular murakkab tizimlarni yanada realistik va nozik ifodalash imkonini beradi.

Ushbu o'quv qo'llanma KTDT sohasida chuqur bilim olishni istagan talabalar, tadqiqotchilar va mutaxassislar uchun mo'ljallangan. Qo'llanma nazariy asoslardan boshlab, KTDTni yechishning turli usullarini va ularning amaliy qo'llanilishiga oid misollarni o'z ichiga oladi.

Qo'llanma quyidagi asosiy bo'limlardan iborat:

Maxsus funksiyalar: Bu bo'limda Eyler integrallari, Gipergeometrik funksiyalar, Mittag-Lefler funksiyalari, Foks funksiyasining asosiy tushunchalari, ularning kelib chiqishi va matematik asosi tushuntiriladi.

Integral almashtirishlar: Bu bo'limda KTDT ni yechishning analitik usullari, jumladan, Laplas almashtirish, Furye almashtirishi, Mellin almshtirishi va boshqa usullar bayon etiladi.

Kasr tartibli hosilalar va integral operatorlar: KTDT ning asosiy tarkibiy qismi bo'lgan kasr tartibli hosilalar va integral operatorlarning ta'riflari, xossalari va turlari keltiriladi.

Kasr tartibli differensial tenglamalar: Kasr tartibli differensial tenglamalar va ularni yechish usullari haqida batafsil ma'lumot beriladi.

Bu qo'llanma kasr tartibli differensial tenglamalar nazariyasini o'rganishda va amaliy masalalarni yechishda sizga yordam beradi deb umid qilamiz. Qo'llanmaning muvaffaqiyatli bo'lishida yordam bergan barcha hamkasblarimizga minnatdorlik bildiramiz. Sizning muvaffaqiyatingiz biz uchun eng katta mukofotdir.

I-BOB. MAXSUS FUNKSIYALAR.

Klassik maxsus funksiyalarni o'rganish matematik tahlilning bir qismi bo'lib, uning barchasi gipergeometrik funksiyaga qaratilgan bo'lib, ularning differensial tenglamalari mos ravishda uchta va ikkita yagona nuqtalarni, shuningdek, maxsus holatlarni taqdim etadi.

Bundan tashqari, bu maxsus funksiyalar differensial tenglamalarning yechimlari ifodalash uchun juda muhim bo'lganligi sababli, biz klassik gamma funksiyalarni, faktorial va beta tushunchasini umumlashtirishni ikkita parametrga qarab, ushbu funksiyalar sinfining bir qismi deb hisoblaymiz.

Biz dastlab Eyler integrallarini taqdim etishdan boshlaymiz, ya'ni gamma funksiya tushunchalari, Pochhammer belgisi bilan tanishamiz, xatolik funksiyasi, qo'shimcha xato funksiyasi va beta funksiyasi. Bundan tashqari, biz gipergeometrik funksiyani, qo'shilgan gipergeometrik funksiyani taqdim etamiz va biz faqat adekvat parametr tanlash sifatida, alohida holatlar sifatida, boshqa maxsus funksiyalarni eslatib o'tamiz.

Beta funksiya va uning xossalari.

Ushbu

$$\int_0^1 x^{a-1}(1-x)^{b-1}dx \qquad (1)$$

Xosmas integral berilgan bo'lsin.

1-ta'rif. (1) integral beta funksiya yoki birinchi tur Eylar integrali deyiladi va B(a,b) kabi belgilanadi, demak

$$B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx, \quad (a>0, b>0)$$

Shunday qilib , $B(a,b)$ funksiya R^2 fazodagi $M=\{(a,b)\in R^2 : a\in (0;+\infty), b\in(0;+\infty)\}$ to'plamda berilgandir.

Endi $B(a,b)$ funksiyaning xossalarini ko'rib chiqamiz.

1^0. $B(a,b)$ integralni olamiz.

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx \qquad (2)$$

Bu integral a va b ga nisbatan simmetrik funksiyalardan iborat, ya'ni

$$B(a,b) = B(b,a) \qquad (3)$$

Isbot. Haqiqatda,

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$$

integralda $x = 1-t$ almashtirish bajarilsa, u holda quyidagiga ega bo'lishihi topamiz.

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx = -\int_1^0 (1-t)^{a-1}t^{b-1}dt$$

$$= \int_0^1 t^{b-1}(1-t)^{a-1}dt = B(b,a)$$

Chunki faqat a va b ning rollari almashadi

2^0 . (1) integral

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$$

ixtiyoriy $M = \{(a,b)\in R^2 : \ a \in (0;\, +\infty), b \in (0;\, +\infty)\}\ (a_0 > 0, b_0 > 0)$ to'plamda tekis yaqinlashuvchi bo'ladi.

Isbot. Berilgan integralni tekis yaqinlashuvchilikka tekshirish uchun uni quyidagicha

$$\int_0^1 x^{a-1}(1-x)^{b-1}dx = \int_0^{1/2} x^{a-1}(1-x)^{b-1}dx + \int_{1/2}^1 x^{a-1}(1-x)^{b-1}dx$$

yozib olamiz.

Ma'lumki, $a > 0$ bo'lganda $\int_0^{1/2} x^{a-1}\,dx$ integral yaqinlashuvchi, $b > 0$ bo'lganda $\int_{1/2}^{1}(1-x)^{b-1}dx$

integral yaqinlashuvchi bo'ladi.

Parametr a ning $a \geq a_0 (a_0 > 0)$ qiymatlari va ixtiyoriy $b > 0$, ixtiyoriy $x \in (0, \frac{1}{2})$ uchun

$$x^{a-1}(1-x)^{b-1} \leq x^{a_0-1}(1-x)^{b-1} \leq 2x^{a_0-1}$$

bo'ladi. Veyershtrass alomatidan foydalanib,

$$\int_0^{1/2} x^{a-1}(1-x)^{b-1}dx$$

integralning tekis yaqinlashuvchiligini topamiz.

Shuningdek paramtr b ning $b \geq b_0 (b_0 > 0)$ qiymatlari va ixtiyoriy $a > 0$, ixtiyoriy $x \in [\frac{1}{2}, 1)$ uchun,

$$x^{a-1}(1-x)^{b-1} \leq x^{a-1}(1-x)^{b_0-1} \leq 2(1-x)^{b_0-1}$$

bo'ladi va yana Veyershtrass alomatiga ko'ra,

$$\int_{\frac{1}{2}}^{1} x^{a-1}\,(1-x)^{b-1}dx$$

Integralning tekis yaqinlashuvchiligi kelib chiqadi.

Demak, $\int_0^1 x^{a-1}\,(1-x)^{b-1}dx$ integral $a \geq a_0 > 0, va\ \ b \geq b_0 > 0$ bo'lganda, ya'ni

$$M_0 = \{(a,b) \in R^2 : a \in [a_0, +\infty), b \in [b_0, +\infty)\}$$

to'plamda tekis yaqinlashuvchi bo'ladi.

3^0. B (a, b) funksiya $\mathrm{M} = \{(\mathrm{a}, \mathrm{b}) \epsilon R^2 : a \epsilon (0; +\infty), b \epsilon [0; +\infty)\}$ to'plamda uzluksiz funksiyadir.

Isbot: Haqiqatdan ham,

$$B(a, b) = \int_0^1 x^{a-1} (1-x)^{b-1} dx$$

Integralning M_0 to'plamda tekis yaqinlashuvchi bo'lishidan va integral ostidagi funksiyaning $\forall (a, b) \in M$ da uzluksizligidan quyidagi teoremaga asosan $B(a, b)$ funksiya $\mathrm{M} = \{(\mathrm{a}, \mathrm{b}) \epsilon R^2 : a \epsilon (0; +\infty), b \epsilon [0; +\infty)\}$ to'plamda uzluksiz bo'ladi.

1-teorema. $f(x, y)$ funksiya M_0 to'plamda uzluksiz va

$$I_1(y) = \int_a^b f(x, y) dx$$

Integral $[c, d]$ da tekis yaqinlashuvchi bo'lsin. U holda $I_1(y)$ funksiya $[c, d]$ oraliqda uzluksiz bo'ladi.

4^0. $B(a, b)$ funksiya quyidagicha ham ifodalanadi

$$B(a, b) = \int_0^{+\infty} \frac{t^{a-1}}{1 + t^{a+b}} dt \qquad (4)$$

Isbot. (1) integralda $x = \frac{t}{1+t}$ almashtirish bajarilsa, u holda

$$B(a, b) = \int_0^1 x^{a-1} (1-x)^{b-1} dx = \int_0^{+\infty} \left(\frac{t}{1+t}\right)^{a-1} (1 - \frac{t}{1+t})^{b-1} \frac{dt}{(1+t)^2}$$

$$= \int_0^{+\infty} \frac{t^{a-1}}{(1+t)^{a+b}} dt$$

bo'ladi.

Xususan, $b = 1 - a$ $(0 < a < 1)$ bo'lganda

$$B(a,1-a)=\int_0^{+\infty}\frac{t^{a-1}}{1+t}dt=\frac{\pi}{sina\pi} \qquad (5)$$

Isbot. (5) integral chegaralanmagan funksiyaning chegarasi cheksiz

xosmas integral bolib, a parametrga bog'liqdir.

$$I(a)=\int_0^{+\infty}\frac{t^{a-1}}{1+t}dt$$

Bu integralni quyidagi ikki qismga ajratib

$$I(a)=\int_0^{+\infty}\frac{t^{a-1}}{1+t}dt=\int_0^{1}\frac{t^{a-1}}{1+t}dt+\int_1^{+\infty}\frac{t^{a-1}}{1+t}dt=I_1(a)+\ I_2(a)$$

Ularning har birini alohida – alohida yaqinlashuvchilikka tekshiramiz.

$0<t<1$ da quyidagi

$$\frac{1}{2}t^{a-1}\le\frac{t^{a-1}}{1+t}<t^{a-1}$$

tengsizlik o'rinli va $\int_0^1 t^{a-1}dt$ integral $a>0$ da yaqinlashuvchi, $a\le 0$ da uzoqlashuvchi,

$$I_1(a)=\int_0^{1}\frac{t^{a-1}}{1+t}dt$$

Integral $a>0$ da yaqinlashuvchi, $a\le 0$ da uzoqlashuvchi, bo'ladi. $t>1$ da quyidagi

$$\frac{1}{2}t^{a-2}\le\frac{t^{a-1}}{1+t}<t^{a-2}$$

tenglik o'rinli va $\int_1^{+\infty} t^{a-2}dt$ integral $a<1$ da yaqinlashuvchi, $a\ge 1$ da uzoqlashuvchi,

$$I_2(a)=\int_1^{+\infty}\frac{t^{a-1}}{1+t}dt$$

integral $a < 1$ da yaqinlashuvchi, $a \geq 1$ da uzoqlashuvchi bo'ladi. Shunday qilib, berilgan

$$I(a) = \int_0^{+\infty} \frac{t^{a-1}}{1+t} dt$$

integral $0 < a < 1$ da yaqinlashuvchi bo'lishini topamiz. Endi $I(a)$ integralni hisoblaymiz.

Ma'lumki $0 < t < 1$ da

$$\frac{t^{a-1}}{1+t} = \sum_{k=0}^{\infty} (-1)^k t^{a+k-1} \qquad (5')$$

bo'lib bu qator $[a_{0,}, b_0]$ $(0 < a_0 \leq t \leq b_0 \leq 1)$ da tekis yaqinlashuvchi bo'ladi.

$(5')$ darajali qatorning qismiy yig'indisi

$$S_n(t) = \sum_{k=0}^{n-1} (-1)^k t^{a+k-1} = \frac{t^{a-1}[1-(-t)^n]}{1+t}$$

bo'ladi. Agar $\forall\, n \in N$ va $\forall\, t \in (0,1)$ uchun,

$$\frac{t^{a-1}[1-(-t)^n]}{1+t} \leq t^{a-1}$$

tengsizlikning o'rinli bo'lishini hamda

$$\int_0^1 t^{a-1} dt \quad (0 < a < 1)$$

integralning yaqinlashuvchiligini e'tiborga olsak, unda Veyershtrass alomatiga ko'ra integral $\int_0^1 S_n(t)dt$ tekis yaqinlashuvchi bo'ladi, ikkinchi teoremaga ko'ra,

2- teorema. $f(x, y)$ funksiya

1. y o'zgaruvchining E dan olingan har bir tayin qiymatida x o'zgaruvchining funksiyasi sifatida $[a, b)$ da uzluksiz.

2. y→ y_0 ixtiyoriy $[a,t)$ $(a < t < b)$ oraliqda $\varphi(x)$ limiti

Agar

$$I_1(y) = \int_a^b f(x,y)dx$$

integral E to'plamda tekis yaqinlashuvchi bo'lsa, u holda $y \to y_0$ da $I_1(y)$ funksiya limitga ega va

$$\lim_{y\to y_0} I_1(y) = \lim_{y\to y_0} \int_a^b f(x,y)dy = \int_a^b \left[\lim_{y\to y_0} f(x,y)\right] dx = \int_a^b \varphi(x)dx$$

bo'ladi.

$$\lim_{n\to\infty} \int_0^1 S_n(t)dt = \int_0^1 [\lim_{n\to\infty} S_n(t)]dt$$

ya'ni

$$\lim_{n\to\infty} \int_0^1 \sum_{k=0}^{n-1} [(-1)^k t^{a+k-1}]dt = \int_0^1 [\lim_{n\to\infty} \sum_{k=0}^{n-1} (-1)^k t^{a+k-1}]dt = \int_0^1 \frac{t^{a-1}}{1+t} dt$$

bo'ladi.

Bu tenglikdan quyidagini topamiz.

$$I_1(a) = \int_0^1 \frac{t^{a-1}}{1+t} dt = \lim_{n\to\infty} \sum_{k=0}^{n-1} [(-1)^k t^{a+k-1}\, dt]$$

$$= \sum_{k=0}^{\infty} [\int_0^1 (-1)^k t^{a+k-1} dt] = \sum_{k=0}^{\infty} \frac{(-1)^k}{a+k}$$

Demak,

$$I_1(a) = \sum_{k=0}^{\infty} \frac{(-1)^k}{a+k}$$

Agar

$$I_2(a) = \int_1^{+\infty} \frac{t^{a-1}}{1+t} dt$$

Integral t=1\p almashtirish bajarsak, u holda

$$I_2(a) = \int_0^1 \frac{p^{-a}}{1+p}\, dp = \int_0^1 \frac{p^{(1-a)-1}}{1+p}\, dp$$

bo'ladi. Yuqoridagi yo'l bilan

$$I_2(a) = \sum_{k=0}^{\infty} \frac{(-1)^k}{a-k}$$

bo'lishini topamiz. Demak,

$$I(a) = I_1(a) + I_2(a) = \sum_{k=0}^{\infty} \frac{(-1)^k}{a+k} + \sum_{k=0}^{\infty} \frac{(-1)^k}{a-k} = \frac{1}{a} + \sum_{k=1}^{\infty} (-1)^k$$

$$= \frac{1}{a} + \sum_{k=1}^{\infty} (-1)^k \ (\frac{1}{a+k} + \frac{1}{a-k})$$

bo'ladi.

Agar

$$\frac{1}{a} + \sum_{k=1}^{\infty} (-1)^k \left(\frac{1}{a+k} + \frac{1}{a-k}\right) = \frac{\pi}{sina\pi} \ (0 < a < 1)$$

bo'lishini e'tiborga olsak, unda

$$I(a) = \frac{\pi}{sina\pi}$$

ekanligi kelib chiqadi. Demak,

$$\int_0^{+\infty} \frac{t^{a-1}}{1+t} dt = \frac{\pi}{sina\pi} \ (0 < a < 1) \ (5)$$

Bo'ladi. (5) munosabatdan quyidagini topamiz.

$$B(\frac{1}{2},\frac{1}{2}) = B(\frac{1}{2},1-\frac{1}{2}) = \frac{\pi}{sin\frac{\pi}{2}} = \pi$$

Beta funksiyaning xususiy holda uzluksiz hosilalarga ega, ya'ni

$$\mathrm{B(a, 1-a)} = \int_0^{+\infty} \frac{t^{a-1}}{1+t} dt = \frac{\pi}{sina\pi}$$

$$B'_a(a,1-a) = \int_0^{+\infty} \frac{t^{a-1}}{1+t} lntdt = \left(\frac{\pi}{sina\pi}\right) = \frac{\pi^2 cosa\pi}{sin^2 a\pi}$$

$$B''_a(a,1-a) = \int_0^{+\infty} \frac{t^{a-1}}{1+t} ln^2tdt = (\frac{\pi}{sina\pi})'' = \frac{\pi^3 sin^3 a\pi + 2\pi^3 sina\pi cos^2 a\pi}{sin^4 a\pi}$$

va hokazo,

$$B^{(n)}(a,1-a) = \int_0^{+\infty} \frac{t^{a-1}}{1+t} ln^n tdt = (\frac{\pi}{sina\pi})^{(n)} \quad (n = 1,2,3,...$$

bo'ladi.

5^0 . $\forall(a,b) \in M' (M' = \{(a,b) \in R2: a \in (0;+\infty), b \in (1;+\infty)\}$ uchun

$$B(a,b) = \frac{b-1}{a+b-1} B(a,b-1) \tag{6}$$

bo'ladi.

Isbot. (1) integralni bo'laklab integrallaymiz.

$$B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx=\int_0^1(1-x)^{b-1}d(\frac{x^a}{a})$$

$$=\frac{1}{a}x^a(1-x)^{b-1}I_0^1+\frac{b-1}{a}\int_0^1 x^a(1-x)^{b-2}dx \ \ (a>0, \ \ b>1)$$

Agar

$$x^a(1-x)^{b-2}=x^{a-1}[1-(1-x)](1-x)^{b-2}$$
$$=x^{a-1}(1-x)^{b-2}-x^{a-1}(1-x)^{b-1}$$

ekanligigni e’tiborga olsak, uholda

$$\int_0^1 x^a(1-x)^{b-2}dx=\int_0^1 x^{a-1}(1-x)^{b-2}dx-\int_0^1 x^{a-1}(1-x)^{b-2}dx$$
$$=B(a,b-1)-B(a,b)$$

bo’lib, natijada

$$B(a,b)=\frac{b-1}{a}[B(a,b-1)-B(a,b)]$$

bo’ladi. bu tenglikdan esa

$$B(a,b)=\frac{b-1}{a+b-1}B(a,b-1)\ (a>0,b>1)$$

bo’lishini topamiz.

Xuddi shunga o’xshash $\forall(a,b)\in$ M” uchun $M''=\{(a,b)\ \in R2: a\in(1,\infty)$ b∈(0, +∞)})

$$B(a,b)=\frac{a-1}{a+b-1}B(a-1,b)$$

bo’ladi.

Isbot. (1) integralni bo’laklab integrallaymiz.

$$B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx=\int_0^1 x^{a-1}d\left(\frac{(1-x)^b}{b}\right)=$$

$$=-\frac{1}{b}x^{a-1}(1-x)^b I_0^1$$

$$+\frac{a-1}{b}\int_0^1 x^{a-2}(1-x)^b dx=\frac{a-1}{b}\int_0^1 x^{a-2}(1-x)^b dx,$$

$$(a>1),(b>0)$$

$$x^{a-2}(1-x)^b=x^{a-2}(1-x)^{b-1}(1-x)=$$

$$=x^{a-2}(1-x)^{b-1}-x^{a-1}(1-x)^{b-1}$$

$$\int_0^1 x^{a-2}(1-x)^b dx=$$

$$=\int_0^1 x^{a-2}(1-x)^{b-1}dx-\int_0^1 x^{a-1}(1-x)^{b-1}dx=B(a-1,b)$$

$$-B(a,b)$$

$$B(a,b)=\frac{a-1}{a+b-1}B(a-1,b)\ (\mathrm{a}>1,b>0)$$

bo’lishini topamiz.

Xususan , $b=n\ (n\in N)$ bo’lganda

$$\mathrm{B(a,b)}=\mathrm{B(a,n)}=\frac{n-1}{a+n-1}B(a,n-1)$$

bo’lib, (6) formulani takror qo’llab, quyidagini topamiz.

$$B(a,n)=\frac{n-1}{a+n-1}*\frac{n-2}{a+n-2}*\dots*\frac{1}{n+1}B(a,1)$$

Ma’lumki,$B(a,1)=\int_0^1 x^{a-1}dx=\frac{1}{a}$, demak

$$B(a,n)=\frac{1*2*...*(a-1)}{a(a+1)(a+2)...(a+n-1)}. \qquad (7)$$

Agarda (7) da $a=m$ $(m\in N)$ bo'lsa, u holda

$$B(m,n)=\frac{1*2*...*(n-1)}{m(m+1)...(m+n-1)}=\frac{(n-1)!\,(m-1)!}{(m+n-1)!}$$

Mustaqil bajarish uchun topshiriqlar.

1. Betta funksiya xossalaridan foydalanib quyidagi xosmas integralni hisoblang. $\int_0^1 \sqrt{x-x^2}dx$
2. Betta funksiya xossalaridan foydalanib quyidagi xosmas integralni hisoblang. $\int_0^{+\infty} \frac{\sqrt[4]{x}}{(1-x)^2}dx$
3. Betta funksiya xossalaridan foydalanib quyidagi xosmas integralni hisoblang. $\int_0^a x^2\sqrt{a^2-x^2}\,dx$.
4. Betta funksiya xossalaridan foydalanib quyidagi xosmas integralni hisoblang.$\int_0^{+\infty} \frac{x^{m-1}}{1+x^n}dx$
5. Betta funksiya xossalaridan foydalanib quyidagi xosmas integralni hisoblang.$\int_0^1 x^2(1-x)^3\,dx$

Gamma funksiya va uning xossalari.

Biz

$$\int_0^{+\infty} x^{a-1}\,e^{-x}dx, \qquad (1)$$

xosmas integralni qaraylik.Bu chegaralanmagan funksiyaning (a<1 da x = 0 maxsus nuqta) cheksiz oraliq bo'yicha olingan xosmas integrali bo'lishi bilan birga a ga (parametrga) ham bog'liqdir.

1-ta'rif: (1) integral gamma funksiya yoki II tur Eyler integrali deb ataladi va Γ(a) kabi belgilanadi. Demak,

$$\Gamma(a) = \int_0^{+\infty} x^{a-1}\, e^{-x} dx; \qquad (2)$$

Shunday qilib, Γ(a) funksiya (0; +∞) da berilgandir.Endi Γ(a) funksiyaning xossalarini o‘rganaylik.

1°. (1) integral$\Gamma(a)=\int_0^{+\infty} x^{a-1}\, e^{-x}$dx ixtiyoriy$[a_0, b_0]$(0<$a_0 < b_0 < +\infty$) oraliqda tekis yaqinlashuvchi bo’ladi.

Isbot: Integralni quyidagi 2 qismga ajratib,

$\int_0^{+\infty} x^{a-1}\, e^{-x}$dx=$\int_0^{1} x^{a-1}\, e^{-x}$dx +$\int_1^{+\infty} x^{a-1}\, e^{-x}$dx

Ularning har birini alohida-alohida tekis yaqnlashuvchilikka tekshiramiz.Agara_0($a_0 > 0$) sonini olib , parameter a ning$a \geq a_0$qiymatlari qaralsa , unda barcha $x \in (0,1]$uchun

$$x^{a-1}e^{-x} \leq \frac{1}{x^{1-a_0}}$$

bo′lib, ushbu Veyrshtrass alomatiga ko’ra

$$\int_0^1 x^{a-1}\, e^{-x} dx$$

ushbu integral tekis yaqinlashuvchi bo’ladi. Agar b_0 ($b_0 > 0$) sonni olib,parametr a ning $a \leq b_0$qiymatlari qaraladigan bo’lsa, unda barcha $x \geq 1$uchun

$$x^{a-1}e^{-x} \leq x^{b_0-1}e^{-x} \leq \left(\frac{b_0+1}{e}\right)^{b_0+1} \frac{1}{x^2}$$

bo’lib,

$$\int_1^{\infty} \frac{1}{x^2} dx$$

Integralning yaqinlashuvchi ekanligidan, yana Veyershtass alomatiga ko'ra

$$\int_0^{+\infty} x^{a-1}\, e^{-x} dx$$

integralning tekis yaqinlashuvchi ekanligini topamiz.

Shunday qilib,

$$\Gamma(a) = \int_0^{+\infty} x^{a-1}\, e^{-x} dx$$

$[a_0, b_0](0<a_0 < b_0 <+\infty)$ da tekis yaqinlashuvchi bo'ladi.

2°. $\Gamma(a\ )$ funksiya (0;+∞) da uzluksiz hamda barcha tartibdagi uzluksiz hosilalarga ega va

$$\Gamma^{(n)}(a) = \int_0^{+\infty} x^{a-1}\, e^{-x}(lnx)^n dx\ (n = 1,2, \dots.)$$

Isbot: $\forall a \in (0;+\infty)$ nuqtani olaylik. Unda shunday $[a_0, b_0](0<a_0 < b_0 < +\infty)$ oraliq topiladiki , $a \in [a_0, b_0]$ bo'ladi. Ravshanki,

$$\Gamma(a) = \int_0^{+\infty} x^{a-1}\, e^{-x} dx$$

integral ostidagi $f(x,a) = x^{a-1}\, e^{-x}$funksiya

$$M = \{(x,a) \in R^2 : x \in (0;+\infty)\ , a \in (0;+\infty)\}$$

to'plamda uzluksiz funksiyadir. (1) integral esa $[a_0, b_0]$da tekis yaqinlashuvchi.

U holda teoremaga asosan $\Gamma(a)$ funksiya $[a_0, b_0]$da binobarin, anuqtada uzluksiz bo'ladi.

(1)integral ostidagi $f(x,a) = x^{a-1}\, e^{-x}$ funksiya

$$f_a{}'(x,a) = x^{a-1}e^{-x}lnx$$

hosilasi M to'plamda uzluksiz funksiya.

Endi

$$\int_0^{+\infty} f_a{}'(\mathrm{x},\mathrm{a})\,dx = \int_0^{+\infty} x^{a-1}\, e^{-x} lnxdx$$

integralni $[a_0,\, b_0]$da tekis yaqinlashuvchi bo'lishini ko'rsatamiz.. Ushbu

$$\int_0^1 x^{a-1}\, e^{-x} lnxdx$$

Integral ostidagi $x^{a-1}\, e^{-x} lnx$ funksiya uchun

$0 < x \leq 1$ da$\int_0^1 x^{\frac{a}{2}-1}\, dx$ o'rinlidir. $\psi(x) = x^{\frac{a_0}{2}}|\mathrm{lnx}|$ funksiya $0 < x \leq 1$ da chegaralanganligidan va$\int_0^1 x^{\frac{a}{2}-1}\, dx$ integralning tekis yaqinlashuvchiligini topamiz.

Shunga o'xshsh quyidagi

$$\int_1^{+\infty} x^{a-1}\, e^{-x} lnxdx$$

integralda , integral ostidagi $x^{a-1}\, e^{-x} lnx$funksiya uchun barcha $x \geq 1$ da

$$x^{a-1}e^{-x} lnx \leq x^{b_0-1}e^{-x} lnx \leq (\frac{b_0+1}{e})^{b_0+1} \cdot \frac{1}{x^2}$$

Bo' lib,

$$\int_1^{+\infty} \frac{dx}{x^2}$$

Integralning yaqinlashuvchanligidan, yana Veyrshtrass alomatiga ko'ra

$\int_1^{+\infty} x^{a-1}\, e^{-x} lnx$dx ning tekis yaqinlashuvchiligi kelib chiqadi. Demak$[a_0,\ b_0]$da $\int_1^{+\infty} x^{a-1}\, e^{-x} lnx$dx integral tekis yaqinlashuvchi. Unda teoremaga asosan

$$f'(a) = \left(\int_0^{+\infty} x^{a-1}\, e^{-x} \mathrm{dx}\right)' = \int_0^{=\infty} (x^{a-1}\, e^{-x})'\, dx = \int_0^{+\infty} x^{a-1}\, e^{-x} lnxdx$$

bo'ladi va Γ(a)$[a_0, b_0]$da , shu bilan birga a nuqtada ham uzluksizdir.Xuddi shu yo'l bilan Γ(a) funksiyaning ikkinchi, uchinchi va boshqa tartibli hosilalarining mavjudligi, uzluksizligi hamda

$$\Gamma^{(n)}(a) = \int_0^{+\infty} x^{a-1}\, e^{-x} (lnx)^n dx \ (n = 1,2, \dots .)$$

bo'lishi ko'rsatiladi.

3°.Γ(a) funksiya uchun ushbu $\Gamma(a+1) = a\Gamma(a)$ $(a > 0)$ formula o'rinli.

Haqiqatan ham,

$$\Gamma(a) = \int_0^{+\infty} x^{a-1}\, e^{-x} dx = \int_0^{+\infty} e^{-x}\ d\left(\frac{x^a}{a}\right)$$

Integralni bo'laklab integrallasak,

$$\Gamma(a) = e^{-x} \frac{x^a}{a} |_0^{+\infty} + \int_0^{+\infty} e^{-x} \frac{x^a}{a} dx = \frac{1}{a}\Gamma(a+1)$$

bolib, bundan

$$\Gamma(a+1) = a\Gamma(a) \qquad (7)$$

kelib chiqadi. Bu formula yordamida $\Gamma(a+n)$ ni topish mumkin.

Darhaqiqat, (7) formulani takror qo'llab

$$\Gamma(a+2) = \Gamma(a+1)(a+1)$$

$$\Gamma(a+3) = \Gamma(a+2)(a+2)$$

$$\dots \dots \dots \dots \dots \dots \dots \dots .$$

$$\Gamma(a+n) = \Gamma(a+n-1)(a+n-1)$$

bo'ishini, ulardan esa

$$\Gamma(a+n) = (a+n-1)(a+n-2)\ldots\ (a+2)(a+1)a\Gamma(a)$$

ekanligini topamiz. Xususan, a=1 bo'lganda

$$\Gamma(n+1) = n(n-1)\ldots 2\cdot 1\cdot \Gamma(1)$$

bo'ladi. Agar

$$\Gamma(1) = \int_0^{+\infty} e^{-x}\,dx = 1$$

bo'lishini e'tiborga olsak, unda

$$\Gamma(n+1) = n!$$

ekanligi kelib chiqadi.

Mustaqil bajarish uchun topshiriqlar.

1. Ushbu

$$\Gamma(x)\Gamma(x+\frac{1}{2}) = \sqrt{\pi}2^{1-2x}\Gamma(2x)$$

Lejandr formulasini isbotlang.

2.

$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

Puasson integralini keltirib chiqaring.

3.$\Gamma(\frac{1}{2})$ ni hisoblang.

4. Faraz qilaylik $a > 0$ va $a + b > 0$ bo'lsin. U holda ushbu tenglik bajarilishini isbotlang.

$$\int_{0}^{+\infty} \xi^{a-1}\Gamma(b,\xi)\,d\xi = \frac{\Gamma(a+b)}{a}.$$

Bessel funksiyalari va ularning xossalari.

Birinchi tur Bessel funksiyasi: $J_\nu(z)$ birinchi tur Bessel funksiyasi o'zgaruvchining darajasi shaklidagi qator orqali ifodalanadi:

$$J_\nu(z) := \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\nu+1)} \left(\frac{z}{2}\right)^{2k+\nu} \qquad (1)$$

Bu yerda , z kompleks o'zgaruvchi , ν esa haqiqiy yoki kompleks qiymatlarni qabul qiluvchi parametr. ν butun son bo'lganda, Bessel funksiyasi analitik funksiya bo'ladi; bu holatda

$$J_{-n}(z) = (-1)^n J_n(z)\,, \qquad n = 1,2,\ldots \qquad (2)$$

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+n)!} \left(\frac{z}{2}\right)^{2k+n}$$

$$J_{-n}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k-n)!} \left(\frac{z}{2}\right)^{2k-n} = \sum_{s=0}^{\infty} \frac{(-1)^{n+s}}{(s+n)!\,s!} \left(\frac{z}{2}\right)^{2s+n}$$

Agar ν butun son bo'lmasa, Bessel funksiyasi $\left(\frac{z}{2}\right)^\nu$ ko'paytuvchi tufayli $z = 0$ nuqtada turli xil qiymatlarni qabul qiladi. Shuning uchun , kompleks sohaning manfiy yarim o'qini kesib , z $|arg\ (z)| < \pi$ shart bilan olinadi. Trikomi tavsiyasiga asosan , biz $\left(\frac{z}{2}\right)^\nu$ singulyar ko'paytuvchini (1) dan ajratib olishimiz mumkin:

$$J_\nu^T(z) := \left(\frac{z}{2}\right)^{-\nu} J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\nu+1)} \left(\frac{z}{2}\right)^{2k} \qquad (3)$$

Bessel funksiyalari odatda, Fucks-Frobeniusning ikkinchi tartibli differensial tenglamasining ildizi sifatida qaraladi:

$$\frac{d^2}{dz^2}u(z) + p(z)\frac{d}{dz}u(z) + q(z)u(z) = 0 \qquad (4)$$

Bu yerda, $p(z)$ va $q(z)$ lar ma'lum analitik funksiyalar.Agar biz $p(z)$ va $q(z)$ larni

$$p(z) = \frac{1}{z} \text{ va } q(z) = 1 - \frac{\nu^2}{z^2} \qquad (5)$$

kabi tanlab olsak, va (4) differensial tenglamani ishlasak, (1) ko'rinishidagi qatorga ega bo'lamiz. Natijada , 1-tur Bessel funksiyasi quyidagi differensial tenglamani qanoatlantiradi deyishimiz mumkin:

$$u''(z) + \frac{1}{z}u'(z) + \left(1 - \frac{\nu^2}{z^2}\right)u(z) = 0 \qquad (6)$$

(6) differensial tenglama odatda, **Bessel differensial tenglamasi** deb ataladi.

1-tur Bessel funksiyasining xossalari:

1. $\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x)$
2. $\frac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x)$
3. $x J_p'(x) + pJ_p(x) = x J_{p-1}(x)$
4. $x J_p'(x) - pJ_p(x) = -x J_{p+1}(x)$
5. $J_{p-1}(x) - J_{p+1}(x) = 2 J_p'(x)$
6. $J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x)$
7. $\int x^{p+1} J_p(x)dx = x^{p+1} J_{p+1}(x) + C$
8. $\int x^{-p+1} J_p(x)dx = -x^{-p+1} J_{p-1}(x) + C$

Misol: $J_{1/2}(z)$ va $J_{-1/2}(z)$ funksiyalarning qiymatlarini hisoblang.

Yechim: (1) formulaga asosan hisoblaymiz:

$$J_{1/2}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\frac{1}{2}+1)}\left(\frac{z}{2}\right)^{2k+\frac{1}{2}}$$

Bu yerda

$$\Gamma\left(k+\frac{1}{2}+1\right)=\Gamma\left(k+\frac{3}{2}\right)=\left(k+\frac{1}{2}\right)\Gamma\left(k+\frac{1}{2}\right)=\frac{(2k+1)(2k-1)}{2^2}\Gamma\left(k-\frac{1}{2}\right)$$
$$=\frac{1*3*5*\ldots*(2n-1)*(2n+1)}{2^{n+1}}\Gamma\left(\frac{1}{2}\right)$$

Bu yerda , $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$ ekanligidan foydalansak,

$$J_{1/2}(x)=\left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{2^k k!1*2*3*\ldots*(2k+1)}=\left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

Bu yerda oxirgi yig'indi $sinx$ funksiyasining Makloren qatoriga yoyilmasidan iboratdir. Demak,

$$J_{1/2}(x)=\left(\frac{2}{\pi x}\right)^{\frac{1}{2}}sinx$$

Xuddi shunga o'xshash, $J_{-\nu}(z)\coloneqq\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(k+1)\Gamma(k-\nu+1)}\left(\frac{z}{2}\right)^{2k-\nu}$ formuladan foydalansak,

$$J_{-1/2}(x)=\left(\frac{2}{\pi x}\right)^{\frac{1}{2}}cosx$$

tenglikni hosil qilamiz.

ν butun son bo'lmaganda, Bessel tenglamasiningumumiy integrali

$$u(z)=\gamma_1 J_\nu(z)+\gamma_2 J_{-\nu}(z) \qquad (7)$$

Bu yerda $J_\nu(z)$ va $J_{-\nu}(z)$ lar Wronskian orqali chiziqli erkli.

$$W\{J_\nu(z),J_{-\nu}(z)\}=-\frac{2}{\pi z}\sin(\pi\nu) \qquad (8)$$

Eslatma: $W\{f(z),g(z)\}\coloneqq f(z)g'(z)-f'(z)g(z)$.

Mashqlar: Quyidagilarni isbot qiling:

1. $\left(\frac{d}{xdx}\right)^m \left[x^p J_p(x)\right] = x^{p-m} J_{p-m}(x)$

2. $\left(\frac{d}{xdx}\right)^m \left[\frac{J_p(x)}{x^p}\right] = (-1)^m \frac{J_{p+m}(x)}{x^{p+m}}$

Ikkinchi tur Bessel funksiyasi: $Y_\nu(z)$.

$\nu = n$, $(n = 0, \pm 1, \pm 2, \dots)$ da J_ν dan chiziqli erkli bo'lgan (6) ikkinchi tartibli differensial tenglamaning yechimini olish uchun , 2-tur Bessel funksiyasini kiritamiz:

$$Y_\nu(z) := \frac{J_{-\nu}(z)\cos(\nu\pi) - J_{-\nu}(z)}{sin(\nu\pi)} \tag{9}$$

Butun ν sonlari uchun (9) tenglikning o'ng tomoni no'malum bo'lib qoladi, shuning uchun biz $Y_n(z)$ ni limit ko'rinishida quyidagicha keltiramiz:

$$Y_n(z) := \lim_{\nu\to n} Y_\nu(z) = \frac{1}{\pi}\left[\frac{\partial J_\nu(z)}{\partial \nu}\Big|_{\nu=n} - (-1)^n \frac{\partial J_{-\nu}(z)}{\partial \nu}\Big|_{\nu=n}\right] \tag{10}$$

(10) dan quyidagi kelib chiqadi:

$$Y_{-n}(z) = (-1)^n Y_n(z) \tag{11}$$

ixtiyoriy haqiqiy son bo'lganda , (6) ikkinchi tartibli differensial tenglamaning umumiy yechimi quyidagicha bo'ladi:

$$u(z) = \gamma_1 J_\nu(z) + \gamma_2 Y_\nu(z)\,,\;\; \gamma_1, \gamma_2 \in C \tag{12}$$

va bunga mos Wronskiy quyidagicha bo'ladi:

$$W\{J_\nu(z), Y_\nu(z)\} = \frac{2}{\pi z} \tag{13}$$

Uchinchi tur Bessel funksiyalari: $H_\nu^{(1)}, H_\nu^{(2)}$.

1- va 2-tur Bessel funksiyalariga qo'shimcha ravishda , 3-tur Bessel funksiyalari yoki Hankel funksiyalari quyidagicha kiritiladi:

$$H_\nu^{(1)} := J_\nu(z) + iY_\nu(z)\;\;,\;\; H_\nu^{(2)} := J_\nu(z) - iY_\nu(z) \tag{14}$$

Bu funksiyalar Wronskian orqali chiziqli erkli

$$W\{H_\nu^{(1)}, H_\nu^{(2)}\} = -\frac{4i}{\pi z} \tag{15}$$

(14) formulalarda $Y_\nu(z)$ larning o'rniga (9) formuladan foydalansak, quyidagilarga ega bo'lamiz:

$$\begin{cases} H_\nu^{(1)}(z) := \frac{J_{-\nu}(z) - e^{-i\nu\pi} J_\nu(z)}{i\sin(\nu\pi)} \\ H_\nu^{(2)}(z) := \frac{e^{+i\nu\pi} J_\nu(z) - J_{-\nu}(z)}{i\sin(\nu\pi)} \end{cases} \tag{16}$$

(16) dan quyidagi muhim formulalarga ega bo'lamiz:

$$H_{-\nu}^{(1)}(z) = e^{+i\nu\pi} H_\nu^{(1)}(z), \quad H_{-\nu}^{(2)}(z) = e^{-i\nu\pi} H_\nu^{(2)}(z) \tag{17}$$

Mustaqil ishlash uchun topshiriqlar:

Quyida berilgan integrallarni soddalashtiring:

1. $\int x\, J_0(x) dx$
2. $\int x^4 J_3(x) dx$
3. $\int J_1(x) dx$
4. $\int x^{-2} J_3(x) dx$

Quyidagilarni isbotlang.

1. $J_{\frac{-1}{2}}(x) = \sqrt{\frac{2}{\pi x}} cosx$
2. $J_{-\frac{3}{2}}(x) = -\sqrt{\frac{2}{\pi x}}\left(\frac{cosx}{x} + sinx\right)$
3. $J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{3-x^2}{x^2} - \frac{3}{x} cosx\right)$
4. $\int_0^x sJ_0(s)ds = xJ_1(x)$.

Gipergeometrik funksiya va uning xossalari

Ushbu

$$x(1-x)y'' + [c-(a+b+1)x]y' - aby = 0$$

gipergeometrik funksiya yoki Gauss tenglamasi deb ataluvchi tenglamani tekshiramiz. Bu yerda a, b, c- 3 ta ∀ parametr bo‘lib, haqiqiy yoki kompleks qiymatlar qabul qiladi.

Bulardan (a, b) ikkitasi tenglamada simmetrik ishtirok etadi.

(1) tenglamaning yechimini

$$y = \sum_{n=0}^{\infty} A_n\, x^n$$

darajali qator ko‘rinishida izlaymiz. Bundan

$$y' = \sum_{n=1}^{\infty} n\, A_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)\, A_{n+1} x^n$$

$$y'' = \sum_{n=1}^{\infty} n\,(n+1) A_{n+1} x^{n-1} = \sum_{n=0}^{\infty} (n+2)\,(n+1) A_{n+2} x^n$$

Bu hosilalarning qiymatini va y ni (1) tenglamaga qo‘yamiz. U holda

$$\sum_{n=0}^{\infty} x\,(1-x)(n+2)(n+1)A_{n+2}x^n + \sum_{n=0}^{\infty} (c-(a+b+1)x)\,(n+1)A_{n+1}x^n - ab\sum_{n=0}^{\infty} A_n\, x^n$$

$$A_{n+2}(n+2)(n+1)x^{n+1} - A_{n+2}(n+2)(n+1)x^{n+2} + c(n+1)A_{n+1}x^n - (a+b+1)(n+1)A_{n+1}x^n - abA_n x^n = 0$$

$$A_{n+1}(n+1)n - A_n n(n-1) + c(n+1)A_{n+1} - (a+b+1)nA_n - abA_n = 0$$

tenglikni hosil qilamiz.

$$A_{n+1}(n+1)(c+1) = (ab + an + bn + n + n^2 - n)A_n = a(b+n) + n(b+n)$$

$$A_{n+1} = \frac{(a+n)(b+n)}{(c+1)(n+1)} A_n$$

rekurent formulaga ega bo‘lamiz.

Bu yerda

$A_0 = 1; c \neq 0, -1, -2, \dots, -n,$ ... deb hisoblaymiz.

(1) gipergeometrik tenglamaning 1-xususiy yechimi y_1 ni $F(a,b,c,x)$ orqali belgilab, A_n koeffitsiyentlarning topilgan qiymatlarini (2) qatorga qo‘yamiz.

U holda

$$y_1 = 1 + \frac{ab}{c}x + \frac{(a+1)(b+1)}{(c+1)2}\frac{ab}{c}x^2 + \cdots = 1 + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} x^n$$

$$A_0 = 1 \qquad A_2 = \frac{(a+1)(b+1)}{(c+1)2}\frac{ab}{c}$$

$$A_1 = \frac{ab}{c} \qquad A_3 = \frac{(a+2)(b+2)}{(c+2)3}\frac{(a+1)(b+1)ab}{2(c+1)c}$$

$$y_1 = F(a,b,c,x) = 1 + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} x^n$$

Bunda

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma((a)} = a(a+1)(a+2)\dots(a+n-1)$$

$$(a)_0 = 1; \qquad n = 1,2,3,\dots$$

Xususiy holda

$$(1)_n = n!$$

(3) qator gipergeometrik qator, bu qatorning yig‘indisi bo‘lgan $F(a,b,c,x)$ funksiya esa, gipergeometrik funksiya deyiladi.

Dalamber prinsipiga asosan

$$\lim_{n} \to \infty \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n} -\to \infty \left|\frac{(a+n)(b+n)}{(n+1)(c+n)}x\right| = |x|$$

Demak (3) qator $|x| < 1$ da absolyut yaqinlashuvchi, |x|>1 da esa uzoqlashuvchi bo‘ladi.

$x = 1$ bo‘lganda, agar $c - a - b \leq 0$ bo‘lsa, uzoqlashuvchi.

$x = -1$ bo‘lganda esa

a) agar $c - a - b > 0$ bo‘lsa, absolyut yaqinlashuvchi.

b) agar $-1 < c - a - b \leq 0$ bo‘lsa, absolyut bo‘lmay yaqinlashuvchi.

c) agar $c - a - b \leq -1$ bo‘lsa uzoqlashuvchi bo‘ladi.

Agar (1) formulada $b = c$ bo‘lsa, $\quad C_n^{-a} = \frac{n!}{(n-a)!n!}$

$$(a)_n = (-1)^n(-a)(-a-1)\dots(-a-n+1) = (-1)^n \binom{-a}{n} n!$$

ga asosan $F(a,b,b,x) = 1 + \sum_{n=1}^{\infty}(-1)^n \binom{-a}{n} x^n = (1-x)^{-a}$ binomial qator hosil bo‘ladi.

Agar $a = 1; b = c$ bo‘lsa (1) formula ushbu

$$F(1,b,b,x) = 1 + \sum_{n=1}^{\infty} x^n = \frac{1}{1-x}$$

ko‘rinishga ega bo‘ladi, ya‘ni $a = 1, b = c$ bo‘lgan holda gipergeometrik qator geometrik progressiyaga aylanadi. Shuning uchun ham u gipergeometrik qator deyiladi.

(1) tenglamaning 2-xususiy, umuman aytganda, (3) ga chiziqli bog‘liq bo‘lmaganda yechimini topish uchun (1) tenglamada

$$y = x^{\rho}\eta$$

almashtirish bajaramiz. U holda (1) tenglama quyidagi ko‘rinishda yoziladi:

$$x(1-x)\eta'' + [(c+2\rho) - (a+b+1+2\rho)x]\eta'$$
$$-\left[ab + \rho(a+b+\rho) - \frac{\rho(\rho+c+1)}{x}\right]\eta = 0$$

$$y' = \rho \cdot x^{\rho-1}\eta + x^{\rho} \cdot \eta'$$

$$y'' = \rho(\rho-1)x^{\rho-2}\eta + \rho \cdot x^{\rho-1}\eta' + \rho \cdot x^{\rho-1}\eta' + x^{\rho}\eta''$$

$$= \rho(\rho-1)x^{\rho-2}\eta + 2\rho \cdot x^{\rho-1}\eta' + x^{\rho}\eta''$$

$$x(1-x)x^{\rho}\eta'' + \rho(\rho-1)x^{\rho-1}\eta - \rho(\rho-1)x^{\rho}\eta + 2\rho x^{\rho}\eta' - 2\rho x^{\rho-1}\eta'$$
$$+ [c - (a+b+1)x](\rho x^{\rho-1}\eta + x^{\rho}\eta') - abx^{\rho}\eta = 0$$

$$x(1-x)\eta'' + [(c+2\rho) - (a+b+1+2\rho)x]\eta'$$
$$-\left[ab + \rho(a+b+\rho) - \frac{\rho(\rho+c-1)}{x}\right]$$

Bu tenglama (1) tenglamalar tipiga tegishli bo‘lishi uchun yoki $\rho = 0$ yoki $\rho = 1 - c$ bo‘lishi kerak . U holda 42

$$x(1-x)\eta'' + [(2-c) - [(a-c+1) + (b-c+1) + 1]x]\eta'$$
$$-(a-c+1)(b-c+1)\eta = 0$$

tenglamaga ega bo‘lamiz. Shunday qilib, $\rho = 1 - c$ bo‘lganda $y = x^{\rho}\eta$ almashtirish (1) tenglamani xuddi shu ko‘rinishdagi tenglamaga o‘tkazadi, faqat a, b, c larni mos ravishda

$$a - c + 1 \qquad b - c + 1 \qquad 2 - c$$

larga almashtirish zarur. Demak, berilgan (1) tenglama y_1 ga chiziqli bog‘liq bo‘lmagan

$$y_2 = x^{1-c}F(a-c+1, b-c+1, 2-c; x)$$

yechimga ega bo‘ladi. Shu bilan birga y_2

$$2-c\neq 0,1,-2,\dots,-n,\dots$$

bo‘lgandagina ma‘noga ega bo‘ladi. Shunday qilib (1) tenglamaning umumiy yechimini quyidagi ko‘rinishda yozish mumkin.

$$y=c_1F(a,b,c;x)+c_2x^{1-c}F(a-c+1,b-c+1,2-c;x)$$

bu yerda c_1 va c_2–∀ o‘zgarmaslar.

Agar gipergeometrik funksiya simmetrik bo‘lib kirgan a va b parametrlardan bittasi manfiy butun son $-n$ ga teng bo‘lsa, (3) gipergeometrik qator uzulib qoladi va u n-darajali ko‘phadga aylanadi.

Agarda $a=-n_1, b=-n_2$, bunda $n_1>0; n_2>0-$ butun sonlar bo‘lsa, u holda gipergeometrik qator ko‘phadga aylanib, uning darajasi n_1, n_2 sonlarning kichigiga teng bo‘ladi. (3) qatorni hadlab differensiallash natijasida darhol ushbu

$$F(a,b,c;x)=\frac{ab}{c}F(a+1,b+1,c+1;x)$$

formulani hosil qilamiz.

(3) qatorni avval x^a, x^b yoki x^{c-1} ga ko‘paytirib, so‘ngra hadlab differensiallashdan quyidagi formulalar kelib chiqadi.

$$\begin{cases}\frac{d}{dx}[x^aF(a,b,c;x)]=ax^{a-1}F(a+1,b,c,x)\\ \frac{d}{dx}[x^bF(a,b,c;x)]=bx^{b-1}F(a,b+1,c,x)\\ \frac{d}{dx}[x^{c-1}F(a,b,c;x)]=(c-1)x^{c-2}F(a,b,c-1,x)\end{cases}$$

Mittag-Leffler funksiyalari

Oldingi mavzularda biz butun tartibli hisob bilan bog’liq funksiyalarni tashkil etuvchi klassik gipergeometrik funksiyalarni, xususan, gamma funksiyasi bilan faktoriy tushunchani umumlashtirishni taqdim etdik. Shunga o'xshab, kasr hisobi nima uchun ko'plab tabiiy hodisalarning tavsifini aniqlashtirish uchun muhim vosita ekanligini tushunishimiz mumkin, xususan, xotira effekti mavjud bo'lgan vaqtinchalik bog'liqlik, u bilan bog’liq funksiyalar umumlashtiradigan shaklni tushunishdan. butun sonlar tartibi hisobiga nisbatan funksiyalar.

Doimiy koeffitsientli oddiy chiziqli differensial tenglamaning yechimi eksponensial funksiya asosida berilgan. Boshqa tomondan, doimiy koeffitsientlarga ega bo’lgan kasrli differensial tenglama ko'p hollarda Mittag-Leffler deb ataladigan funksiyada berilgan yechimga ega. Shu ma’noda aytishimiz mumkinki, Mittag-Lefler funksiyasi ko‘rsatkichli funksiyani kasrli umumlashtirishdir.

Shunday qilib, Mittag-Leffler funksiyasining eksponensial funksiyani umumlashtirish usulini tushunishda biz, ba’zi hollarda, nega butun sonli differensial tenglama berilgan hodisaning tegishli butun songa nisbatan adekvat tavsifini berishini tushunishimiz mumkin. tartibli differentsial tenglama, butun son bo'lmagan differensial tenglama tartibiga nisbatan cheklovchi holat sifatida olingan.

Mittag-Leffler funksiyasi

1903 yilda Mittag-Lefler tomonidan kiritilgan, bitta parametrni o'z ichiga olgan va bugungi kunda o’z nomini olgan funksiyani eksponensial funksiyani umumlashtirish deb hisoblash mumkin, chunki parametr unitar bo’lganda u kamayadi. Bu murakkab dalillar uchun aniqlangan bo’lsada, bu kitobda biz faqat haqiqiy o’zgaruvchining holatini muhokama qilamiz. Bu yerda, biz faqat o’zi tomonidan kiritilgan Mittag-Leffler funksiyasini va ikki va uchta parametrli Mittag-Leffler funksiyasini taqdim etamiz va muhokama qilamiz.

Ta’rif 1. α parametrga ($Re\{\alpha\} > 0$)bog’liq bo’lgan quyidagi darajali qator bir parametrli Mittag-Leffler funksiyasi deyiladi va $E_\alpha(x)$ ko’rinishda belgilanadi:

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)} \quad \alpha \in \mathbb{C}, Re(\alpha) > 0$$

yoki quyidagi ko'rinishda ham aniqlanadi:

$$E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\Gamma(\alpha k + 1)} \quad \alpha \in \mathbb{C}, Re(\alpha) > 0.$$

Bir parametrli Mittag-Leffler funksiyasining xususiy hollarini keltirib o'tamiz.

1)α=0 da

$$E_0(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(1)} = \sum_{k=0}^{\infty} x^k = \begin{cases} \frac{1}{1-x}, & |x| < 1 \\ \infty, & |x| \geq 1 \end{cases}.$$

2)α=1 da

$$E_1(x) = \sum_{n=0}^{\infty} \frac{x^k}{\Gamma(k + 1)} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = e^x.$$

3)α=2 da

$$E_2(x) = \sum_{n=0}^{\infty} \frac{x^k}{\Gamma(2k + 1)} = 1 + \frac{x}{2!} + \frac{x^2}{4!} + \frac{x^3}{6!} + \cdots = ch\sqrt{x} = \frac{e^{\sqrt{x}} + e^{-\sqrt{x}}}{2}.$$

Mittag-Leffler funksiyasining bir nechta turlari mavjud bo'lin, ular bir-biridan parametrlar va o'zgaruvchilar soni bilan farq qiladi. Ikki parametrli Mittag Leffler funksiyasini 1905-yilda Wiman tomonidan o'rganilgan.

Ta'rif 2. Faraz qilaylik $x \in \mathbb{C}$ va ikkita parametrlar $\alpha \in \mathbb{C}, \beta \in \mathbb{C}, \mathbb{R}e(\alpha) > 0, Re(\beta) > 0$ bo'lsin. U holda ikki parametrli Mittag Leffler funksiyasi ikki parametrlar bilan quyidagidarajali qator ko'rinishida aniqlanadi:

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}, \qquad \alpha, \beta \in \mathbb{C}, \quad Re(\alpha) > 0, Re(\beta) > 0.$$

Bu ikki parametrli Mittag-Leffler funksiyasi bir parametrli Mittag-Leffler funksiyaning umumlashmasidir.

Ikki parametrli Mittag-Leffler funksiyasiga doir misollar

1) α=1,β=2;

$$E_{1,2}(x)=\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(n+2)}=1+\frac{x}{2!}+\frac{x^2}{3!}+\frac{x^3}{4!}+\cdots=\frac{1}{x}(1+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots-1)=$$

$$=\frac{1}{x}(\sum_{n=0}^{\infty}\frac{x^n}{n!}-1)=\frac{1}{x}(e^x-1)$$

2) α=2,β=2

$$E_{2,2}(x)=\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(2n+2)}=1+\frac{x}{3!}+\frac{x^2}{5!}+\frac{x^3}{7!}=\frac{1}{\sqrt{x}}(1+\frac{\sqrt{x}^3}{3!}+\frac{\sqrt{x}^5}{5!}+\cdots)=$$

$$=\frac{1}{2\sqrt{x}}\left(1+\sqrt{x}+\frac{\sqrt{x}^2}{2!}+\frac{\sqrt{x}^3}{3!}-\frac{\sqrt{x}^4}{4!}+\cdots+(-1)+\sqrt{x}-\frac{\sqrt{x}^2}{2!}+\frac{\sqrt{x}^3}{3!}+\frac{\sqrt{x}^4}{4!}\dots\right)$$

$$=\frac{1}{2\sqrt{x}}\left(e^{\sqrt{x}}-e^{-\sqrt{x}}\right)=\frac{sh\sqrt{x}}{\sqrt{x}}$$

3) α=1,β=3

$$E_{1,3}(x)=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(k+3)}=\frac{1}{2!}+\frac{x}{3!}+\frac{x^2}{4!}+\frac{x^3}{5!}+\cdots=\frac{1}{x^2}(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+$$

$$+(-1-x))=\frac{1}{x^2}(\sum_{k=0}^{\infty}\frac{x^k}{k!}-(1+x))=\frac{1}{x^2}(e^x-x-1)$$

Mittag –Leffler funksiyasini Mellin-Barnes tipidagi integral orqali ham ifodalash mumkin va u quyidagicha ko’rinishda bo’ladi:

- Bir parametrli Mittag –Leffler funksiyasi:

$$E_\alpha(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{t^{\alpha-1}e^t}{t^\alpha - z}dt\,;$$

- Ikki parametrli Mittag –Leffler funksiyasi:

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{t^{\alpha-\beta}e^t}{t^\alpha - z}dt\,;$$

Bir parametrli Mittag –Leffler funksiyasining n-tartibli hosilasi quyidagicha aniqlanadi:

$$\left(\frac{d}{dx}\right)^k E_k(x^k) = E_k(x^k), k \in N.$$

$Isbot$:

$$\left(\frac{d}{dx}\right)^k E_k(x^k) = \sum_{n=1}^{\infty}\frac{1}{\Gamma(kn+1)}\left(\frac{d}{dx}\right)^k (x^{kn}) =$$

$$= \sum_{n=1}^{\infty}\frac{1}{\Gamma(kn+1)}\frac{\Gamma(kn+1)}{\Gamma(kn-k+1)}(x^{kn-k}) = \sum_{n=1}^{\infty}\frac{x^{kn-k}}{\Gamma(kn-k+1)}$$

n→n+1 o’zgarish kiritish orqali quyidagi tenglikka ega bo’lamiz

$$\left(\frac{d}{dx}\right)^k E_k(x^k) = \sum_{n=0}^{\infty}\frac{x^{kn}}{\Gamma(kn+1)} = E_k(x^k)$$

Ikki parametrli Mittag –Leffler funksiyasining $\mu > 0$ va $1 \le n \le k$ uchun n -tartibli hosilasi quyidagicha aniqlanadi:

$$\left(\frac{d}{dx}\right)^k \left[x^{n-1}E_{k,n}(\mu x^k)\right] = \mu x^{n-1}E_{k,n}(\mu x^k).$$

Umumlashgan Mittag-Leffler funksiyasi

Umumlashgan Mittag-Leffler (3 parametrli Mittag-Leffler)funksiyasi quyidagi ko'rinishda aniqlanadi:

$$E_{\alpha,\beta}^{\rho}(z)=\sum_{k=0}^{\infty}\frac{(\rho)_k}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!}\quad z\in\mathbb{C}\,;\alpha,\beta,\rho\in\mathbb{C}.$$

Bu yerda $(\rho)_k$- Pochhammer simvoli va u quyidagicha aniqlanadi

$$(\rho)_k=\rho(\rho+1)(\rho+2)\dots(\rho+k-1);$$

Bu funksiya ikki parametrli Mittag-Leffler funksiyasining umumlashlasidir, ya'ni ρ=1 bo'lsa, $E_{\alpha,\beta}^{1}(z)=E_{\alpha,\beta}(z)$ bo'ladi.

Eslatma. Faraz qilaylik $x\in\mathbb{C}$, $\alpha,\beta\in\mathbb{C}, Re\{\alpha\}>0, Re\{\beta\}>0$ bo'lsin va $k=0,1,2,\dots$. U holda ikki va uch parametrli Mittag-Leffler funksiyalari orasida quyidagi munosabat o'rinlidir:

$$\frac{d^k}{dx^k}E_{\alpha,\beta}(x)=k!\,E_{\alpha,\beta+\alpha k}^{k+1}(x).$$

Misol. $\alpha>0$ uchun ushbu tenglik bajarilishini isbotlang?

$$E_{2\alpha}(x^2)-xE_{2\alpha,\alpha+1}(x^2)=E_{\alpha}(-x)$$

Isbot:

$$E_{2\alpha}(x^2)-xE_{2\alpha,\alpha+1}(x^2)=\sum_{k=0}^{\infty}\frac{(\llbracket x^2)\rrbracket^k}{\Gamma(2\alpha k+1)}-x\sum_{k=0}^{\infty}\frac{(\llbracket x^2)\rrbracket^k}{\Gamma(2\alpha k+\alpha+1)}=$$

$$=\left(1+\frac{x^2}{\Gamma(2\alpha+1)}+\frac{x^4}{\Gamma(4\alpha+1)}+\frac{x^6}{\Gamma(6\alpha+1)}+\cdots\right)-$$

$$-\left(1+\frac{x}{\Gamma(\alpha+1)}+\frac{x^3}{\Gamma(3\alpha+1)}+\frac{x^5}{\Gamma(5\alpha+1)}+\cdots\right)=$$

$$=\left(1+\frac{(-x)}{\Gamma(\alpha+1)}+\frac{(-x)^2}{\Gamma(2\alpha+1)}+\frac{(-x)^3}{\Gamma(3\alpha+1)}+\frac{(-x)^4}{\Gamma(4\alpha+1)}\right)+$$

$$+\left(\frac{(-x)^5}{\Gamma(5\alpha+1)}+\frac{(-x)^6}{\Gamma(6\alpha+1)}+\cdots\right)=\sum_{n=1}^{\infty}\frac{(-x)^k}{\Gamma(k\alpha+1)}$$

Bundan kelib chiqadiki

$$E_{2\alpha}(x^2)-xE_{2\alpha,\alpha+1}(x^2)=E_\alpha(-x).$$

Mustaqil yechish uchun topshiriqlar.

Quyidagi tengliklarni isbotlang.

1) $$E_{1,2}(x)=1+E_{1,3}(x)$$

2)Faraz qilaylik $\alpha > 0$ va $x \in R$ bo'lsin. U holda Mittag-Leffler funksiyasi uchun ushbu tenglik bajarilishini isbotlang.

$$\frac{1}{2}\left[E_\alpha(\sqrt{x})+E_\alpha(-\sqrt{x})\right]=E_{2\alpha}(x)$$

3) Hisoblang.

$$xE_{2,2}(-x^2)=sinx$$

4)

$$E_{-\alpha,\beta}(x)=\frac{1}{\Gamma(\beta)}-E_{\alpha,\beta}\left(\frac{1}{x}\right)$$

5)

$$E_{-2,1}\left(-\frac{1}{x^2}\right)=1-\cos x$$

6)

$$E_{\alpha,\beta}(-x)=\frac{1}{\alpha\Gamma(\beta-\alpha)}\int_0^1\left(1-\varepsilon^{\frac{1}{\alpha}}\right)^{\beta-\alpha-1}E_{\alpha,\alpha}(-x\varepsilon)d\epsilon$$

7)

$$E^{\rho}_{\alpha,\beta-\alpha}(x)-E^{\rho-1}_{\alpha,\beta-\alpha}(x)=xE^{\rho}_{\alpha,\beta}(x)$$

8)

$$\int_0^x \varepsilon^{\beta-1} E^{\rho}_{\alpha,\beta-\alpha}(\mu\varepsilon^{\alpha})d\varepsilon = x^{\beta} E^{\rho}_{\alpha,\beta+1}(\mu x^{\alpha})$$

10) $\beta > 0$ uchun ushbu tenglik bajarilishini isbotlang.

$$\frac{d}{dz}E_{\beta}(z) = \frac{E_{\beta,\beta}(z)}{\beta}$$

Foks funksiyasi va uning xossalari.

Foksning H-funksiyasi kasr tartibli hisobning maxsus funksiyalaridan bo'lib, xususiy hollarda Mittag-Leffler funksiyasini ifodalaydi. H funksiya Fox tomonidan kiritilgan bo'lib Meyer funksiyasining umumlashmasini ifodalaydi. Ushbu ishda [28] adabiyotda berilgan ta'rif va xossalardan foydalanamiz. Bundan tashqari H-funksiya Mellin-Braus tipidagi integral bilan quyidagicha aniqlangan.

$$H^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p)\\(b_1,B_1),\ldots,(b_q,B_q)\end{matrix}\right.\right]=H^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_{\Omega} \vartheta\, z^{-s}ds$$

bunda:

$$\vartheta(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j+B_js)\prod_{k=1}^{n}\Gamma(1-a_k-A_ks)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-B_js)\prod_{k=n+1}^{p}\Gamma(a_k+A_ks)}$$

bu yerda:

$i = (-1)^{\frac{1}{2}}, z \neq 0, va\ z^{-s} = \exp\{-s[\ln|z| + i arg z]\}.$

$0 \le n \le p, 0 \le m \le q, A_e, B_j \in R_+, a_e, B_j \in \mathbb{C}(R), e = 1,2,\ldots,p, j = 1,2,\ldots,q.$

Integral yaqinlashuvchi bo'ladi agarda quyidagi shartlar bajarilsa.

1.) $C > 0, |argz| < \frac{1}{2}\pi c\ va\ z \neq 0;$

2.) $C = 0, pD + Re(E) < -1, argz = 0\ va\ z \neq 0$

bunda

$$C := \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j$$

$$D := \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \; E := \sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j + \frac{p-q}{2}$$

Foks ning H-funksiyasi haqida to'liqroq ma'lumotlar [28] adabiyotda mavjud. Biz ushbu ishda quyida keltiriladigan Foksning H-funksiyasida bazi xossalardanva shakl almashtirishdan foydalanamiz.

H-funksiya xossalari [28;b.11-B]

Quyidagi rekurent formulalar o'rinli :

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p)\\(b_1,B_1),\ldots,(b_{q-1},B_{q-1}),(a_1,A_1)\end{matrix}\right.\right]=H_{p-1,q-1}^{m,n-1}\left[z\left|\begin{matrix}(a_2,A_2),\ldots,(a_p,A_p)\\(b_1,B_1),\ldots,(b_{q-1},B_{q-1})\end{matrix}\right.\right]$$

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_p,A_p)\\(b_q,B_q)\end{matrix}\right.\right]=H_{p,q}^{n,m}\left[\frac{1}{z}\left|\begin{matrix}(1-b_q,B_q)\\(1-a_p,A_p)\end{matrix}\right.\right]$$

1-Xossa.(1.1.1)H-funksiya (a_1,α_1),...,(a_n,α_n);(a_{n+1},α_{n+1}),...,(a_p,α_p)((b_1,β_1),...,(b_m,β_m) va(b_{m+1},β_{m+1}),...,(b_q,β_q) juftliklar to'plamida simetrikdir.

2-Xossa. Agar (a_i,α_i) dan biri (i=$\overline{1,n)}$ dan biri ((b_j,β_j) (j=m+1,q) yoki ((a_i,α_i) dan biri (i=n+1,p) (b_j,β_j) (j=1,m) dan biriga teng bo'lsa)lardan biriga teng bo'lsa u holda H-funksiya biror quyi tartibga kamaydi, ya'ni p,q va n(yoki m) birlik bilan kamayadi. Bunday qisqartirish formulalariga ikkita miso1 keltiramiz.

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q-1},(a_1,\alpha_1)\end{matrix}\right.\right]=H_{p-1,q-1}^{m,n-1}\left[z\left|\begin{matrix}(a_i,\alpha_i)_{2,p}\\(b_j,\beta_j)_{1,q-1}\end{matrix}\right.\right] \quad (1)$$

talab etiladi $n \geq 1$ va $q > m$; va

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_i,\alpha_i)_{1,p-1},(b_1,\beta_1)\\(b_j,\beta_j)_{1,q}\end{matrix}\right.\right]=H_{p-1,q-1}^{m-1,n}\left[z\left|\begin{matrix}(a_i,\alpha_i)_{1,p-1}\\(b_j,\beta_j)_{2,q}\end{matrix}\right.\right], \quad (2)$$

$m \geq 1\ va\ \ p > n\ da$

3- Xossa. Quyidagi munosabat mavjud

$$H_{p,q}^{m,n}\left[\frac{1}{z}\left|\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q},\end{matrix}\right.\right] = H_{p,q}^{m,n}\left[z\left|\begin{matrix}(1-b_j,\beta_j)_{1,p}\\(1-a_i,\alpha_i)_{1,p}\end{matrix}\right.\right] \qquad (3)$$

2.4. Xossa. $k > 0$ uchun

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q},\end{matrix}\right.\right] = kH_{p-1,q-1}^{m,n-1}\left[z^k\left|\begin{matrix}(a_i,k\alpha_i)_{1,p}\\(b_j,k\beta_j)_{1,q}\end{matrix}\right.\right], \qquad (4)$$

munosabat mavjud.

5- Xossa. $\sigma \in C$ uchun

$$z^{\sigma}H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right.\right] = H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_i+\alpha_i,\alpha_i)_{2,p}\\(b_j+\sigma\beta_j,\beta_j)_{1,q-1}\end{matrix}\right.\right], \qquad (5)$$

Keyingi 6 formula (1) H-funksiyaning ta’rifi va Gamma funksiyaning aks etish formulasi

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(z\pi)} \qquad (6)$$

dan kelib chiqadi.

6- Xossa. $c \in C, \alpha > 0$ va $k = 0, \pm 1, \pm 2, \ldots,$ lar uchun

$$H_{p+1,q+1}^{m,n+1}\left[z\left|\begin{matrix}(c,\alpha),(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q},(c+k,\alpha)\end{matrix}\right.\right] = (-1)^k H_{p+1,q+1}^{m+1,n}\left[z\left|\begin{matrix}(a_i,\alpha_i)_{1,p},(c,\alpha)\\(c+k,\alpha),(b_j,\beta_j)_{1,q}\end{matrix}\right.\right], \qquad (7)$$

$$H_{p+1,q+1}^{m+1,n}\left[z\left|\begin{matrix}(a_i,\alpha_i)_{1,p},(c,\alpha)\\(c+k,\alpha),(b_j,\beta_j)_{1,q}\end{matrix}\right.\right] = (-1)^k H_{p+1,q+1}^{m+1,n}\left[z\left|\begin{matrix}(c,\alpha),(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q},(c+k,\alpha)\end{matrix}\right.\right], \qquad (8)$$

munosabat o’rinli.

7-Xossa. $a, b \in C$ sonlar uchun quyidagi munosabatlar mavjud:

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a,0),(a_i,\alpha_i)_{2,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right.\right] = \Gamma(1-\alpha)H_{p-1,q}^{m,n-1}\left[z\left|\begin{matrix}(a_i,\alpha_i)_{2,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right.\right], \qquad (9)$$

bunda $Re(1-\alpha) > 0$ va $n \geq 1$;

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b,0),(b_j,\beta_j)_{2,q}\end{matrix}\right.\right] = \Gamma(b)H_{p,q-1}^{m-1,n}\left[z\left|\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{2,q}\end{matrix}\right.\right], \qquad (10)$$

bunda $Re(b) > 0$ va $m \geq 1$;

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q-1},(b,0)\end{matrix}\right.\right] = \frac{1}{\Gamma(1-b)}H_{p,q-1}^{m,n}\left[z\left|\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q-1}\end{matrix}\right.\right], \qquad (11)$$

bunda $Re(1-b)>0$ va$p>n$.

Differensiallash formulalari

Bu yerda H-funksiyaning differensiallanishi shuningdek xuddi shunday katta tartibli funksiyaning berilishi isbotlangan. H-funksiyaning ta'rifidan biz quyidagi munosabatlarni osongina olamiz:

8- Xossa. $(\frac{d}{dz})^k\{z^w H_{p,q}^{m,n}\left[cz^\sigma|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right]=$

$$=z^{w-k}H_{p+1,q+1}^{m,n+1}\left[cz^\sigma|_{(b_j,\beta_j)_{1,q},(k-w,\sigma)}^{(-w,\sigma),(a_i,\alpha_i)_{2,p}}\right] \quad (12)$$

$$=(-1)z^{w-k}H_{p+1,q+1}^{m+1,n}\left[cz^\sigma|_{(k-w),(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p},(-w,\sigma)}\right], \quad (13)$$

$w,c\in\mathbb{C}$ va $\sigma>0$ uchun;

$w,a,c_j\in\mathbb{C}\ (j=1,\dots k),\sigma>0$ uchun

$$\prod_{j=1}^{k}\left(z\frac{d}{dz}-c_j\right)\{z^w\,H_{p,q}^{m,n}\left[az^\sigma|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right]\}=$$

$$=z^w H_{p+k,q+k}^{m,n+k}\left[az^\sigma|_{(b_j,\beta_j)_{1,q},\,(c_j+1-w,\sigma)_{1,k}}^{(c_j-w,\sigma)_{1,k},(a_i,\alpha_i)_{1,p}}\right], \quad (14)$$

$$=(-1)^k z^w H_{p+k,q+k}^{m+k,n}\left[az^\sigma|_{(c_j+1-w,\sigma)_{1,k},(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p},(\,c_j\,-w,\sigma)_{1,k}}\right] \quad (15)$$

va $c,d\in\mathbb{C},\sigma>0$ lar uchun

$$\left(\frac{d}{dz}\right)^k H_{p,q}^{m,n}\left[(cz+d)^\sigma|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right]=$$

$$\frac{c^k}{(cz+d)^k}H_{p+1,q+1}^{m,n+1}\left[(cz+d)^\sigma|_{(b_j,\beta_j)_{1,q},(k,\sigma)}^{(0,\sigma),(a_i,\alpha_i)_{1,p}}\right], \quad (16)$$

$$\left(\frac{d}{dz}\right)^k H_{p,q}^{m,n}\left[\frac{1}{(cz+d)^\sigma}|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right]=\frac{c^k}{(cz+d)^k}H_{p+1,q+1}^{m,n}\left[\begin{matrix}(a_i,\alpha_i)_{1,p},(1-k,\sigma)\\(1,\sigma),(b_j,\beta_j)_{1,q}\end{matrix}\right], \quad (17)$$

munosabatlar o'rinli.

Boshqa differensiallash formulalari H-funksiyaning bir xil tartibdagi, ammo turli parametrlarga bog'laydi:

9-Xossa. $m\geq 1$ va $\sigma=\beta$, bo'lganda $k>1$ uchun

$$\left(\frac{d}{dz}\right)^k (z^{-\sigma bq}/_q H_{p,q}^{m,n}\left[z^{\sigma}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right] =$$

$$(\frac{\sigma}{\beta_q})^k z^{-k-\sigma bq}/_{\beta_q}\ H_{p,q}^{m,n}\left[z^{\sigma}\Big|_{(b_j,\beta_j)_{1,q-1},(b_q+k,\beta_q)}^{(a_i,\alpha_i)_{1,p}}\right], \qquad (18)$$

$n \geq 1$ va $\sigma = \alpha$, bo'lganda $k > 1$ uchun

$$\left(\frac{d}{dz}\right)^k (z^{-\sigma(1-a_1}/_{\alpha_1} H_{p,q}^{m,n}\left[z^{-\sigma}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right] =$$

$$(-\frac{\sigma}{\beta_q})^k z^{-k-\sigma(1-\alpha_1)}/_{\alpha_1}\ H_{p,q}^{m,n}\left[z^{-\sigma}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_1-k,\alpha_1),(a_i,\alpha_i)_{2,p}}\right], \qquad (19)$$

$p > n$ va $\sigma = \alpha_p$ bo'lganda $k > 1$ uchun

$$\left(\frac{d}{dz}\right)^k (z^{-\sigma(1-a_p}/_{\alpha_p} H_{p,q}^{m,n}\left[z^{-\sigma}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right] =$$

$$(-\frac{\sigma}{\alpha_q})^k z^{-k-\sigma(1-\alpha_p)}/_{\alpha_p}\ H_{p,q}^{m,n}\left[z^{-\sigma}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p-1},(a_p-k,\alpha_p)}\right], \qquad (20)$$

bo'ladi.

(17)-(20) munosabatlar agar 2-xossani inobatga olsak,(21) va (22) ifodadan kelib chiqadi.

10- Xossa. $n \geq 1$ uchun

$$z\frac{d}{dz}\{H_{p,q}^{m,n}\left[z^{\sigma}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right]\} = \frac{\sigma(a_1-1)}{\alpha_1} H_{p,q}^{m,n}\left[z^{\sigma}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right] +$$

$$+\frac{\sigma}{\alpha_1} H_{p,q}^{m,n}\left[z^{p}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_1-1,\alpha_1)(a_i,\alpha_i)_{2,p}}\right]; \qquad (21)$$

$n \leq p - 1$ uchun

$$z\frac{d}{dz}\{H_{p,q}^{m,n}\left[z^{\sigma}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right]\} = \frac{\sigma(a_p-1)}{\alpha_p} H_{p,q}^{m,n}\left[z^{\sigma}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right]$$

$$-\frac{\sigma}{\alpha_p} H_{p,q}^{m,n}\left[z^{\sigma}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{i,p-1},(a_{p-1},\alpha_p)}\right]; \qquad (22)$$

$m \geq 1$ uchun

$$z\frac{d}{dz}\{H_{p,q}^{m,n}\left[z^{\sigma}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right]\}=\frac{\sigma b_1}{\beta_1}H_{p,q}^{m,n}\left[z^{\sigma}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right]$$

$$-\frac{\sigma}{\alpha_p}H_{p,q}^{m,n}\left[z^{\sigma}\Big|_{(b_1+1,\beta_1),(b_j,\beta_j)_{2,q}}^{(a_i,\alpha_i)_{1,\beta}}\right]; \qquad (23)$$

va $m\le q-1$ lar uchun

$$z\frac{d}{dz}\{H_{p,q}^{m,n}\left[z^{\sigma}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,q}}\right]\}=\frac{\sigma bq}{\beta_q}H_{p,q}^{m,n}\left[z^{\sigma}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right]$$

$$+\frac{\sigma}{\beta_q}H_{p,q}^{m,n}\left[z^{\sigma}\Big|_{(b_j,\beta_j)_{1,q-1},(b_q+1,\beta_q)}^{(a_i,\alpha_i)_{1,p}}\right]; \qquad (24)$$

bo’ladi.

(11)-(14) formulalar quyidagi munosabatlar asosida o’rnatiladi:

$$-\alpha_1 s\Gamma(1-a_1-\alpha_1 s)=(a_1-1)\Gamma(1-a_1-\alpha_1 s)+\Gamma(2-a_1-\alpha_1 s);$$

$$-\frac{\alpha_p s}{\Gamma(a_p+\alpha_p s)}=\frac{a_p-1}{\Gamma(a_p+\alpha_p s)}-\frac{1}{\Gamma(a_p-1+\alpha_p s)};$$

$$-\beta_1 s\Gamma(b_1+\beta_1 s)=b_1\Gamma(b_1+\beta_1 s)-\Gamma(b_1+1+\beta_1 s);$$

$$-\frac{\beta_q s}{\Gamma(1-b_q-\beta_q s)}=\frac{b_q}{\Gamma(1-b_q-\beta_q s)}+\frac{1}{\Gamma(-b_q-\beta_q s)}$$

mos ravishda

$$z\Gamma(z)=\Gamma(z+1) \qquad (25)$$

munosabatlar kelib chiqadi.

Qaytarilish (Takrorlanish) munosabatlari va kengaytirish formulalari.

H-funksiyaning chiziqli kombinatsiyasini bir xil m,n,p va q bilan taqdim etamiz, bunda ba’zi a_i va b_i lar $a_i\pm 1$ va $b_j\pm 1$ $(i=1,\dots,p, j=1,\dots,q)$ bilan almashtiriladi. Bunday munosabatlar Srivastava, Gupta va Goyal tomonidan qo’shma munosabatlar deb ataladi. Agar (2.2.15)ni hisobga olsak, (1.1.1) va (1.1.2) da berilgan H-funksiyaning ta’rifi orqali bu munosabatlar to’g’ridan-to’g’ri chiqariladi.

11-Xossa. $m\ge 1$ va $1\le n\le p-1$ uchun

$$(b_1\alpha_p-a_p\beta_1+\beta_1)H_{p,q}^{m,n}\left[z^{\sigma}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right]=\alpha_p H_{p,q}^{m,n}\left[z^{\sigma}\Big|_{(b_1+1,\beta_1)(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right]$$

$$-\beta_1 H_{p,q}^{m,n}\left[z^\sigma\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{i,p-1},(a_{p-1},\alpha_p)}\right]; \qquad (26)$$

$n \geq 1$ va $1 \leq m \leq q-1$ uchun

$$(b_q\alpha_1 - a_1\beta_q + \beta_q)H_{p,q}^{m,n}\left[z^\sigma\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right] = \beta_q H_{p,q}^{m,n}\left[z^\sigma\Big|_{(b_j,\beta_j)_{1,q}}^{(a_1-1,\alpha_1)(a_i,\alpha_i)_{2,p}}\right]$$

$$-\alpha_1 H_{p,q}^{m,n}\left[z^\sigma\Big|_{(b_j,\beta_j)_{1,q-1},(b_q+1,\beta_q)}^{(a_i,\alpha_i)_{i,p}}\right]; \qquad (27)$$

bo'ladi.

Izoh. H-funksiyaning qo'shma munosabatlarining to'la ro'yhatini Buschman [2] da toppish mumkin.

(26) dan quyidagi sonli qator munosabatlariga kelamiz;

12-Xossa. Har qanday $r \in N, m \geq 1$ va $1 \leq n \leq p-1$ uchun,

$$\frac{\beta_1}{\alpha_p}\sum_{k=1}^{r}\frac{1}{\Gamma\left(b_1+k-\frac{(a_p-1)\beta_1}{\alpha_p}\right)}H_{p,q}^{m,n}\left[z\Big|_{(b_1+k-1,\beta_1),(b_j,\beta_j)_{2,q}}^{(a_i,\alpha_i)_{1,p-1},(a_p-1,\alpha_p)}\right] =$$

$$= \frac{1}{\Gamma(b_1+r-\frac{(a_p-1)\beta_1}{\alpha_p})}H_{p,q}^{m,n}\left[z\Big|_{(b_1+r,\beta_1),(b_j,\beta_j)_{2,q}}^{(a_i,\alpha_i)_{1,p}}\right] -$$

$$-\frac{1}{\Gamma(b_1-\frac{(a_p-1)\beta_1}{\alpha_p})}H_{p,q}^{m,n}\left[z\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right]; \qquad (28)$$

$$\frac{\alpha_p}{\beta_1}\sum_{k=1}^{r}\frac{(-1)^k}{\Gamma\left(k-[a_p-1]+\frac{b_1\alpha_p}{\beta_1}\right)}H_{p,q}^{m,n}\left[z\Big|_{(b_1+1,\beta_1),(b_j,\beta_j)_{2,q}}^{(a_i,\alpha_i)_{1,p-1},(a_p-k+1,\alpha_p)}\right] =$$

$$= \frac{(-1)^r}{\Gamma(r-[a_p-1]+\frac{b_1\alpha_p}{\beta_1})}H_{p,q}^{m,n}\left[z\Big|_{(b_1+k,\beta_1),(b_j,\beta_j)_{2,q}}^{(a_i,\alpha_i)_{1,p}}\right] =$$

$$= \alpha_p^r H_{p,q}^{m,n}\left[z\Big|_{(b_1+r+1,\beta_1)(b_j,\beta_j)_{2,q}}^{(a_i,\alpha_i)_{1,p-1},(a_p+r,\alpha_p)}\right] - \beta_1^r H_{p,q}^{m,n}\left[z\Big|_{(b_1+1,\beta_1),(b_j,\beta_j)_{2,q}}^{(a_i,\alpha_i)_{1,p}}\right]; \quad (29)$$

Isbot.(28) formula (26) asosida isbotlangan. Soddalashtirish uchun determinantni belgilaymiz:

$$d(b_1, a_p - 1) = \begin{vmatrix} b_1 & a_p - 1 \\ \beta_1 & \alpha_p \end{vmatrix} = b_1\alpha_p - (a_p - 1)\beta_1 . \quad (30)$$

H −funksiyani H sifatida yozamiz va boshqa barcha parametrlarni o'zgartirmagan holda $H[b_1 + 1]$ va $H[a_p - 1]$ belgisidan foydalanib, b_1 va a_p ni mos ravishda $b_1 + 1$ va $a_p - 1$ bilan almashtiramiz. So'ng (28) va (30) munosabatlar quyidagicha soddalashtiriladi.

$$d(b_1, a_p - 1)H = \alpha_p H[b_1 + 1] - \beta_1 H[a_p - 1] \quad (31)$$

va

$$\frac{\beta_1}{\alpha_p}\sum_{k=1}^{r}\frac{H[b_1+k-1, a_p-1]}{\Gamma\left(b_1+k-[a_p-1]\frac{\beta_1}{\alpha_p}\right)} = \frac{H(b_1+r, a_p)}{\Gamma(b_1+r-[a_p-1]\frac{\beta_1}{\alpha_p})} - \frac{H}{\Gamma(b_1-[a_p-1]\frac{\beta_1}{\alpha_p})} .$$

Endi (29) bilan $b_1$ni $b_1 + k - 1$ ga almashtilib, (15) munosabatdan foydalanib, (8)ni nazarda tutib va (29) dan

$$\frac{\beta_1}{\alpha_p}\sum_{k=1}^{r}\frac{H[b_1+k]}{\Gamma\left(b_1+k-[a_p-1]\frac{\beta_1}{\alpha_p}\right)} = \frac{1}{\alpha_p}\sum_{k=1}^{r}\frac{\alpha_p H[b_1+k] - d(b_1+k-1, a_p-1)H[b_1+k-1]}{\Gamma\left(b_1+k-[a_p-1]\frac{\beta_1}{\alpha_p}\right)} =$$

$$= \sum_{k=1}^{r}\frac{H[b_1+k]}{\Gamma\left(b_1+k-[a_p-1]\frac{\beta_1}{\alpha_p}\right)} - \sum_{k=0}^{r-1}\frac{H[b_1+k]}{\Gamma\left(b_1+k-[a_p-1]\frac{\beta_1}{\alpha_p}\right)}$$

(30) munosabat ham xuddi shunday (32) ning boshqa foydalanilishi orqali isbotlangan. (31) ga kelsak, (32) ga ko'ra chap tomon quyidagi ko'rinishni oladi:

$$\sum_{k=1}^{r}\alpha_p^{k-1}\beta_1^{r-k} d(b_1 + k, a_p + k - 1)H[b_1 + k, a_p + k].$$

(32) ni qo'llab va yana yig'indining tartibini almashtirib

$$\sum_{k=1}^{r}\alpha_p^{k-1}\beta_1^{r-k} d(b_1 + k, a_p + k - 1)H[b_1 + k, a_p + k] =$$

$$= \sum_{k=1}^{r}\alpha_p^{k-1}\beta_1^{r-k}\left(\alpha_p H[b_1 + k + 1, a_p + k] - \beta_1 H[b_1 + k, a_p + k - 1]\right) =$$

.

$$\sum_{k=1}^{r}\alpha_p^k\beta_1^{r-k}H\left[b_1+k+1,a_p+k\right]-\sum_{k=0}^{r-1}\alpha_p^k\beta_1^{r-k}H\left[b_1+k+1,a_p+k\right]=$$

$$=\alpha_p^r H\left[b_1+r+1,a_p+r\right]-\beta_1^r H\left[b_1+1,a_p\right],$$

ni topamiz va (31) isbotlandi.

13-Xossa. Har qanday $r\in N,\ n\geq 1$ va $1\leq m\leq q-1$ uchun quyidagilar o‘rinli:

$$\frac{\alpha_1}{\beta_q}\sum_{k=1}^{r}\frac{(-1)^k}{\Gamma\left(1+k-\alpha_1+b_q\frac{\alpha_1}{\beta_q}\right)}H_{p,q}^{m,n}\left[z\Big|_{(b_j,\beta_j)_{1,q-1},(b_q+1,\beta_q)}^{(a_p-k+1,\alpha_1)(a_i,\alpha_i)_{2,p}}\right]=$$

$$=\frac{1}{\Gamma(1+r-a_1+b_q\frac{\alpha_1}{\beta_q})}H_{p,q}^{m,n}\left[z\Big|_{(b_j,\beta_j)_{1,q}}^{(a_1-r,\alpha_1),(a_i,\alpha_i)_{2,p}}\right]-$$

$$=\frac{1}{\Gamma(1-a_1+b_q\frac{\alpha_1}{\beta_q})}H_{p,q}^{m,n}\left[z\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right]; \qquad (34)$$

$$\frac{\beta_q}{\alpha_1}\sum_{k=1}^{r}\frac{(-1)^k}{\Gamma\left(b_q+k-[\alpha_1-1]\frac{\alpha_1}{\beta_q}\right)}H_{p,q}^{m,n}\left[z\Big|_{(b_j,\beta_j)_{1,q-1},(b_q+k-1,\beta_q)}^{(a_p-1,\alpha_1)(a_i,\alpha_i)_{2,p}}\right]-$$

$$=\frac{(-1)^r}{\Gamma(b_q+r-[a_1-1]\frac{\beta_q}{\alpha_1})}H_{p,q}^{m,n}\left[z\Big|_{(b_j,\beta_j)_{1,q-1},(b_q-r,\beta_q)}^{(a_i,\alpha_i)_{1,p}}\right]-$$

$$=\frac{1}{\Gamma(b_q-[a_1-1]\frac{\beta_q}{\alpha_1})}H_{p,q}^{m,n}\left[z\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right]; \qquad (35)$$

$$\sum_{k=1}^{r}\alpha_1^{k-1}\beta_q^{r-k}[(b_q+k-1)\alpha_1-(a_1+k-2)\beta_q]\,H_{p,q}^{m,n}\left[z\Big|_{(b_j,\beta_j)_{1,q-1},(b_q+k-1,\beta_q)}^{(a_p+k-1,\alpha_1),(a_i,\alpha_i)_{2,p}}\right]$$

$$=\beta_q^r H_{p,q}^{m,n}\left[z\Big|_{(b_j,\beta_j)_{1,q}}^{(a_1-1,\alpha_1),(a_i,\alpha_i)_{2,p}}\right]-\alpha_1^r H_{p,q}^{m,n}\left[z\Big|_{(b_j,\beta_j)_{1,q-1},(b_q+r,\beta_q)}^{(a_1+r-1,\alpha_1),(a_i,\alpha_i)_{2,p}}\right] \qquad (36)$$

Shuningdek, H-funksiya uchun ko’paytirish teoremalari deb nomlanuvchi kengayishlarni ham beramiz:

1-Teorema:.$\lambda\in\mathbb{C}$ va (1.1.6) shartlari qanoatlantirsin. U holda quydagi munosabatlar mavjud:

$$H_{p,q}^{m,n}\left[\lambda z\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{2,p}}\right] = \lambda^{b_1/\beta_1} \sum_{k=1}^{\infty} \frac{(1-\lambda^{1/\beta_1})^k}{k!} H_{p,q}^{m,n}\left[z\Big|_{(b_1+k,\beta_1),(b_j,\beta_j)_{2,q},}^{(a_i,\alpha_i)_{1,p}}\right] \qquad (37)$$

bunda $m > 0, \left|\lambda^{1/\beta_1} - 1\right| < 1$ va $m > 1$.

$$H_{p,q}^{m,n}\left[\lambda z\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right] = \lambda^{b_q/\beta_q} \sum_{k=0}^{\infty} \frac{(\lambda^{1/\beta_q}-1)^k}{k!} H_{p,q}^{m,n}\left[z\Big|_{(b_j,\beta_j)_{2,q},(b_q+k,\beta_q)}^{(a_i,\alpha_i)_{1,p}}\right], \qquad (38)$$

bunda $q > m$ va $\left|\lambda^{1/\beta_q} - 1\right| < 1$;

$$H_{p,q}^{m,n}\left[\lambda z\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right] = \lambda^{(a_1-1)/\alpha_1} \sum_{k=1}^{\infty} \frac{(1-\lambda^{-1/\alpha_1})^k}{k!} H_{p,q}^{m,n}\left[z\Big|_{(b_j,\beta_j)_{1,q}}^{(a_1-k,\alpha_1),(a_i,\alpha_i)_{2,p}}\right] \qquad (39)$$

bunda $n > 0$ va $Re\left(\lambda^{1/\alpha_1}\right) > \frac{1}{2}$;

$$H_{p,q}^{m,n}\left[\lambda z\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right] =$$
$$\lambda^{(a_p-1)/\alpha_p} \sum_{k=0}^{\infty} \frac{(\lambda^{-1/\alpha_p}-1)^k}{k!} H_{p,q}^{m,n}\left[z\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p-1},(a_p-k,\alpha_p),}\right] \qquad (40)$$

bunda $p > n$ va $Re\left(\lambda^{1/\alpha_p}\right) > \frac{1}{2}$;

Isbot. 1-Teoremaga ko‘ra $z^{b_1} H_{p,q}^{m,n}\left[z^{\beta_1}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right]$ funksiya $z \in \mathbb{C}$ uchun analitikdir. $(z \neq 0)$

Shuning uchun $|\eta| < |z|$ uchun Teylor formulasi

$$(z+\eta)^{-b_1} H_{p,q}^{m,n}\left[(z+\eta)^{\beta_1}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right] =$$

$$=\sum_{k=0}^{\infty} \frac{\eta^k}{k!}\left(\frac{d}{dz}\right)^k \{(z+\eta)^{-b_1} H_{p,q}^{m,n}\left[(z+\eta)^{\beta_1}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right]\}$$ mavjud.

(2..2.7) bilan $\sigma = \beta_1$ ni qo‘llab,

$$(z+\eta)^{-b_1} H_{p,q}^{m,n}\left[(z+\eta)^{\beta_1}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right] =$$

$$=\sum_{k=0}^{\infty} \frac{(-\eta)^k}{k!} \{z^{-b_1-k} H_{p,q}^{m,n}\left[z^{\beta_1}\Big|_{(b_1+k,\beta_1),(b_j,\beta_j)_{2,q}}^{(a_i,\alpha_i)_{1,p}}\right]\}$$

ni olamiz.

$\eta = z(1-\lambda^{1/\beta_1})$ va $\left|1-\lambda^{1/\beta_1}\right| < 1$ uchun

$$H_{p,q}^{m,n}\left[\lambda z^{\beta_1}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right] = \lambda^{b_1/\beta_1}\sum_{k=0}^{\infty}\frac{(1-\lambda^{1/\beta_1})^k}{k!}H_{p,q}^{m,n}\left[z^{\beta_1}\Big|_{(b_1+k,\beta_1),(b_j,\beta_j)_{2,q}}^{(a_i,\alpha_i)_{1,p}}\right]$$

bo‘ladi. (12) da z^{β_1} ni z orqali almashtiramiz. (13)-(15) munosabatlar shunday isbotlangan.

Foks funksiyaning asimptotikalari haqida teorema

Ta’rif va belgilashlar.

$$H_{p,q}^{m,n}(z) \equiv H_{p,q}^{m,n}\left[z\left|\begin{matrix}[a_p,A_p]\\ [b_q,B_q]\end{matrix}\right.\right] \equiv H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,A_1),\dots(a_p,A_p)\\ (b_1,B_1),\dots,(b_q,B_q)\end{matrix}\right.\right]$$: H-funksiya,

Bu yerda $0 \le n \le p, 0 \le m \le q, A_l, B_j \in \mathbb{R}_+, a_l, b_j \in \mathbb{C}(\mathbb{R}), l =, \dots, j = 1, \dots, q$.

1-Ta’rif. Mellin-Barnes ma’nosidagi ushbu

$$H_{p,q}^{m,n}(z) = \frac{1}{2\pi i}\int_{\Omega} H_{p,q}^{m,n}(s)z^{-s}ds, \qquad (1)$$

integralga H-funksiya deyiladi, bu yerda $\Omega - \mathbb{C}$ kompleks tekislikdagi trayektoriya, $i = \sqrt{-1}, z \ne 0, z^{-s} = e^{-s(\ln|z|+iargz)}$,

$$H_{p,q}^{m,n}(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j+B_js)\prod_{l=1}^{n}\Gamma(1-a_l-A_ls)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-B_js)\prod_{l=n+1}^{p}\Gamma(a_l+A_ls)} = \frac{A(s)B(s)}{C(s)D(s)} \qquad (2)$$

Agar $m = 0$ bo’lsa, u holda

$$H_{p,q}^{0,n}\left[z\left|\begin{matrix}[a_p,A_p]\\ [b_q,B_q]\end{matrix}\right.\right] = 0$$

deb hisoblanadi.$[8.3, 528-531\ bb]^1$. Shuning uchun, biz bundan keyin $1 \le m \le q$ deb qaraymiz. (0.2) ko’paytmada quyidagi trival munosabatlar o’rinli

$$n = 0 \Leftrightarrow B(s) = 1, m = 1 \Leftrightarrow C(s) = 1, n = p \Leftrightarrow D(s) = 1. \qquad (3)$$

U holda (0.1) integral

$$H_{p,q}^{m,n}(z) = \frac{1}{2\pi i}\int_{\Omega}\prod_{j=1}^{m}\Gamma(b_j+B_js)\,z^{-s}ds,$$

ko’rinishni oladi. Agar, $m = 1, b_1 = 0, B_1 = 1$ deb olsak, u holda oxirgi integral

$$H_{0,1}^{1,0}(z) = \frac{1}{2\pi i}\int_{\Omega}\Gamma(s)z^{-s}ds, \qquad (4)$$

ko'rinishni oladi.

(1)ko'rinishdagi integralni hisoblashning asosi Ω konturning tanlanishida. Shu maqsadda Ω komturni $\Gamma(b_j + B_j s)$ funksiyaning

$$b_{jv} = -\frac{b_j+v}{B_j}, \quad j = 1, \dots, m, v = 0,1,2, \dots \tag{5}$$

va $\Gamma(1 - a_l - A_{ls})$ ning

$$a_{lk} = \frac{1-a_l+k}{A_l}, \; l = 1, \dots, n, k = 0,1,2, \dots, \tag{6}$$

qutblarini ajralish yo'li orqali integrallanadi, ya'ni

$$b_{jv} \neq a_{lk}, \quad j = 1, \dots, m, l = 1, \dots, n, v, k = 0,1,2, \dots . \tag{7}$$

(7) ni hisobga olsak, Ω kontur mavjud. Keyin hisoblashlarda kontur mavjud deb qaraymiz. Ω kontur uchun, quyidagi uch hol ro'y berishi mumkin:

i.1) $\Omega_{-\infty}$, i.2) $\Omega_{+\infty}$, i.3) $\Omega_{i\gamma\infty}$.

Bu konturlar quyidagicha ta'riflanadi:

1) $\Omega_{-\infty,}$ $\Gamma\left(b_j - B_{js}\right), j = 1, \dots, m$ funksiyaning barcha qutb nuqtalarni o'z ichiga olib, $-\infty$ da tugaydi va yo'nalish musbat bo'lib, $\Gamma(1 - a_l - A_{ls}), l = 1, \dots, n$ ning birorta ham qutb nuqtalarini o'zida saqlamaydi. Bu holda (0.1) integral quyidagi shartlar bajarilganda yaqinlashuvchi bo'ladi:

i.1.1)$\mu > 0$ va $z \neq 0$, i.1.2) $\mu = 0$ va $0 < |z| < \beta$,

i.1.3) $\mu = 0, |z| = \beta$ va $Re(\delta) < -1$, (8)

bu yerda

$$\beta: \prod_{l=1}^{q} A_l^{-A_l} \times \prod_{j=1}^{q} B_j^{B_J} \tag{9}$$

$$\mu := \sum_{j=1}^{q} B_j - \sum_{l=1}^{p} A_l \tag{10}$$

$$\delta := \sum_{j=1}^{q} \beta_j - \sum_{l=1}^{p} a_l + \frac{p-q}{2}. \tag{11}$$

2) $\Omega_{+\infty}$: $\Gamma(1 - a_l - A_{ls}), l = 1, \dots, n$ funksiyaning barcha qutb nuqtalarini o'z ichiga olib, $+\infty$ da tugaydi va yo'nalish manfiy bo'lib, $\Gamma\left(b_j + B_{js}\right), j = 1, \dots, m$ ning birorta ham qutb nuqtalarini o'zida saqlamaydi. Bu holda (0.1) integral quyidagi shartlar bajarilganda yaqinlashuvchi bo'ladi:

2.1)$\mu < 0$ va $z \neq 0$, $i.$2.2)$\mu = 0$ va $|z| > \beta$, $i.$2.3) $i.$1.3. (12)

3) $\Omega_{i\gamma\infty}$: $\Gamma\left(b_j + B_{js}\right), j = 1, \dots, m$ funksiyaning barcha qutb nuqtalari $\Gamma(1 - a_l - A_{ls}), l = 1, \dots, n$ ning qutb nuqtalaridan ajratadi va Ω kontur $\gamma - i\infty$ dan boshlanib $\gamma + i\infty$ nuqtada tugaydi. Bu holda (0.1) integral quyidagi shartlar bajarilganda yaqinlashuvchi bo'ladi:

$$3.1)\ \alpha > 0, |argz| < \frac{\pi\alpha}{2},\ \ z \neq 0, \qquad (13)$$

va

$$3.2)\ \alpha = 0,\ \gamma\mu + Re(\delta) < -1, |argz| = 0,\ z \neq 0, \qquad (14)$$

bu yerda

$$\alpha := \sum_{l=1}^{n} A_l - \sum_{l=n+1}^{p} A_l + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j. \qquad (15)$$

***H* −funksiya uchun mavjudlik shartlari.**

1-Teorema. z ning analitik funksiyasi bo'lgan H-funksiya quyidagi hollarda mavjud:

c.1)$q \geq 1, \mu > 0$ bo'lsa u holda $\forall z \neq 0$ lar uchun H-funksiya mavjud; (1)

c.2)$q \geq 1, \mu = 0$ bo'lsa, u holda $0 < |z| < \beta$ lar uchun funksiya mavjud; (2)

c.3)$q \geq 1, \mu = 0\ Re(\delta) < -1$, bo'lsa, u holda$|z| = \beta$lar uchun H-f.m.; (3)

c.4)$p \geq 1, \mu < 0$ bo'lsa u holda barcha $z \neq 0$ lar uchun H-f.m.; (4)

c.5)$p \geq 1, \mu = 0$ bo'lsa $|z| > \beta$ lar uchun H-funksiya mavjud; (5)

c.6)$p \geq 1, \mu = 0\ Re(\delta) < -1$ bo'lsa, u holda $|z| = \beta$ lar uchun H-f.m.; (6)

c.7)$\alpha > 0, |argz| < \frac{\pi\alpha}{2}$ bo'lsa, u holda ixtiyoriy $z \neq 0$ lar uchun H-f.m.; (7)

c.8)$\alpha = 0, \gamma\mu + Re(\delta) < -1$ bo'lsa, u holda $z \neq 0$ va $argz \neq 0$lar uchun H (8)

Isbot. Mavjudlik shartlarini olish uchun $H_{p,q}^{m,n}(s)$ funksiyaning cheksizlikdagi holatini o'rganib (0.1) integralning yaqinlashish holatlari o'rganiladi. Natijalarni olishda $\Gamma(z), z = x + iy, x, y \in \mathbb{R}$ funksiyaning asimptotikasidan va o'qlardagi asimptotikalaridan foydalanib topiladi, ya'ni

$$|\Gamma(x + iy)| \sim \sqrt{2\pi}|x|^{x-\frac{1}{2}} e^{-x-\pi[1-sign(x)]^{y}/2} \quad (|x| \to \infty) \qquad (9)$$

va

$$|\Gamma(x + iy)| \sim \sqrt{2\pi}|x|^{x-\frac{1}{2}} e^{-x-{\pi y}/2} \quad (|y| \to \infty) \qquad (10)$$

Mos ravishda yuqoridagi baholar quyidagi Stirling formulasidan foydalanib ko'rsatish mumkin:

$$\Gamma(z) \sim \sqrt{2\pi}\, z^{z-\frac{1}{2}}\, e^{-z}\,, z \to \infty. \qquad (11)$$

1-Teoremani isbotlash uchun dastlab quyidagi ikkita lemmani to'g'riligini ko'rsatamiz. Bu lemmalar quyidagi

$$l_1 := \{t + i\varphi_1 : t \in \mathbb{R}\},\; l_2 := \{t + i\varphi_2 : t \in \mathbb{R}\}, l_\gamma := \{\gamma + it : t \in \mathbb{R}\}, \qquad (12)$$

bu yerda $\varphi_1, \varphi_2, \gamma \in \mathbb{R}$, chiziqlarning cheksizlikdagi asimptotikalarini topish uchun qo'llaniladi.

1-Lemma. $t, \sigma \in \mathbb{R}$ lar uchun quyidagi asimptotik baholar o'rinli:

$$\left|H_{p,q}^{m,n}(t + i\sigma)\right| \sim A\left(\frac{e}{t}\right)^{-\mu t} \beta^t t^{Re(\delta)}, \quad t \to +\infty \qquad (13)$$

bu yerda

$$A := (2\pi)^{c^*} e^{q-m-n} \frac{\prod_{j=1}^{q} B_j^{Re(b_j)-\frac{1}{2}} e^{-Re(b_j)} \prod_{l=1}^{n} e^{\pi(\sigma A_l + Im(a_l))}}{\prod_{l=1}^{p} A_l^{Re(a_l)-\frac{1}{2}} e^{-Re(a_l)} \prod_{j=m+1}^{q} e^{\pi(\sigma B_j + Im(b_j))}}, \qquad c^* :=$$

$$m + n - \frac{p+q}{2}, \qquad (14)$$

va

$$\left|H_{p,q}^{m,n}(t + i\sigma)\right| \sim B\left(\frac{e}{|t|}\right)^{\mu|t|} \beta^{-|t|} t^{Re(\delta)}, \quad t \to +\infty, \qquad (15)$$

bu yerda

$$B := (2\pi)^{c^*} e^{q-m-n} \frac{\prod_{j=1}^{q} B_j^{Re(b_j)-\frac{1}{2}} e^{-Re(b_j)} \prod_{l=n+1}^{p} e^{\pi(\sigma A_l + Im(a_l))}}{\prod_{l=1}^{p} A_l^{Re(a_l)-\frac{1}{2}} e^{-Re(a_l)} \prod_{j=1}^{m} e^{\pi(\sigma B_j + Im(b_j))}}, \qquad (16)$$

va β, μ, δ o'zgarmaslar (0.9)-(0.11) da aniqlangan.

2 -Lemma. $t, \sigma \in \mathbb{R}$ lar uchun quyidagi asimptotik baho o'rinli:

$$\left|H_{p,q}^{m,n}(\sigma + it)\right| \sim C|t|^{\mu\sigma + Re(\delta)} e^{-\frac{\pi}{2}(|t|\alpha + Im(\omega)sign(t)} \quad |t| \to +\infty, \qquad (17)$$

baho $\mathbb{R}$ dagi biror chegaralangan intervaldagi σ larda tekis bajariladi, bu yerda

$$C := (2\pi)^{c^*} e^{-c^* - \mu\sigma - Re(\delta)} \prod_{l=1}^{p} A_l^{\frac{1}{2} - a_l} \prod_{j=1}^{q} B_j^{b_j - \frac{1}{2}}, \qquad \omega := \sum_{l=1}^{n} a_l - \sum_{l=n+1}^{p} a_l +$$

$$\sum_{j=1}^{m} b_j - \sum_{j=m+1}^{q} b_j. \qquad (0.18)$$

1va 2-Lemmalarning isboti (2), (6) va (7) dan kelib chiqadi. 1- va 2-Lemmalardan foydalanib (0.1) integral ostidagi funksiyani cheksizlikda quyidagi asimptotik munosabatlar o'rinli bo'lishini ko'rsatish qiyin emas:

$t \to -\infty$ da

$$\left|H_{p,q}^{m,n}(s)z^{-s}\right| \sim \boldsymbol{B}_k e^{\varphi_k argz} (\frac{e}{|t|})^{\mu|t|} (\frac{|z|}{\delta})^{|t|} |t|^{Re(\delta)}, (s = t + i\varphi_k \in l_k; k = 1,2) \quad (19)$$

$t \to -\infty$ da

$$\left|H_{p,q}^{m,n}(s)z^{-s}\right| \sim \boldsymbol{A}_k e^{\varphi_k argz} (\frac{e}{|t|})^{\mu t} (\frac{|\delta|}{z})^{t} |t|^{Re(\delta)}, (s = t + i\varphi_k \in l_k; k = 1,2) \quad (20)$$

va $t \to \infty$ da

$$\left|H_{p,q}^{m,n}(s)z^{-s}\right| \sim \boldsymbol{C}_l e^{-\gamma \ln|z| + \frac{\pi}{2} Im(\omega) sgn(t)} |t|^{\mu\gamma + Re(\delta)} |t|^{\mu\gamma + Re(\delta)} e^{\frac{\pi}{2}|t|\alpha + targz}, \left(s = \gamma + it \in l_\gamma\right) \quad (21)$$

Bu yerda A_1 va A_2, B_1 va B_2 mos ravishda (14) va (16) dan σ ni $\varphi_k, k = 1,2$ larga almashtirish orqali aniqlanadi.

Ba'zi maxsus funksiyalarni Foks funksiyasi yordamida olish.

Misollardan na'munalar.

H-funksiya o'z ichiga olgan soda misollardan eksponensial, Mittag-Leffler vaumumlashgan Mittag-Leffler funksiyalardir.

1-misol. Quyidagi

$$f(z) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \Gamma(s) z^{-s} ds, |argz| < \frac{\pi}{2}, z \neq 0 \quad (1)$$

integrallarni hisoblang, bu yerda integrallash yo'li $\Gamma(s)$ ning $s = -v, v = 0,1,2, \ldots$ qutb nuqtalaridan o'ngda joylashgan $Re(s) = \gamma, \gamma > 0$ to'g'ri chiziq bo'ylab olinadi, natijani H-funksiya orqali ifodalang

Yechish.Qoldiqlar nazaryasi bo'yicha hisoblaymiz

$$f(z) = \sum_{v=0}^{\infty} \lim_{s \to -v} (s + v)\Gamma(s) z^{-s} = \sum_{v=0}^{\infty} \lim_{s \to -v} \frac{(s+v)(s+v-1)\ldots s}{(s+v-1)\ldots s} \Gamma(s) z^{-s} =$$

$$\sum_{v=0}^{\infty} \lim_{s \to -v} \frac{\Gamma(s+v+1)}{(s+v-1)\ldots s} z^{-s} = \sum_{v=0}^{\infty} \frac{(-1)^v}{v!} = e^{-z}. \quad (2)$$

(0.4) ga ko'ra (2.1) integral H-funksiya orqali quyidagicha

$$e^{-z} = H_{0,1}^{1,0}\left[z|_{(0,1)}\right], \qquad (3)$$

ifodalanadi.

2-misol. Quyidagini isbotlang

$$(1-z)^{-a} = \frac{1}{2\pi i\Gamma(a)}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(-s)\Gamma(s+a)(-z)^s ds, \quad |\arg(-z)| < \pi, \qquad (4)$$

Bu yerda $0 < Re(\gamma) < Re(a)$ va kontur $\Gamma(-s)$ ning qutb nuqtalaridan $\Gamma(s+a)$ qutb nuqtalarini ajratib, $Re(s) = \gamma$ to'g'ri chiziq bo'ylab harakatlanadi.

Yechim.3.1-Misolni yechish jarayonini bu misolda ham qo'llaymiz:

$$\frac{1}{2\pi i\Gamma(a)}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(s+a)(-z)^s ds = \frac{1}{\Gamma(a)}\sum_{v=0}^{\infty}\frac{(-1)^v\Gamma(v+a)}{v!}(-z)^v = \sum_{v=0}^{\infty}\frac{(a)_v}{v!}z^v =$$

$$F_0(a;;z) = (1-z)^{-a}, \quad |z| < 1, \qquad (5)$$

bu yerda $(a)_k, a \in \mathbb{C}$, $k \in \mathbb{N}_0$ Pochhammer simvoli yoki faktorialning o'zgarishi bo'lib, $\Gamma(a)$ mavjud bo'lganda u quyidagicha aniqlangan

$$a_0 = 1, \quad (a)_k = a(a+1)\dots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}, a \neq 0. \qquad (6)$$

(6) dan foydalanib

$$(1-z)^{-a} = \frac{1}{\Gamma(a)}H_{1,1}^{1,1}\left[-z|_{(0,1)}^{(1-a,1)}\right] \qquad (7)$$

olamiz.

3-Misol. Mellin-Barnes integralini hisoblang

$$f(z) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(s)\Gamma(1-s)}{\Gamma(1-\alpha s)}(-z)^{-s}ds, \quad |argz| < \pi, \qquad (8)$$

bu yerda $\alpha \in \mathbb{R}_+$ va $f(z)$-Mittag-Leffler funksiyasi ekanligini ko'rsating.

Yechim. Quyidagiga egamiz

$$f(z) = \sum_{v=0}^{\infty}\lim_{s\to -v}\frac{(s+v)\Gamma(s)\Gamma(1-s)}{\Gamma(1-\alpha s)}(-z)^{-s} = \sum_{v=0}^{\infty}\frac{z^v}{\Gamma(av+1)} = E_\alpha(z) =$$

$$H_{1,2}^{1,1}\left[-z|_{(0,1),(0,\alpha)}^{(0,1)}\right]. \qquad (9)$$

4-Misol.Quyidagi Mellin-Barnes integrali umumlashgan $E_{\alpha,\beta}(z)$-Mittag-Leffler funksiyasini ifodalanishini ko'rsating:

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\beta-\alpha s)}(-z)^{-s}ds, \quad |argz| < \pi, \qquad (10)$$

bu yerda $\alpha \in \mathbb{R}_+, \beta \in \mathbb{C}, Re(\beta) > 0$.

Yechim. Qoldiqlar nazariyasidan foydalanib

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\beta-\alpha s)}(-z)^{-s}ds =$$

$$=\sum_{v=0}^{\infty}\lim_{s\to -v}\frac{(s+v)\Gamma(s)\Gamma(1-s)}{\Gamma(\beta-\alpha s)}(-z)^{-s}=\sum_{v=0}^{\infty}\frac{z^{v}}{\Gamma(av+\beta)}=$$

$$E_{\alpha,\beta}(z)=H_{1,2}^{1,1}\left[-z\Big|_{(0,1),(1-\beta,\alpha)}^{(0,1)}\right]. \tag{11}$$

ekanligini topamiz.

<u>5- Misol:</u> $H_{0,1}^{1,0}\left[z|_{(0,1)}\right]=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(s)z^{-s}ds=$

$$=\sum_{\nu=0}^{\infty}\lim_{s\to-\nu}(s+\nu)\ \Gamma(s)z^{-s}$$

$$=\sum_{\nu=0}^{\infty}\lim_{s\to-\nu}\frac{(s+\nu)\cdot(s+\nu-1)\cdot(s+\nu-2)\dots s\cdot\ \Gamma(s)z^{-s}}{(s+\nu-1)\cdot(s+\nu-2)\dots s}=$$

$$=\sum_{\nu=0}^{\infty}\lim_{s\to-\nu}\frac{\Gamma(s+\nu+1)z^{-s}}{(s+\nu-1)\cdot(s+\nu-2)\cdot\dots\cdot s}=\sum_{\nu=0}^{\infty}\lim_{s\to-\nu}\frac{(-1)^{\nu}}{\nu!}z^{\nu}=e^{-z}$$

<u>5-misol:</u> $H_{1,2}^{1,1}\left[z\Big|_{(0,1);(0,a)}^{(0,1)}\right]=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(s)\Gamma(1-s)ds}{\Gamma(1-as)}=$

$$\left|\begin{matrix} a\in R_{+} & \nu=0,1,2\dots \\ 1-s=-\nu & 1-as=-\nu \\ s=\nu+1 & s=\dfrac{(1+\nu)}{a}\end{matrix}\right|=\sum_{\nu=0}^{\infty}\lim_{s\to-\nu}\frac{(s+\nu)\Gamma(s)\Gamma(1-s)(-z)^{-s}}{\Gamma(1-as)}=$$

$$\sum_{\nu=0}^{\infty}\lim_{s\to-\nu}\frac{(s+\nu)\cdot(s+\nu-1)\cdot\dots\cdot s\Gamma(s)\Gamma(1-s)(-z)^{-s}}{(s+\nu-1)\cdot\dots\cdot s\Gamma(1-as)}=$$

$$\sum_{\nu=0}^{\infty}\frac{\Gamma(1+\nu)(-z)^{\nu}}{(-1)^{\nu}\nu\Gamma(1+a\nu)}=\sum_{\nu=0}^{\infty}\frac{z^{\nu}}{\Gamma(1+a\nu)}=E_{a}(z)$$

Misol: $H_{1,2}^{1,1}\left[z\Big|_{(0,1);(1-\beta,\alpha)}^{(0,1)}\right]=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(s)\Gamma(1-s)ds}{\Gamma(1-1+\beta-\alpha s)}=$

$$\left|\begin{array}{ccc} \alpha\in R_+ & \beta\in C & Re(\beta)>0 \\ -s=\nu & 1-s=-\nu & \beta-\alpha s=-\nu \\ s=-\nu & s=1+\nu & s=\frac{\beta+\nu}{\alpha} \end{array}\right|$$

$$=\sum_{\nu=0}^{\infty}\lim_{s\to-\nu}\frac{(s+\nu)\cdot(s+\nu-1)\cdot\ldots\cdot s\Gamma(s)\Gamma(1-s)(-z)^{-s}}{(s+\nu-1)\cdot\ldots\cdot s\Gamma(\beta-\alpha s)}$$

$$=\sum_{\nu=0}^{\infty}\lim_{s\to-\nu}\frac{\Gamma(s+\nu+1)\Gamma(1-s)(-z)^{-s}}{(s+\nu-1)\cdot\ldots\cdot s\Gamma(\beta-\alpha s)}$$

$$=\sum_{\nu=0}^{\infty}\frac{\Gamma(1+\nu)(-z)^{\nu}}{(-1)^{\nu}\nu!\,\Gamma(\beta+\alpha\nu)}=\sum_{\nu=0}^{\infty}\frac{z^{\nu}}{\Gamma(\beta+\alpha\nu)}=E_{\alpha,\beta}(z)$$

6- misol: $H_{1,2}^{1,1}\left[-x\Big|_{(0,1);(1-\beta,\alpha)}^{(1-\rho,1)}\right]=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(s)\Gamma(1-1+\rho-s)(-x)^{-s}ds}{\Gamma(1-1+\beta-\alpha s)}=$

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(s)\Gamma(\rho-s)(-x)^{-s}ds}{\Gamma(\beta-\alpha s)}=\quad\left|\begin{array}{l} s=-\nu \\ s=\rho+\nu \\ s=\dfrac{\beta+\nu}{\alpha} \end{array}\right|=$$

$$\sum_{\nu=0}^{\infty}\lim_{s\to-\nu}\frac{\Gamma(s+\nu+1)\Gamma(\rho-s)(-z)^{-s}}{(s+\nu-1)\cdot\ldots\cdot s\Gamma(\beta-\alpha s)}=\sum_{\nu=0}^{\infty}\frac{\Gamma(1)\Gamma(\rho+\nu)(-x)^{\nu}}{(-1)^{\nu}\Gamma(\beta+\alpha\nu)}=$$

$$\sum_{\nu=0}^{\infty}\frac{\Gamma(\rho+\nu)x^{\nu}\Gamma(\rho)}{\Gamma(\beta+\alpha\nu)\nu!\,\Gamma(\rho)}=\Gamma(\rho)E_{\alpha,\beta}^{\rho}(x)$$

II-BOB. INTEGRAL ALMASHTIRISHLAR

Umuman olganda, tenglama (tenglamalar sistemasi) masalaning geometriyasidan va/yoki muammoning fizikasidan kelib chiqadigan shartlar, shu jumladan chegaraviy shartlar bilan birga keladi. va ko'p hollarda, vaqtga bog’liq o'zgaruvchi mavjud bo'lganda, dastlabki holat (boshlang'ich tezlik).

Tenglamalarga nisbatan, o'ziga xos tasniflash bilan bog'liq ko'plab imkoniyatlar mavjud, chunki ular qatorida differensial yoki integral yoki integro differentsial bo'lishi mumkin. Bundan tashqari, u chiziqli yoki chiziqli bo'lmagan, bir jinsli yoki bir jinsli bo'lmagan, oddiy yoki xususiy hosilali bo'lishi mumkin. Tenglama turi ma'lum bo'lgach, shartlarning navbati keladi. Bu erda ham bizda bir nechta imkoniyatlar bor, lekin biz o'zimizni dastlabki shartlar va chegara shartlari bilan cheklaymiz. Keyin, masalani, ya'ni tenglama va shartlarni hisobga olgan holda, biz uni yechishning eng yaxshi metodologiyasini izlashimiz kerak. Bu yerda variantlar oralig'i deyarli nazoratsiz ravishda o'sib boradi, chunki har bir tenglama turiga mos usullar mavjud. Metodologiyaga kelsak, biz boshidan farq qilamiz, chunki biz faqat analitik usullar bilan shug'ullanamiz, hisoblash usullari bilan bog'liq emas. Bundan tashqari, Laplas, Furye va Mellin deb ataladigan integral almashtirishlar metodologiyasi e'tibor qaratimiz, chunki ular ma'lum bir masalalarni yechish uchun ishlatiladi.

Ushbu bob quyidagicha tartibga solingan: biz integral almashtirishlar metodologiyasi deb ta'riflagan Furye konvertatsiyasi, keyin Laplas konvertatsiyasi va nihoyat Mellin konvertatsiyasi uchun yadroni, aniqlash oralig'ini va funksiyalar sinfini aniqlaymiz. Ushbu o'zgarishlarning har biri uchun biz ba'zi xossalarni, shuningdek, tegishli teskari almashtirishni taqdim etamiz. Bob davomida biz ba'zi misollarni keltiramiz.

Furyening integral almashtirishi.

Dastlab bir o’zgaruvchili $\varphi(t), t\epsilon R$ funksiyaning Furye almashtirishidan boshlaymiz.

Ta'rif. *Haqiqiy o'zgaruvchili* $\varphi(t) \epsilon L_1$ *(-∞, +∞) funksiyaning Fure almashtirishi deb*

$$F[\varphi](\lambda) = \widehat{\varphi(\lambda)} := \int_{-\infty}^{+\infty} e^{i\lambda t}\varphi(t)dt, \quad \lambda \in R \qquad (1)$$

Integralga aytiladi. $g \in L_1(-\infty, +\infty)$ *funksiyaning teskari Fure almashtirishi deb esa*

$$F^{-1}[g](A) = \frac{1}{2\pi i}\hat{g}(-\lambda) := \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\lambda t}g(t)dt, D \in R \qquad (2)$$

integralga aytiladi.

$\varphi, g \in L_1(R)$–funksiyalar uchun (1) va (2) integrallar absolyut yaqinlashuv bo'ladi. Xususan yaxshi $\xi\ \varphi, g$ funksiyalar uchun yuqoridagi almashtirishlar biri ikkinchisiga teskaridir.

$$F^{-1}F[\varphi] = \varphi,\ FF^{-}[g] = g \qquad (3)$$

va quyidagi oddiy munosabat o'rinli bo'ladi.

$$FF[\varphi] = \varphi(-x) \qquad (4)$$

Agar $\varphi(t) \in L_1(R)$ bo'lsa u holda $F[\varphi](\lambda)$-uzluksiz chegaralangan funksiya bo'ladi va Riman- Lebeg teoremmasiga ko'ra $|\lambda| \to \infty$ da $F[\varphi](\lambda) \to 0$

$$\lim_{|\lambda|\to\infty} \int_{-\infty}^{+\infty} e^{i\lambda t}\varphi(t)dt = 0, \quad \varphi(t) \in L_1(R) \qquad (5)$$

$F[\varphi](\lambda)$ funksiyaning cheksizlikdagi kamayish tartibi $\varphi(t)$ funksiyaning silliqligiga bog'liq. Bu bog'liqlik quyidagi tenglik yordamida berilgan:

$$F[\varphi^k(t)] = (-i\lambda)^k \mathrm{F}[\varphi](\lambda), \qquad (k \in N) \qquad (6)$$

va

$$\frac{d^k}{d\lambda^k}\big(F[\varphi](\lambda)\big) = (it)^k F[\varphi(t)](\lambda) \quad (k \in N) \qquad (7)$$

Bu tengliklar yetarlicha silliq φ funksiyalar uchun o'rinli misol uchun

$$\varphi^{(1)}(t) \in L_1(R)\ (j = 0, k), \varphi(t) \in C(R)$$

Ikkita h va φ funksiyalarni Fure o'rama operatori

$$h * \varphi = (h * \varphi)(\lambda) = \int_{-\infty}^{+\infty} h(\lambda - t)\varphi(t)dt, \lambda \in R \qquad (8)$$

Integral yordamida aniqlanadi va $h * \varphi = \varphi * h$ komunikativlik xossasi o'rinli. O'rama operatorning $L_p(R)$ sinfda chegaralanganligi Yupe teoremmasi orqali berilgan.

1-teorema. *Agar* $h(t) \in L_1(R)$ *va* $\varphi(t) \in L_p(R)$ *bo'lsa, u holda* $(h * \varphi)(\lambda) \in L_p(R)$ *(*$1 \leq p \leq \infty$*) munosabat va*

$$\|h * \varphi\|_p \leq \|h\|_1 \|\varphi\|_p \tag{9}$$

tengsizlik o'rinli.

Xususan agar $h(t) \in L_1(R)$ va $\varphi(t) \in L_2(R)$ *bo'lsa u holda* $(h * \varphi)(\lambda) \in L_2(R)$ *va*

$$\|h * \varphi\|_{L_2(R)} \leq \|h\|_1 \|\varphi\|_2 \tag{10}$$

tengsizlik o'rinli bo'ladi.

Agar $h(t) \in L_2(R)$ va $\varphi(t) \in L_2(R)$ bo'lsa, u holda (8) o'rama quyidagi hossaga ega $(h * \varphi)(\lambda)$ – uzluksiz chegaralangan funksiya bo'lib cheksizlikda nolga intiladi.

(8) o'ramaning Fure almashtirishi Furening o'rama teoremasi yordamida berilgan.

2-teorema. *Aytaylik* $h(t) \in L_1(R)\ va\ \varphi(t) \in L_1(R)\ yoki$ $h(t) \in L_1(R)\ va\ \varphi(t) \in L_2(R)$ *yoki*

$h(t) \in L_2(R)\ va\ \varphi(t) \in L_2(R)$ *bo'lsin.*

U holda $h * \varphi$ *o'ramaning Fure almashtirishi*

$$F[h * \varphi](\lambda) = F[h](\lambda) * F[\varphi](\lambda) \tag{11}$$

tenglik yordamida beriladi.

$\varphi(t) t \in R^h$ funksiyaning n –o'lchamli Fure almashtirishini qaraymiz.

$$\mathrm{F}[\varphi](\lambda) = \widehat{\varphi(\lambda)} = \int_{R^n} e^{i\lambda t} \varphi(t)dt \quad t \in R^n \tag{12}$$

Mos ravishda teskari Fure almashtirishi esa

$$F^{-1}[g(t)](\lambda) = \frac{1}{(2\pi)^n} \widehat{g}(-\lambda) \coloneqq \widehat{\frac{1}{(2\pi)^n}} \int_{R^n} e^{-\iota\lambda - t} g(t)dt \quad \lambda \in R^n \tag{13}$$

Integral yordamida beriladi, bu yerda $\lambda t = \lambda_1 t_1 + \lambda_2 t_2 + \cdots + \lambda_n t_n$ (12) va (13) integrallar (1), (2) ga o'xshash xossalarga ega.

$$\lim_{|x| \to \infty} \int_{R^n} e^{i\lambda t} \varphi(t)dt = 0 \qquad \varphi(t) \in L_1(R^n) \tag{14}$$

$$F[D^k \varphi(t)](\lambda) = (-i\lambda)^k F[\varphi](\lambda), \quad (\lambda \in R^n, k \in N^n) \tag{15}$$

$$D^k[F[\varphi](\lambda) = F[(it)^k \varphi(t)](\lambda), \quad (\lambda \in R^n, k \in N^n) \tag{16}$$

Agar $\Delta - \pi - o'lchamli$ Lamos almashtirishi bo'lsa

$$\Delta := \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2} \qquad (17)$$

u holda (15) dan

$$F[\Delta\varphi](\lambda) = -|\lambda|^2 \, \mathrm{F}[\varphi](\lambda) \qquad (\lambda \in R^n \qquad (18)$$

Kelib chiqadi.

(8) ga o'sish ikkita h(t), $\varphi(t)$ $t \in R^n$ funksiyalarning Fure o'ramli operatori

$$(h * \varphi)(\lambda) = \int_{R^n} h(\lambda - t)\varphi(t)dt, \qquad (\lambda \in R^n) \qquad (19)$$

Ko'rinishda aniqlanadi va bu yerda ham kommuntativlik xossasi o'rganish R^n da aniqlangan φ, h funksiyalarning Fure almashtirishlari uchun ham Th_1 va Th_2 lar o'rinlidir.

Furye almashtirishining xossalari

1. (*chiziqlilik xossasi*) a,b ∈ $\mathbb{R}$.Furye almashtirishi chiziqlidir.

$$F(af_1 + bf_2) = aF(f_1) \pm bF(f_2)$$

2. (*Furye o'ramasi*) Furye o'ramasi $*$ kabi belgilanadi. $f(t)$ va $g(t)$ t ∈ $\mathbb{R}$ da

Direxle shartini qanoatlantirsin.

$$f(t) * g(t) := \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau = \int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau$$

$$f * g(t) = g * f(t)$$

3. (*o'ramaning Furye almashtirishi*)

$$F(f * g)(x) = F(f)(x)F(g)(x)$$

4. (*Ko'paytmaning Furye almashtirishi*)

$$F(f \cdot g) = \frac{1}{2\pi} F(f)(x) * F(g)(x)$$

5. (*Hosilaning Furye almashtirishi*)

n ∈ N, n- tartibli hosilaning Furye va teskari Furye almashtirishi

$$F\left(f^{(n)}\right)(x) = (ix)^n F(f)(x); \quad va$$

$$F^{-1}\left(f^{(n)}\right)(x) = (ix)^n f(x)$$

6. $F^{-1}Ff = f \quad va \; F^{-1}Fg = g$

Unda kerak bo'ladigan bir qancha maxsus funksiyalarning Fure almashtirishini keltirib o'tamiz:

Misol: $f(x) = e^{-a|x|}$

$$\hat{f}(ʒ) = \int_{-\infty}^{\infty} e^{-a|x|} e^{-ixʒ} dx = \int_{-\infty}^{0} e^{x(a-iʒ)} dx + \int_{0}^{\infty} e^{-x(a+iʒ)} dx =$$

$$= \frac{e^{x(a-iʒ)}}{a-iʒ}\Bigg|_{-\infty}^{0} + \frac{e^{-x(a+iʒ)}}{e^{-(a+iʒ)}}\Bigg|_{0}^{\infty} = \frac{1}{a-iʒ} + \frac{1}{a+iʒ} = \frac{2a}{a^2+ʒ^2}$$

Misol : $\varphi(t) = \delta(t), t \in R,$ F[φ] =?.

Yechim: Bu yerda $\delta(\cdot) - Dirakning$ delta funksiya va u

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases} \quad \int_R \delta(t - t_0)\varphi(t)dt = \varphi(t_0)$$

xossalarga ega (1.3.1) tenglikga ko'ra

F$[\delta(t)](\lambda) = \int_R e^{-i\lambda t} \delta(t)dt = e^{-i\lambda t} \Big|_{t=0}^{t=1}$ ya'ni F$[\delta]$ $(\lambda) = 1, \lambda \in R$

munosabatga binoan

$1 = F[\delta(t)]\ (\lambda) \quad F[1] = \delta(-t)$ ya'ni $F[1] = 2\pi\delta(\lambda)$.

Misol. $F\left[e^{iat} f(t)\right](\omega) = F(\omega - a)$

Agar $f(x) \in L_1(\mathbb{R})$ bo'lsa , u holda $F[f](s)$ chegaralangan uzluksiz funksiya

va Riman-Liuvil teoremasiga ko'ra

$$\lim_{\xi \to \infty} e^{ix\xi} f(x)dx = 0; \;\; f(x) \in L_1(\mathbb{R})$$

$$e^{-ix\xi} = \cos x\xi - \sin x\xi$$

Natija:$\mathbb{R}$ sonlar o'qida absolyut integrallanuvchi bo'lgan har qanday funksiya uchun quyidagi o'rinli

$$\lim_{\lambda \to \infty} \int_{-\infty}^{\infty} f(x) \cos \lambda x dx = 0 \quad va \quad \lim_{\lambda \to \infty} \int_{-\infty}^{\infty} f(x) \sin \lambda x \, dx = 0$$

Natija(2):$[a; b] \subset \mathbb{R}$ kesmada Riman bo'yicha integrallanuvchi bo'lgan har qanday f funksiya uchun

$$\lim_{\lambda \to \infty} \int_{a}^{b} f(x) \cos \lambda x \, dx = \lim_{\lambda \to \infty} \int_{a}^{b} f(x) \sin \lambda x \, dx = 0$$

Tenglik o'rinli bo'ladi.

Mustaqil yechish uchun topshiriqlar.

1. $f(t) = e^{-|t|}$ funksiyaning Furye almashtirishini aniqlang?
2. $f(t) = e^{-t^2}$ funksiyaning Furye almashtirishini aniqlang?
3. $f(t) = \sin t$ funksiyaning Furye almashtirishini aniqlang?
4. Faraz qilaylik $a \in R_+$ bo'lsin. U holda ushbu tenglik bajarilishini isbotlang $\mathcal{F}[f(t-a)](\omega) = e^{i\omega a} F(\omega)$, bu yerda $F(\omega)$- $f(t)$ funksiyaning Fure almashtirishi.
5. Faraz qilaylik $a \in R$ bo'lsin. U holda ushbu tenglik bajarilishini isbotlang. $\mathcal{F}\left[e^{iat} f(t)\right](\omega) = F(\omega - a)$, bu yerda $F(\omega)$- $f(t)$ funksiyaning Fure almashtirishi.

6. Ushbu tenglik bajarilishini isbotlang $f(-\omega) = \mathcal{F}[F(t)](\omega)$, bu yerda $F(\omega)$-$f(t)$ funksiyaning Fure almashtirishi.
7. $F(\omega) = \mathcal{F}[f(t)](\omega)$ va $G(\omega) = \mathcal{F}[g(t)](\omega)$ Furye almashtirishlari $f(t)$ va $g(t)$ funksiyalarning. Ushbu tenglik bajarilishini isbotlang:
$\int_{-\infty}^{\infty} F(\omega)g(\omega)e^{i\omega t}d\omega = \int_{-\infty}^{\infty} f(\xi)G(\xi - t)d\xi.$
8. $(\omega) = \mathcal{F}[f(t)](\omega)$ va $G(\omega) = \mathcal{F}[g(t)](\omega)$ Furye almashtirishlari $f(t)$ va $g(t)$ funksiyalarning. Ushbu tenglik bajarilishini isbotlang:
$\int_{-\infty}^{\infty} F(\omega)G(\omega)d\omega = \int_{-\infty}^{\infty} f(-\xi)g(\xi)d\xi.$
9. Ushbu tenglik bajarilishini isbotlang $F[f(t-a)](\omega) = e^{i\omega a}F(\omega),\quad F(\omega)\ f(t)$ ning Furye almashtirishi
10. Ushbu tenglik bajarilishini isbotlang $F[f(at)](\omega) = \frac{1}{|a|}F\left(\frac{\omega}{a}\right)$,
$F(\omega)\ f(t)$ ning Furye almashtirishi

Laplas almashtirishi.

Haqiqiy o'zgaruvchili $\varphi(t), t \in (0 + \infty)$ funksiyaning Laplas almashtirishi

$$L^{-1}\{\varphi\}(s) = \widetilde{\varphi}\,(s) \coloneqq \int_0^{+\infty} e^{-st}\,\varphi(t)dt, \qquad (s \in \mathbb{C}) \qquad (20)$$

integral yordamida aniqlanadi. Agar (20) integral $s_0 \in \mathbb{C}$ nuqtada yaqinlashuvchi bo'lsa, u holda $Res > Res_0$ shartni qanoatlantiruvchi $s \in \mathbb{C}$ larda absolyut yaqinlashuvchi bo'ladi. (20). Laplas integrali aniqlashuvchi bo'ladigan s larning infinumi $\sigma_\varphi -$ yaqinlashish absissasi deyiladi. Shuning uchun (28) integral $Re\, s > \sigma_\varphi$ da yaqinlashuvchi $Res < \sigma_\varphi$ da esa uzoqlashuvchi bo'ladi.

Teskari Laplas almashtirish esa

$$L^{-1}\{g(s)\}(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st}\, g(s)ds, \quad (\gamma = Re(s) > \sigma_\varphi\,, \qquad (21)$$

Integral yordamida beriladi.

Yetarlicha yaxshi $\varphi\ va\ g$ funksiyalar uchun to'g'ri va teskari Laplas almashtirishlari o'rtasida quyidagi tengliklar o'rinli.

$$L^{-1}L\{\varphi\} = \varphi \quad \text{va} \quad LL^{-1}\{g\} = g \tag{22}$$

Dastlab Laplas almashtirishining oddiy xossalarini keltirib o'tamiz:

$$\mathrm{L}\{\{\varphi^k(t)\}(s) = s^k L\{\varphi\}(s) \qquad (\mathrm{k} \in N) \tag{23}$$

$$\frac{d^h}{ds^h}\big(L\{\varphi\}(s)\big) = (-1)^h L\{t^k\varphi(t)\}(s) \qquad (\mathrm{k} \in N) \tag{24}$$

Bu tengliklar yetarlicha yaxshi φ funksiyalar uchun o'rinli.

Agar $\varphi \in C^k(R^t)$ bo'lsa $\mathrm{L}\{\varphi\}(s)$ va $\mathrm{L}\{D^k\varphi(t)\}(s)$ Laplas almashtirishlari mavjud va j=a,k-1 lar uchun $\lim_{t\to+\infty} (\mathrm{D}\,\varphi)^{(j)}(t) = 0$ tenglik o'rinli bo'lsa, u holda (23) tenglikni quyidagicha umumlashtrish mumkin:

$$\mathrm{L}\{D^h\varphi(t)\}(s) = s^h L\{\varphi\}(s) - \sum_{j=0}^{k-1} s^{k-j-1}\,(D^j\,\varphi)(0) \qquad (\mathrm{k} \in N) \tag{25}$$

R^tda berilgan h(t) va $\varphi(t)$ funksiyalarning Laplas o'rama opertsiyasi

$$h *\ \varphi = (h * \varphi) \coloneqq \int_0^t h(t-\tau)\varphi(\tau)d\tau \qquad (\mathrm{t} \in R^1) \tag{26}$$

integral yordamida aniqlanadi va $h *\ \varphi = \varphi * h$ kommutativlik xossasi o'rinli.(26) uchun 2-teoremani qo'llasak

$$L\{\,h *\ \varphi\}(s) = L\{h\} * L\{\varphi\} \tag{27}$$

tenglikni yetarlicha yaxshi h, φ funksiyalar uchun olamiz.

Laplas almashtirishining xossalari

1. $\mathcal{L}(af_1 \pm bf_2) = a\mathcal{L}(f_1) \pm b\mathcal{L}(f_2)$ a,b∈ℝ
2. a∈ℝ . $\mathscr{L}[f]$(s)=$F(s)$ bo'lsa,

$$\mathcal{L}[e^{-at}f(t)] = F(s+a)$$

3. a>0 $H(t-a)$ Heaviside funksiyasi

$$\mathcal{L}[f(t-a)H(t-a)] = e^{-as}F(s)$$

yoki

$$\mathcal{L}[f(t)H(t-a)] = e^{-as}\mathcal{L}[f(t)]$$

4. Funksiya differensialining Laplas almashtirishi $f\colon \mathbb{R} \to \mathbb{R}$ va (n-1)-tartibli hosilagacha yopiq $[0;c] \subset \mathbb{R}$ intervalda uzluksiz bo'lsin.U holda ∃M>0 soni mavjudki $t_0 > 0$ uchun

$|f(t)| \leq Me^{at}$, $\left|\frac{d}{dt}f(t)\right| \leq Me^{at}$,... , $\left|\frac{d^{n-1}}{dt^{n-1}}f(t)\right| \leq \; Me^{bt}$

$Re(s) > b$ baholar o’rinli bo’lsin.

$$\mathcal{L}\left[\frac{d^n}{dt^n}f\right](s) = s^n\mathcal{L}[f](s) - \sum_{k=0}^{n-1}\left[\left(\frac{d^{n-1-k}}{dt^{n-1-k}}f_{t=0}\right)\right]$$

5. Laplas almashtirishining o’ramasi

$$\mathcal{L}[f(t) * g(t)](s) = \mathcal{L}[f](s)\,\mathcal{L}[g](s) = F(s)G(s)$$

$$f(t) * g(t) \coloneqq \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t g(t-\tau)f(\tau)d\tau$$

6. Laplas almashtirishining hosilasi

$$\mathcal{L}[t^n f(t)](s) = (-1)^n\frac{d^n}{ds^n}F(s); \quad n = 0,1,2,3,\dots$$

7. Laplas almashtirishining integrali

$$\mathcal{L}\left[\frac{f(t)}{t}\right](s) = \int_s^\infty F(\xi)d\xi$$

8. Integralning Laplas almashtirishi

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\}(s) = \frac{F(s)}{s}$$

Teskari Laplas almashtirishi

$$\mathcal{L}^{-1}[F](t) = \frac{1}{2\pi i}\lim_{\tau\to\infty}\int_{\sigma-i\tau}^{\sigma+i\tau} e^{st}F(s)ds. \quad\quad Re(s) = \sigma > 0$$

Fure almashtirishiga o’xshash Laplas almashtirishini t$\in R^n$ uchun umumlashtirish mumkin. Laplas almashtirishga oid bir nechta misollar ko’rib o’tamiz.

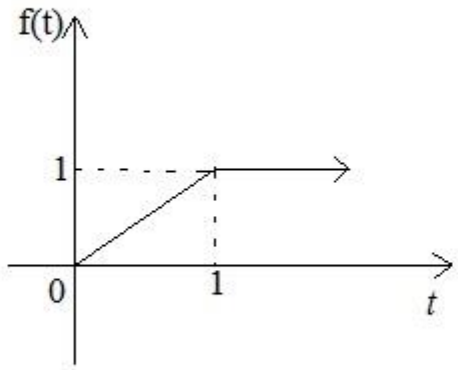
Misol. $f(t) = \mathrm{e}^{2\mathrm{t}} \cos \mathrm{t}$ funksiyaning Laplas almashtirishini hisoblang.

$$\mathcal{L}[\mathrm{e}^{2\mathrm{t}} \cos \mathrm{t}] = \int_0^\infty e^{-st} e^{2t} \cos t dt = \int_0^\infty e^{-t(s-2)} \cos t dt$$

$$\mathcal{L}[\mathrm{e}^{2\mathrm{t}} \cos \mathrm{t}] = Re\{e^{-(s-2-i)t} dt\} = Re\left\{\frac{1}{s-2-i}\right\} = Re\left\{\frac{s-2+i}{(s-2)^2+i}\right\}$$

$$\mathcal{L}[\mathrm{e}^{2\mathrm{t}} \cos \mathrm{t}] = \frac{s-2}{(s-2)^2+1}$$

Misol: Funksiya f(t)=$\begin{cases} t & 0 \le t \le 1 \\ 1 & t > 1 \end{cases}$ bo'lsin L{f(t)}(s)=?



Yechim: tarifga ko'ra

L{f(t)}(s)=$\int_0^{+\infty} e^{-st} f(t) dt = \int_0^1 e^{-st}\, t dt + \int_1^{+\infty} e^{-st} 1 dt = \int_0^1 e^{-st} dt +$

$\lim\limits_{\tau\to\infty} \int_1^\tau e^{-st} dt = -\frac{te^{-st}}{3}\Big|_0^1 + \frac{1}{3}\int_\tau^1 e^{-st} dt + \lim\limits_{\tau\to\infty}\left(\frac{e^{-st}}{3}\right)\Big|_1^\tau = \frac{1-e^{-s}}{s^2}$ (Re(s)> 0

Agar f(t)=$a_0 + a_1 t + \ldots + a_n t^n$ –formula ko'phad bo'lsa u holda

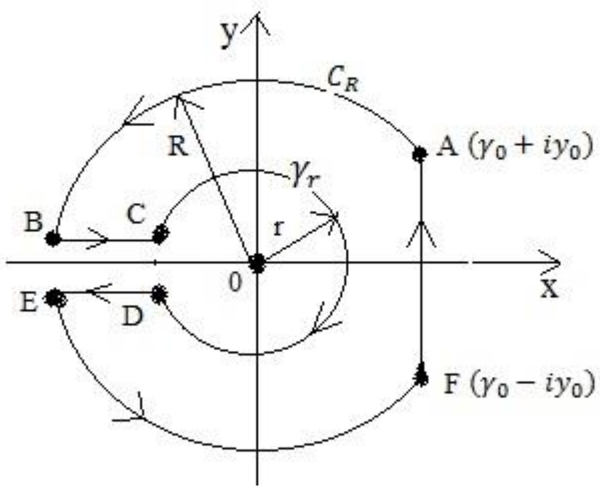
L{f(a)}(1)=L{$\sum_{k=0}^n a_k\, t^k$}=$\sum_{k=0}^n a_k L\{t^k\}(3)$ $= \sum_{k=0}^\infty a_k \frac{K!}{s^{K+L}}$ =$\sum_{k=0}^\infty a_k \frac{\Gamma(k+1)}{s^{k+1}}$

Misol: F(s)=$\frac{1}{\sqrt{s}}$ funksiyani teskari Laplas almashtirishini toping. Bunda s=0 nuqta maxsus nuqtasi hisoblanadi.

Yechim: $L^{-1}\left\{\frac{1}{\sqrt{s}}\right\}(\delta) = \frac{1}{2\pi i}\int_{\gamma=i\infty}^{\gamma+i\infty} e^{st} s^{-1} ds$ (28) integralini hisoblashimiz kerak. (28) integralni 1.2 da keltirilganma'lumotlardan foydalanib hisoblashga harakat qilamiz. Quyidagi konturni qaraylik

$C\ R = ABCDEFA$ bunda AB va EF lar

R radiusli aylana markazi 0 nuqtada.



CD esa markazi 0 nuqtada 2 ga teng $\gamma_r$aylanadan iborat. $\omega = \sqrt{s}$ uchun bir qiymatli analitk tarmoqni qaraymiz. U holda F(s)=$1/\sqrt{s}$ funksiya C_R da va ichida analitik demak Koshi teoremasiga ko'ra

$$I = ?$$

Undan tashqari

$$0 = \frac{1}{2\pi i}\int_{\gamma_{0-iy_o}}^{\gamma_{0+iy_o}} \frac{e^{ts}}{\sqrt{3}} ds +$$

$$\frac{1}{2\pi i}\int_{AB} \frac{e^{ts}}{\sqrt{s}} ds + \frac{1}{2\pi i}\int_{BC} \frac{e^{ts}}{\sqrt{s}} ds + \frac{1}{2\pi i}\int_{\gamma_r} \frac{e^{ts}}{\sqrt{s}} ds +$$

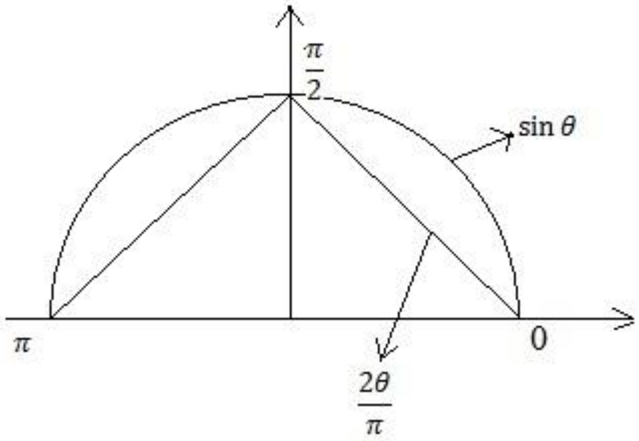
$$+\frac{1}{2\pi i}\int_{DE}\frac{e^{ts}}{\sqrt{s}} ds + \frac{1}{2\pi i}\int_{EF}\frac{e^{ts}}{\sqrt{s}} ds \quad (29)$$

Endi har bir integralni hisoblaymiz.

$$I_2 = \int_{AB}\frac{e^{ts}}{\sqrt{s}} ds = \left|\begin{matrix} 3 = Re^{i\theta} & 0 \le G \le \tau \\ ds = Rie^{i\theta} d\partial & \end{matrix}\right| = \int_0^{\tau} \frac{e^{tRe^{i\theta}}}{\sqrt{R}e^{i\theta/2}} Rie^{i\theta} d\theta =$$

$$\sqrt{R}\ i\int_0^{\pi} e^{tR(\cos\partial\sin\partial)} e^{i\partial/2} d\partial = i\sqrt{R\ \int_0^{\pi} e \quad e^{+\partial/2} d\theta}$$

$$|I_2| = \sqrt{R}\int_0^{\pi} e^{-tR\cos\theta}\, d\theta => 2\sqrt{R}\int_0^{\pi} e^{-tR\sin\theta}\, d\theta$$



$$\sin\theta \geq \frac{2\theta}{\pi}$$

$$|I_2| \leq 2\sqrt{R}\int_0^{\frac{\pi}{2}} e^{-R\frac{2\theta}{\pi}}\,d\theta = \frac{-\pi}{\sqrt{R}}e^{-\frac{2}{\pi}R\theta}\Bigg|_0^{\frac{\pi}{2}} = \frac{\pi}{\sqrt{R}}(1-e^{-R})$$

$$\lim_{R\to\infty}|I_2| = 0$$

Xuddi shunday ko'rsatish mumkin.

$\lim_{R\to\infty}|I_\delta| = 0$ endi $I_3 = \int_{\gamma_r}\frac{e^{ts}}{\sqrt{s}}$ ds integralni $r \to 0$ dagi hisoblaymiz.

Yuqoridagiga o'xshash $s = re^{i\theta}$ d$s = rie^{i\theta}d\theta$

$$I_3 = \int_{\gamma_r}\frac{e^{ts}}{\sqrt{s}}ds = \int_{\pi}^{-\pi}\frac{e^{tre^{i\theta}}}{\sqrt{r}\,e^{i\theta/2}}\,rie^{i\theta}\,d\theta = \int_{\pi}^{-\pi}\frac{e^{tr(\cos\theta+\sin\theta)}}{\sqrt{r}}ire^{i\theta}d\theta$$

$$|I_3| \leq \sqrt{r}\left|\int_{+\pi}^{-\pi}e^{-trs\theta}d\theta\right|\begin{matrix}\to 0\\ r\to 0\end{matrix}$$

(20) tenglikda faqat BC va CE kesimlar bo'yicha integrallash qoldi.

$$I_\gamma = \int_{BC}\frac{e^{ts}}{\sqrt{s}}ds = \left|\begin{matrix} s = xe^{i\pi} & ds = e^{iF}dx \\ \sqrt{s} = \sqrt{x}\,e^{i\pi/2} = i\sqrt{x} & e^{i\pi=-1} \\ -R \leq s \leq -r & r \leq x \leq R\end{matrix}\right| = \int_{-R}^{+R}\frac{e^{txe^{i\pi}}}{i\sqrt{x}}\,e^{i\pi}dx$$

$$= -\frac{1}{i\pi}\int_{+R}^{-R}\frac{e^{-tx}}{\sqrt{n}}dx = \frac{1}{i}\int_r^R\frac{e^{-tx}}{n}dn$$

Xuddi shunday DE: $s = xe^{-i\pi}$; $\sqrt{s} = \sqrt{n}e^{-\pi/2} = -i\sqrt{a}\,va$

$$I_B = \int_{DE} \frac{e^{ts}}{\sqrt{s}} ds = \int_{-r}^{-R} \frac{e^{ts}}{\sqrt{s}} ds = + \int_{+r}^{+R} \frac{e^{txe^{-i\pi}}}{\sqrt{ni}} e^{-i\pi} dx = \frac{1}{i} \int_{-r}^{-R} \frac{e^{tx}}{\sqrt{a}} dx$$

$I_3\ va\ I_B$ integrallarni birlashtrsak $\frac{1}{2\pi i}\int_{BC}\frac{e^{ts}}{\sqrt{s}}ds + \frac{1}{2\pi i}\int_{DE}\frac{e^{ts}}{\sqrt{s}}ds = -\frac{1}{\pi}\int_r^R \frac{e^{-ta}}{\sqrt{n}}da$ natijaga ega bo'lamiz. (29) da R$\to \infty, r \to 0$ umumiyga o'tsak

$$0=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{ts}}{\sqrt{s}}ds - \frac{1}{\pi}\int_{\partial}^{+\infty}\frac{e^{-tn}}{\sqrt{n}} \text{ da yoki}$$

$$I=\frac{1}{\pi}\int_0^{+\infty}\frac{e^{-tn}}{\sqrt{n}}da = \frac{1}{\sqrt{\pi t}}.$$

Shunday qilib $L^{-1}\left\{s^{-1/2}\right\} = \frac{1}{\sqrt{\pi t}}$

Mustaqil yechish uchun topshiriqlar.

1. Quyidagi funksiyaning Laplas almashtirishini hisoblang. $f(t) = e^{2t}\cos t$.
2. Quyidagi funksiyaning Laplas almashtirishini hisoblang. $f(t) = e^{2t}(1 - at);\ \ a \in R_+$.
3. Quyidagi funksiyaning Laplas almashtirishini hisoblang. $f(t) = \frac{\sin t}{t}$.
4. Quyidagi funksiyaning teskari Laplas almashtirishini hisoblang. $\mathcal{L}^{-1}\left[\frac{s}{(s+a)(s+b)}\right]$, bu yerda $a, b \in R; a \neq b$.
5. Laplas almashtirishidan foydalanib quyidagi oddiy differensial tenglamani yeching. $\begin{cases} \frac{d^2}{dt^2}x(t) + x(t) = t, \\ x(0) = x'(0) = 0. \end{cases}$
6. Laplas almashtirishidan foydalanib quyidagi oddiy differensial tenglamani yeching. $\begin{cases} \frac{d^2}{dt^2}x(t) + 5\frac{d}{dt}x(t) + 6x(t) = e^{2t}, \\ x(0) = x'(0) = 0. \end{cases}$
7. Laplas almashtirishidan foydalanib quyidagi oddiy differensial tenglamalar sistemasini yeching.

$$\begin{cases} \frac{d}{dt}x(t) + y(t) = 1, \\ \frac{d}{dt}y(t) - x(t) = -1, \\ x(0) = y(0) = 2. \end{cases}$$

8. Faraz qilaylik , $a \in R, \alpha > 0, \beta > 0, \gamma > 0$ bo'lsin. U holda ushbu tenglik bajarilishini isbotlang.

$$\mathcal{L}^{-1}\left[\frac{s^{2\alpha-\beta-\gamma}}{s^{2\alpha} - a^2}\right] = t^{\beta+\gamma-1}E_{2\alpha,\beta+\gamma}(a^2t^{2\alpha}).$$

9. Yuqoridagi misolni maxsus hol ya'ni $\alpha = \beta = \gamma$ uchun Laplas almashtirishini isbotlang.
10. $a \in R, \alpha > 0, \beta > 0$ hol uchun quyidagi funksiyaning Laplas almashtirishini hisoblang.

$$t^{\beta-1}E_{\alpha,\beta}(at^{\alpha}).$$

Mellin almashtirishlari

Biz Mellin almashtirishi va uning teskari almashtirishini kompleks Furye almashtirishi va teskari Furye almashtirishidan keltirib chiqaramiz:

$$F\{g(s)\} = G(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-ik\xi}\, g(\xi)d\xi \tag{1}$$

$$F^{-1}\{G(k)\} = g(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{ik\xi}\, G(k)dk \tag{2}$$

$e^{\xi} = x$ va $ik = c - p$ almashtirish bajaramiz va (1) va (2) dan quyidagilarga kelamiz, bu yerda $c = const$

$$G(ip - ic) = \frac{1}{\sqrt{2\pi}}\int_{0}^{+\infty} x^{p-c-1}\, g(lnx)dx \tag{3}$$

$$g(lnx) = \frac{1}{\sqrt{2\pi}}\int_{c-i\infty}^{c+i\infty} x^{c-p}\, G(ip - ic)dp \tag{4}$$

Biz $\frac{1}{\sqrt{2\pi}}x^{-c}\, g(lnx) \equiv f(x)$ va $G(ip - ic) \equiv f(p)$ orqali $f(x)$ ning Mellin almashtirishi va Mellin almashtirishining teskarisini belgilaymiz va quyidagicha yozamiz

$$M\{f(x)\} = \tilde{f}(p) = \int_0^\infty x^{p-1} f(x)dx \qquad (5)$$

$$M^{-1}\{\tilde{f}(p)\} = f(x) = \int_{c-i\infty}^{c+i\infty} x^{-p}\ \tilde{f}(p)dp \qquad (6)$$

Bu yerda $f(x)$ funksiya $(0,\infty)$ da aniqlangan haqiqiy funksiya va Mellin almashtirishining o`zgaruvchisi $p$ kompleks son. Ba`zan $f(x)$ ning Mellin almashtirishi $\tilde{f}(p) = M[f(x), p]$ kabi belgilanadi. M va M^{-1} lar chiziqli integral operatorlar .

Misol: Agar $f(x) = e^{-nx}$ bo`lsa, bunda $n > 0$

$$M\{e^{-nx}\} = \tilde{f}(p) = \int_0^\infty x^{p-1} e^{-nx}\, dx$$

$nx = t$ desak,

$$= \frac{1}{n^p}\int_0^\infty t^{p-1} e^{-t}\, dt = \frac{\Gamma(p)}{n^p} \qquad (7)$$

Misol: $f(x) = \frac{1}{1+x}$ funksiyaning Mellin alshtirishini toping.

$$M\left\{\frac{1}{1+x}\right\} = \tilde{f}(p) = \int_0^\infty x^{p-1} \frac{1}{1+x} dx =$$

$$x = \frac{t}{1-t} \qquad yoki \qquad t = \frac{x}{1+x} \qquad desak$$

$$= \int_0^\infty t^{p-1}(1-t)^{1-p-1}\, dt = B(p, 1-p) = \Gamma(p)\Gamma(1-p) \qquad (8)$$

Misol: $f(x) = (e^x - 1)^{-1}$

$$M\{(e^x-1)^{-1}\} = \tilde{f}(p) = \int_0^\infty x^{p-1}(e^x-1)^{-1}\, dx=$$

Biz $\sum_{n=0}^{\infty} e^{-nx} = \frac{1}{1-e^{-x}}$ dan foydalanamiz, bunnda $\sum_{n=1}^{\infty} e^{-nx} = \frac{1}{e^x-1}$ bo`ladi.

$$= \sum_{n=1}^{\infty} x^{p-1} e^{-nx} dx = \sum_{n=1}^{\infty} \frac{\Gamma(p)}{n^p} = \Gamma(p) \cdot \varsigma(p)$$

$$\varsigma(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}; \quad (Rep > 1)$$

Mustaqil ishlash uchun misollar:

1) $f(x) = \frac{2}{e^{2x}-1}$

2) $f(x) = \frac{1}{e^{x}+1}$

3) $f(x) = \frac{1}{(1+x)^n}$

4) $f(x) = coskx$

5) $f(x) = sinkx$

Yuqoridagi funksiyalarning Mellin almashtirishlarini toping.

Mellin almashtirishlarining xossalari:

Agar $M\{f(x)\} = \tilde{f}(p)$ bo`lsa, u holda quyidagilar o`rinli:

a) $M\{f(ax)\} = a^{-p}\tilde{f}(p), \qquad a > 0,$ (9)

Isbot: $M\{f(ax)\} = \int_0^{\infty} x^{p-1} f(ax)\, dx =$ $\qquad ax = t \;\; desak$

$$= \frac{1}{a^p} \sum_{n=0}^{\infty} t^{p-1} f(t) dt = \frac{\tilde{f}(p)}{a^p}$$

$b)$ $M[x^{\alpha} f(x)] = \tilde{f}(p + a)$ (10)

$c)$ $M[f(x^{\alpha})] = \frac{1}{a}\tilde{f}\left(\frac{p}{a}\right)$ (11)

$M[\frac{1}{x} f(\frac{1}{x})] = \tilde{f}(1 - p)$ (12)

$M[(logx)^n f(x)] = \frac{d^n}{dp^n}\tilde{f}(p)$ n=1,2,3… (13)

5* tenglik $\frac{d}{dp} = (logx)x^{p-1}$ (6*) dan foydalanib isbotlanadi.

$d)$ $M[f'(x)] = -(p-1)\tilde{f}(p-1)$

$M\big[f''(x)\big] = (p-1)(p-2)\tilde{f}(p-2);$

$M\big[f^{(n)}(x)\big] = (-1)^n \frac{\Gamma(p)}{\Gamma(p-n)}\tilde{f}(p-n) = (-1)^n \frac{\Gamma(p)}{\Gamma(p-n)} M[f(x, p-n)];$

Isbot: $M[f'(x)] = \int_0^\infty x^{p-1} f'(x)dx = [x^{p-1}f(x)]_0^\infty - (p-1)\int_0^\infty x^{p-2}f(x)dx =$

$$-(p-1)\tilde{f}(p-1)$$

e) Agar $M[f(x)] = \tilde{f}(p)$ bo`lsa, quyidagilar o`rinli:

$M[xf'(x)] = -p\tilde{f}(p)$

$M\big[x^2 f''(x)\big] = (-1)^2 p(p+1)\tilde{f}(p);$

$M\big[x^n f^{(n)}(x)\big] = (-1)^n \frac{\Gamma(p+n)}{\Gamma(p)}\tilde{f}(p);$

Isbot:

$M[xf'(x)] = \int_0^\infty x^p f'(x)dx = [x^p f(x)]_0^\infty - p\int_0^\infty x^{p-1}f(x)dx =$

$$-p\tilde{f}(p)$$

f) Agar $M[f(x)] = \tilde{f}(p)$ bo`lsa, quyidagilar o`rinli:

$M\left[\left(x\frac{d}{dx}\right)^2 f(x)\right] = M\big[x^2 f''(x) + xf'(x)\big] = (-1)^2 p^2 \tilde{f}(p)$;

$M\left[\left(x\frac{d}{dx}\right)^n f(x)\right] = (-1)^n p^n \tilde{f}(p);$

Isbot: $M\left[\left(x\frac{d}{dx}\right)^2 f(x)\right] = M\big[x^2 f''(x) + xf'(x)\big] = M\big[x^2 f''(x)\big] + M[xf'(x)\big] =$

$$= -p\tilde{f}(p) + p(p+1)\tilde{f}(p) = (-1)^2 p^2 \tilde{f}(p)$$

g) $M\left\{\int_0^x f(t)dt\right\} = \frac{-1}{p}\tilde{f}(p+1)$;

$M\{I_n f(x)\} = M\left\{\int_0^x I_{n-1}f(t)dt\right\} = (-1)^n \frac{\Gamma(p)}{\Gamma(p+n)}\tilde{f}(p+n);$

$I_n f(x) = \int_0^x I_{n-1} f(t)dt.$

Mellin almashtirishining qo`llanishi

Ushbu almashtirishning qo`llanishini misol yordamida tushinib olamiz.

Misol: Chegaraviy masalaning yechimini olamiz.

$$x^2 u_{xx} + x u_x + u_{yy} = 0 \qquad 0 \le x < \infty, \qquad 0 < y < 1;$$

$$u(x,0) = 0, \qquad u(x,1) = \begin{cases} A, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases};$$

Bu yerda $A = const$

x bo`yicha $u(x,y)$ ning Mellin almashtirishini quyidagicha bajaramiz.

$$\tilde{u}(p,y) = \int_0^\infty x^{p-1} u(x,y)dx$$

$$\tilde{u}_{yy} + p^2 \tilde{u} = 0, \qquad 0 < y < 1$$

$$\tilde{u}(p,0) = 0, \qquad \tilde{u}(p,1) = A \int_0^\infty x^{p-1}\, dx = \frac{A}{p}$$

$$\tilde{u}(p,y) = \frac{A}{p} \cdot \frac{sinpy}{sinp}, \qquad 0 < Rep < 1$$

$$u(x,y) = \frac{A}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^p}{p} \frac{sinpy}{sinp} dp;$$

$$u(x,y) = \frac{A}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (-1)^n x^{-n\pi} sinn\pi y;$$

III-BOB. KASR TARTIBLI HISOBGA KIRISH

Kasr hisobi - bu matematik tahlil sohasi bo'lib, u ixtiyoriy tartibdagi integral va hosilalarni tekshirish va qo'llash bilan shug'ullanadi. Kasr atamasi noto'g'ri nom, lekin u keng tarqalgan foydalanishdan keyin saqlanib qolgan..

Kasr hisobi eski, ammo yangi mavzu deb hisoblash mumkin. Bu eski mavzu, chunki Leybnits (1695, 1697) va Eyler (1730)ning ba’zi ishlaridan boshlab, hozirgi kungacha ishlab chiqilgan. Asrimizning o'rtalarigacha ushbu sohaga muhim hissa qo'shgan matematiklar ro'yxati P.S. Laplas (1812), J.B.J. Furye (1822), N. X. Abel (1823-1826), J. Liouvil (1832 -1873), B. Rimann (1847), X. Xolmgren (1865—67), A.K. Grunvald (1867-1872), A.V. Letnikov (1868-1872), X. Loran (1884), P.A. Nekrassov (1888), A. Krug (1890), J. Xadamard (1892), O. Xevisayd (1892-1912), S. Pincherle (1902), G.X. Hardi va J.E.Littlvud (1917-1928), X. Veyl (1917), P. L'evi (1923), A. Marcho (1927), H.T. Devis (1924-1936), A. Zigmund (1935-1945), E.R.Lev (1938-1996), A. Erdelyi (1939-1965), X. Kober (1940), D.V. Vidder (1941), M. Riesz (1949).

Biroq, uni yangi mavzu deb hisoblash mumkin, chunki yigirma yildan sal ko'proq vaqtdan beri u maxsus konferentsiyalar va risolalarning asosiy obyekti bo’lib kelmoqda. Birinchi konferensiya uchun 1974-yil iyun oyida Nyu-Xeyven universitetida kasr hisobi va uni qo‘llash bo‘yicha birinchi konferensiyani B. Ross tashkil etgan va materiallarni tahrir qilgan. Birinchi monografiya uchun K.B. Oldham va J. Spanier 1968 yilda boshlangan qo'shma hamkorlikdan so'ng 1974 yilda kasr hisobiga bag'ishlangan kitobni nashr etishdi. Hozirgi vaqtda kasr hisobiga va uning qo'llanilishiga bag'ishlangan ishlar va ishlarning ro'yxati o'nga yaqin nomlarni o'z ichiga oladi [1-14], ular orasida Samko, Kilbas va Marichevning entsiklopedik risolasi [5] eng ko'zga ko'ringanidir.

So'nggi yillarda kasr hisobiga katta qiziqish ushbu hisobning raqamli tahlilda va fizika va muhandislikning turli sohalarida, shu jumladan fraktal hodisalarda topiladigan ilovalar tufayli rag'batlantirildi.

Grunwald-Letnikov kasr hosilasi.

Grunwald-Letnikov taklif etgan kasr hosila kasr tartibli tenglamalarni sonli tadqiq qilishda muhim rol o'ynaydi.Chunki ushbu kasr hosila bilan aniqlanadigan yig'indining hadlarini masala shartlariga aniqroq yaqinlashish uchun olib tashlash mumkin.

1869 yilda Sonin qo'shimcha differensiallash qoidalari haqida maqola chop etdi. Bu maqola Grunwald va Letnikov tomonidan taklif etgan kasr tartibli hosila haqida edi. Va bu fanda Grunwald-Letnikov kasr hosilasi sifatida tanildi.

$k \in \mathbb{N}$ lar uchun funksiyaning k tartibli hosilasi quyidagi formula yordamida aniqlanadi

$$D^k f(x) = \lim_{h\to 0} \frac{1}{h^k} \sum_{j=0}^{n} (-1)^j \binom{k}{j} f(x - jh)$$

bu yerda $n \geq k$ va $\binom{k}{j} = \frac{k!}{j!(k-j)!}$ binomial koeffisiyent. k manfiy qiymatlari uchun biz binomial koeffisiyentni quyidagicha kiritamiz

$$\binom{-k}{j} = (-1)^j \binom{k}{j}.$$

Va formal ravishda hosilani quyidagicha yozish mumkin

$$D^{-k} f(x) = \lim_{h\to 0} \frac{1}{h^k} \sum_{j=0}^{n} \binom{k}{j} f(x - jh).$$

Biz dastlab $x_0 \in R$ nuqta atrofida va $h = \frac{x-x_0}{n}$ da qaraymiz. Qulaylik uchun $x_0 = 0$ va bu hosila uchun bu qatorda h ni almashiramiz va $n \to \infty$ limitni olamiz.

Natijada Grunwald-Letnikov kasr hosilasi uchun formulani olamiz:

$${}_{0}^{GL}D_x^{\mu} f(x) = \lim_{n\to\infty} \left(\frac{n}{x}\right)^{\mu} \sum_{j=0}^{n} (-1)^j \binom{\mu}{j} f\left(x - \frac{x}{n} j\right).$$

Misol. $f(x) = x$ funksiyani $\frac{1}{2}$ tartibli Grunwald-Letnikov ma'nosidagi kasr hosilasini hisoblang.

Yechish. Yuqoridagi formulaga qo'yib sodda almashtirshdan so'ng biz quyidagi ifodaga ega bo'lamiz:

$${}_{0}^{GL}D_x^{\frac{1}{2}} x = \sqrt{x} \lim_{n\to\infty} \sqrt{n} \sum_{j=0}^{n} (-1)^j \binom{1/2}{j} \left(1 - \frac{j}{n}\right).$$

Qulaylik uchun yig'indini ikki qismga ajratamiz

$$ {}_{0}^{GL}D_x^{\frac{1}{2}} x = \sqrt{x} \left\{ \lim_{n\to\infty} \sqrt{n} \sum_{j=0}^{n} (-1)^j \binom{1/2}{j} - \lim_{n\to\infty} \frac{\sqrt{n}}{n} \sum_{j=1}^{n} (-1)^j \binom{1/2}{j} j \right\}. $$

Ikkichi yig'indin indeksini o'zgartirish uchun $j \to j+1$ almashtirish bajarib quyidagiga ega bo'lamiz:

$$ {}_{0}^{GL}D_x^{\frac{1}{2}} x = \sqrt{x} \left\{ \lim_{n\to\infty} \sqrt{n} \sum_{j=0}^{n} (-1)^j \frac{\Gamma(3/2)}{j!\,\Gamma(-j+3/2)} \right. $$

$$ \left. + \lim_{n\to\infty} \frac{\sqrt{n}}{n} \sum_{j=1}^{n} (-1)^j \frac{\Gamma(3/2)}{j!\,\Gamma(-j+1/2)} \right\}. $$

Ushbu yig'indilarni alohida-alohida hisoblaymiz. Ω_1 orqali birinchi yig'indini belgilaymiz

$$ \Omega_1 = \lim_{n\to\infty} \sqrt{n} \sum_{j=0}^{n} (-1)^j \binom{1/2}{j}. $$

$$ \sum_{j=0}^{n} (-1)^j \binom{a}{j} = (-1)^n \binom{a-1}{n} $$

munosabatdan foydalanib, $a = \frac{1}{2}$ uchun Ω_1 uchun quyidagicha tenglikni olamiz

$$ \Omega_1 = \lim_{n\to\infty} \sqrt{n} \frac{(-1)^n \Gamma\left(\frac{1}{2}\right)}{\Gamma(n+1)\Gamma\left(-n+\frac{1}{2}\right)}. $$

Gamma funksiyaning $\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(-n+\frac{1}{2}\right) = \pi(-1)^n$ xossasidan foydalanib, Ω_1 ni qayta yozib olamiz

$$ \Omega_1 = \frac{1}{\sqrt{\pi}} \lim_{n\to\infty} \sqrt{n} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)}. $$

$z \to \infty$ da Gamma funksiyaning assimptotik bahosidan

$$ \frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} \approx z^{\alpha-\beta} $$

Foydalanib $\alpha = \frac{1}{2}, \beta = 1$ uchun quyidagi natijaga ega bo'lamiz:

$$\Omega_1 = \frac{1}{\sqrt{\pi}} \lim_{n\to\infty} \sqrt{n}\, n^{\frac{1}{2}-1} = \frac{1}{\sqrt{\pi}}.$$

Ω_1 ga o'xshash, Ω_2 ni ya'ni ikkichi yig'indini yozamiz:

$$\Omega_2 = \lim_{n\to\infty} \frac{\Gamma\left(\frac{3}{2}\right)}{\sqrt{n}} \sum_{j=1}^{n} (-1)^j \frac{1}{j!\,\Gamma(-j+1/2)}$$

Yuqoridagi tenglikga $\Gamma\left(\frac{1}{2}\right)$ ni ko'paytirib bo'lsak va binom yoyilma formulasidan foydalanib quyidagi ifodani olamiz

$$\Omega_2 = \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{j=1}^{n-1} (-1)^j \binom{-{}^1\!/_2}{j}.$$

$a = -\frac{1}{2}, n \to n-1$ da

$$\sum_{j=0}^{n} (-1)^j \binom{a}{j} = (-1)^n \binom{a-1}{n}$$

formuladan foydalanib biz quyidagi ifodaga ega bo'lamiz

$$\Omega_2 = \frac{1}{2} \lim_{n\to\infty} \frac{1}{\sqrt{n}} (-1)^{n-1} \binom{-{}^3\!/_2}{n-1} =$$

$$= \frac{1}{\sqrt{\pi}} \lim_{n\to\infty} \frac{1}{\sqrt{n}} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n)}$$

Gamma funksiyaning assimptotik bahosidan foydalanib $\alpha = \frac{1}{2}, \beta = 0$ uchun

$$\Omega_2 = \frac{1}{\sqrt{\pi}} \lim_{n\to\infty} \frac{1}{\sqrt{n}} n^{\frac{1}{2}-0} = \frac{1}{\sqrt{\pi}}.$$

Ω_1 va Ω_2 natijalarini mos ravishda o'rniga keltirib qo'ysak, x funkisyaning ½ tartibli kasr hosilasiga ega bo'lamiz.

$${}_{0}^{GL}D_x^{\frac{1}{2}} x = \sqrt{x} \left\{ \frac{1}{\sqrt{\pi}} + \frac{1}{\sqrt{\pi}} \right\} = 2\frac{\sqrt{x}}{\sqrt{\pi}}.$$

Mustaqil yechish uchun topshiriqlar

Quyidagi funksiyalarning Grunward-Letnikov kasr hosilasini hisoblang.

1. $f(x) = \sqrt{x}$, 2. $f(x) = x^2$; 3. $f(x) = (x+1)^2$;

Rimann-Liuvill ma’nosidagi kasr hosila va kasr integral.

Kasr tartibli hosilalar va integrallar tushunchasi kiritilishi bilan hosila integral orasidagi chegara keskin yo‘qolib ketadi. Integral – differentsial operatorlarning kasr tartibli operatorlariga Koshi tipidagi umumlashgan integral formulasi kasr tartibli differentegrallarining quyidagi ta’riflariga olib keladi:

$$(I_{a+}^{\alpha}f)(x)=\frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}dt \qquad (1)$$

$$(D_{a+}^{\alpha}f)(x)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x \frac{f(t)}{(x-t)^{1+\alpha-n}}dt \qquad (2)$$

bu yerda

$\alpha \in C$, $\mathrm{Re}(\alpha) > 0$ va n=$[\mathrm{Re}(\alpha)] + 1$;

I_{a+}^{α}- α tartibli integral operator

D_{a+}^{α} - α tartibli differensial operator

Kasr tartibli hosilali va integralli operatorning bu ta’rifi yagona emas. Integrodifferensial operatorlarning Veyl, Grunwald-Letnikov, Caputo va boshqa ko‘rinishlari ham ma’lumdir.

$$I^{\alpha}f=\frac{1}{\Gamma(\alpha)}\int_0^x f(t)(x-t)^{\alpha-1}dt \qquad (3)$$

Endi $D^{\alpha}=\frac{d^{\alpha}}{dx^{\alpha}}$, ni aniqlaymiz.

Biz D^{α} ni I^{α} operatorning chap teskari operatori sifatida qaraymiz.

1-tasdiq. $D^{\alpha}I^{\alpha}$ =J ayniyatga ko’ra 0<$\alpha < 1$ bo’lgan hol uchun $D^{\alpha}=DI^{1-\alpha}$

tenglik kasr tartibli differensial operator bo’ladi.

Bu tasdiqni isbotlashdan avval ∀ > α β, 0 sonlar uchun quyidagi tenglikni o‘rinli ekanligidan foydalanamiz. $I^{\alpha}I^{\beta} = I^{\beta}I^{\alpha} = I^{\alpha+\beta}$ (kommuntativlik), (3) formulaga ko‘ra isbot qilinadi.

D^{α} I^{α} operatorning chap teskari operatori ekan demak, $D^{\alpha}I^{\alpha}$ =J ayniyat o‘rinli. D^{α} ning tasdiqda keltirilgan ifodasidan foydalanib ayniyatni tekshirib ko‘raylik:

$D^{\alpha}I^{\alpha} = D^{\alpha}I^{1-\alpha}I^{\alpha} = D^{\alpha}I^{1-\alpha+\alpha}$=*DI=J*

Tasdiq o’rinli bo’ldi.

Endi hozirgina isbotlangan tasdiq hamda (3) formula yordamida 0 $<\alpha<$ 1 bo‘lgan hol uchun kasr tartibli hosila tushunchasini kiritamiz:

$$D^{\alpha}f(x) = DI^{1-\alpha}f(x) = \frac{d}{dx}\frac{1}{\Gamma(1-\alpha)}\int_0^x f(t)(x-t)^{1-\alpha-1}dt$$

$$= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x \frac{f(t)}{(x-t)^{\alpha}}dt \quad (4)$$

$D^{\alpha} = D^{n}I^{n-\alpha}$*-bu operator tenglikni analitik formula ko’rinishida tasvirlaymiz*

$$D^{\alpha}f = D^{n}I^{n-\alpha} = (\frac{d}{dx})^{n}\frac{1}{\Gamma(n-\alpha)}\int_0^x \frac{f(t)}{(x-t)^{\alpha-n+1}}dt$$

Bu formula tartibi ixtiyoriy nomanfiy haqiqiy son bo‘lgan kasr tartibli hosila uchun Riman-Luivill formulasi deyiladi. Ixchamlab yozsak quyidagini olamiz:

$$D^{\alpha}f = (\frac{d}{dx})^{n}\frac{1}{\Gamma(n-\alpha)}\int_0^x \frac{f(t)}{(x-t)^{\alpha-n+1}}dt \quad (5)$$

Masalan: 1)f(x)=x^{μ} funksiyaning Riman-Luivill integralini hisoblang

$I^{\alpha}f=\frac{1}{\Gamma(\alpha)}\int_0^x f(t)(x-t)^{\alpha-1}dt=\frac{1}{\Gamma(\alpha)}\int_0^x t^{\mu}(x-t)^{\alpha-1}dt=$|t=ux|

$\frac{x^{\mu+\alpha}}{\Gamma(\alpha)}\int_0^1 u^{\mu}(1-u)^{\alpha-1}du=\frac{x^{\mu+\alpha}}{\Gamma(\alpha)}B(\mu+1,\alpha) = \frac{x^{\mu+\alpha}\Gamma((\mu+1)}{\Gamma(\mu+1+\alpha)}$

$$I^{\alpha}(x^{\mu}) = \frac{x^{\mu+\alpha}\Gamma(\mu+1)}{\Gamma(\mu+1+\alpha)} \qquad (6)$$

2) $I^{\frac{1}{2}}(x^{\frac{1}{2}})$ ni hisoblang.

Yechish: Buni bajarish uchun (6) formuladan foydalanib hisoblaymiz.

$I^{\frac{1}{2}}(x^{\frac{1}{2}}) = \frac{x^{\frac{1}{2}+\frac{1}{2}}\Gamma(\frac{1}{2}+1)}{\Gamma(\frac{1}{2}+\frac{1}{2}+1)}=\frac{x\Gamma(\frac{3}{2})}{\Gamma(2)}$

$\Gamma\left(\frac{3}{2}\right)$ ni quyidagi formula orqali hisoblaymiz:

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}(2n-1)!!}{2^n}$$

$\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$, $\Gamma(2)=1$

$I^{\frac{1}{2}}(x^{\frac{1}{2}})= \frac{\sqrt{\pi}}{2}x$

3) $I^{\alpha}(e^x)$ ning qiymatini toping.

$I^{\alpha}(e^x) = \frac{1}{\Gamma(\alpha)}\int_0^x e^t\,(x-t)^{\alpha-1}dt=\frac{1}{\Gamma(\alpha)}\int_0^x \sum_{k=0}^{+\infty}\frac{t^k}{k!}(x-t)^{\alpha-1}dt=$

$=\frac{1}{\Gamma(\alpha)}\sum_{k=0}^{+\infty}\frac{1}{k!}\int_0^x t^k(x-t)^{\alpha-1}\,dt=|t=ux|$

$=\sum_{k=0}^{+\infty}\frac{x^{k+\alpha}}{k!\Gamma(\alpha)}\int_0^1 u^k\,(1-u)^{\alpha-1}du=\sum_{k=0}^{+\infty}\frac{x^{k+\alpha}}{k!\Gamma(\alpha)}B(k+1,\alpha)=\sum_{k=0}^{+\infty}\frac{x^{k+\alpha}}{k!\Gamma(\alpha)}\frac{\Gamma(k+1)\Gamma(\alpha)}{\Gamma(\alpha+k+1)}=$

$=x^{\alpha}\sum_{k=0}^{+\infty}\frac{x^k}{\Gamma(k+\alpha+1)}=x^{\alpha}E_{1,\alpha+1}(x)$

Demak

$$I^{\alpha}(e^x) = x^{\alpha}E_{1,\alpha+1}(x)$$

4) f(x)= C= o‘zgarmas sonning $\alpha=\frac{1}{2}$ bo‘lganda Riman-Luivill hosilasini hisoblang

$D^{\frac{1}{2}}$(C)= $\frac{1}{\Gamma(1-\frac{1}{2})}\frac{d}{dx}\int_0^x \frac{C}{(x-t)^{1-1+\frac{1}{2}}}$dt=$\frac{C}{\Gamma(\frac{1}{2})}\frac{d}{dx}(-2(x-t)^{\frac{1}{2}})|_0^x$=$\frac{C}{\Gamma\left(\frac{1}{2}\right)}\frac{d}{dx}(2x^{\frac{1}{2}})$= $\frac{C}{\sqrt{\pi x}}$

$$D^{\frac{1}{2}}(C)= \frac{C}{\sqrt{\pi x}}$$

5) f(x)=x^{μ} funksiyaning n-1<$\alpha < n$ bo‘lganda Riman-Luivill hosilasini hisoblang.

$D^{\alpha}f$=$\frac{1}{\Gamma(n-\alpha)}(\frac{d}{dx})^n\int_0^x \frac{t^{\mu}}{(x-t)^{\alpha-n+1}}$dt|t=xu|=

$$= \frac{1}{\Gamma(n-\alpha)}(\frac{d}{dx})^n x^{\mu+n-\alpha}\int_0^1 u^{\mu}(1-u)^{n-\alpha-1}du =$$

$$\frac{1}{\Gamma(n-\alpha)}(\frac{d}{dx})^n x^{\mu+n-\alpha}B(\mu+1, n-\alpha) = \frac{\Gamma(\mu+1)}{\Gamma(n-\alpha+\mu+1)}(\frac{d}{dx})^n x^{\mu+n-\alpha}$$

$$=\frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}x^{\mu-\alpha}$$

Demak

$D^{\alpha}(x^{\mu})$= $\frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}x^{\mu-\alpha}$

6) $I^{\alpha}\left(t^{\beta-1}E_{\mu,\beta}(\lambda t^{\mu})\right)(x) = x^{\alpha+\beta-1}E_{\mu,\alpha+\beta}(\lambda x^{\mu})$ tenglikni isbotlang.

Yechish:

$$I^{\alpha}\left(x^{\beta-1}E_{\mu,\beta}(\lambda x^{\mu})\right) = \frac{1}{\Gamma(\alpha)}\sum_{k=0}^{+\infty}\frac{\lambda^k}{\Gamma(\mu k+\beta)}\int_0^x \frac{t^{\beta-1+\mu k}}{(x-t)^{1-\alpha}}dt =$$

$$= |t = xu|$$

$= \sum_{k=0}^{+\infty}\frac{\lambda^k x^{\alpha+\beta+\mu k-1}}{\Gamma(\mu k+\beta)\Gamma(\alpha)}B(\beta+\mu k, \alpha)= x^{\alpha+\beta-1}\sum_{k=0}^{+\infty}\frac{(\lambda x^{\mu})^k}{\Gamma(\mu k+\alpha+\beta)}$=

$= x^{\alpha+\beta-1}E_{\mu,\alpha+\beta}(\lambda x^{\mu})$

$Isbotlandi.$

Mustaqil bajarish uchun topshiriqlar.

1. $I^{\frac{3}{2}}(x^2)$ ni hisoblang.

2. $D^{\alpha}\left(t^{\beta-1}E_{\mu,\beta}(\lambda t^{\mu})\right)(x) = x^{\beta-\alpha-1}E_{\mu,\beta-\alpha}(\lambda x^{\mu})$ tenglikni isbotlang

3. $D^{\frac{1}{2}}(12)$ ning qiymatini toping.

4. $f(x) = \cos x$ funksiyaning Riman-Liuvill kasr integralini hisoblang.

Kaputo kasr hosilasi.

$[a,b]$ haqiqiy sonlar o'qidagi chekli interval va $D^{\alpha}_{a+}[y(t)](x) \equiv (D^{\alpha}_{a+}y)(x)$ and $D^{\alpha}_{b-}[y(t)](x) \equiv (D^{\alpha}_{b-}y)(x)$ lar mos ravishda Riemann-Liouvillening $\alpha \in C$ $(Re(\alpha) \geq 0)$ tartibli kasr hosilalari bo'lsin.

Bu yerda

$$(D^{\alpha}_{a+}y)(x) \coloneqq \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^{n}\int_{a}^{x}\frac{y(t)dt}{(x-t)^{\alpha-n+1}} \qquad (n = [Re(\alpha)+1]; x > a) \quad (1)$$

va

$$(D^{\alpha}_{b-}y)(x) \coloneqq \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^{n}\int_{x}^{b}\frac{y(t)dt}{(t-x)^{\alpha-n+1}} \qquad (n = [Re(\alpha)+1]; x < b) \quad (2)$$

$[a,b]$ da $\alpha \in C$ $(Re(\alpha) \geq 0)$ tartibli $({}^{c}D_{a+}^{\alpha}y)(x)$ va $({}^{c}D_{b-}^{\alpha}y)(x)$ kasr hosilalar Riemann-Liouville kasr hosilalari orqali quyidagicha ta'riflanadi:

$$({}^{c}D_{a+}^{\alpha}y)(x) := \left(D_{a+}^{\alpha}\left[y(t) - \sum_{k=0}^{n-1}\frac{y^{(k)}(a)}{k!}(t-a)^k\right]\right)(x) \qquad (3)$$

va

$$({}^{c}D_{b-}^{\alpha}y)(x) := \left(D_{b-}^{\alpha}\left[y(t) - \sum_{k=0}^{n-1}\frac{y^{(k)}(b)}{k!}(b-t)^k\right]\right)(x) \qquad (4)$$

Bu yerda

$$n = [Re(\alpha)] + 1 \text{ for } \alpha \notin N_0;\ n = \alpha \text{ for } \alpha \in N_0 \qquad (5)$$

Bu hosilalar mos ravishda α tartibli *chap va o'ng Kaputo kasr hosilalari* deb ataladi.

Ko'p hollarda , $0 < Re(\alpha) < 1$ bo'lganda, (1) va (2) formulalar quyidagi ko'rinishni oladi:

$$({}^{c}D_{a+}^{\alpha}y)(x) := (D_{a+}^{\alpha}[y(t) - y(a)])(x) \qquad (6)$$

$$({}^{c}D_{b-}^{\alpha}y)(x) := (D_{b-}^{\alpha}[y(t) - y(b)])(x) \qquad (7)$$

Agar $\alpha \notin N_0$ va $y(x)$ funksiya shunday funksiyaki,qaysiki $\alpha \in C$ $(Re(\alpha) \geq 0)$ tartibli $({}^{c}D_{a+}^{\alpha}y)(x)$ va $({}^{c}D_{b-}^{\alpha}y)(x)$ Kaputo kasr hosilalar $(D_{a+}^{\alpha}y)(x)$ va $(D_{b-}^{\alpha}y)(x)$ Riemann-Liouville kasr hosilalari bilan birgalikda mavjud bo'lsa, ular quyidagichi munosabatlar orqali bog'langan:

$$({}^{c}D_{a+}^{\alpha}y)(x) := (D_{a+}^{\alpha}y)(x) - \sum_{k=0}^{n-1}\frac{y^{(k)}(a)}{\Gamma(k-\alpha+1)}(x-a)^{k-\alpha} \quad (n = [Re(\alpha)] + 1) \qquad (8)$$

va

$$({}^cD_{b-}^{\alpha}y)(x) \coloneqq (D_{b-}^{\alpha}y)(x)$$
$$-\sum_{k=0}^{n-1}\frac{y^{(k)}(b)}{\Gamma(k-\alpha+1)}(b-x)^{k-\alpha} \quad (n=[Re(\alpha)]+1) \qquad (9)$$

Ko'p hollarda, $0 < Re(\alpha) < 1$ bo'lganda

$$({}^cD_{a+}^{\alpha}y)(x) \coloneqq (D_{a+}^{\alpha}y)(x) - \frac{y(a)}{\Gamma(1-\alpha)}(x-a)^{-\alpha} \quad (10)$$

va

$$({}^cD_{b-}^{\alpha}y)(x) \coloneqq (D_{b-}^{\alpha}y)(x) - \frac{y(b)}{\Gamma(1-\alpha)}(b-x)^{-\alpha} \quad (11)$$

Agar $\alpha \notin N_0$ bo'lsa, u holda (3) va (4) Kaputo kasr hosilalari (1) va (2) Riemann-Liouville kasr hosilalari bilan quyidagicha bog'langan:

$$({}^cD_{a+}^{\alpha}y)(x) \coloneqq (D_{a+}^{\alpha}y)(x) \qquad (12)$$

agar $y(a) = y'(a) = \cdots = y^{(n-1)}(a) = 0$, $(n = [Re(\alpha)] + 1)$; va

$$({}^cD_{b-}^{\alpha}y)(x) \coloneqq (D_{b-}^{\alpha}y)(x) \qquad (13)$$

agar $y(b) = y'(b) = \cdots = y^{(n-1)}(b) = 0$, $(n = [Re(\alpha)] + 1)$.

Odatda , $0 < Re(\alpha) < 1$ bo'lganda

$$({}^cD_{a+}^{\alpha}y)(x) \coloneqq (D_{a+}^{\alpha}y)(x), \ y(a) = 0 \text{ bo'lsa,} \qquad (14)$$

$$({}^cD_{b-}^{\alpha}y)(x) \coloneqq (D_{b-}^{\alpha}y)(x), \ y(b) = 0 \text{ bo'lsa,} \qquad (15)$$

Agar $\alpha = n \in N_0$ va $y^{(n)}(x)$ n-tartibli hosila mavjud bo'lsa, u holda

$$({}^cD_{b-}^{n}y)(x) = y^{(n)}(x) \quad \text{va} \quad ({}^cD_{b-}^{n}y)(x) = (-1)^n y^{(n)}(x)$$

(16)

$({}^cD_{a+}^{\alpha}y)(x)$ va $({}^cD_{b-}^{\alpha}y)(x)$ Kaputo kasr hosilalar (3) va (4) tengliklarning o'ng tomonidagi Riemann-Liouville kasr hosilalari mavjud bo'lgan y(x) funksiyalari uchun ta'riflanadi. Odatda, ular absolut uzluksiz funksiyalar $AC^n[a,b]$ fazosiga tegishli y(x) funksiyalar uchun ta'riflanadi.

Teorema 1. Faraz qilaylik, $Re(\alpha) \geq 0$ va n (5) tenglik orqali berilgan bo'lsin. Agar y(x) $\in AC^n[a,b]$ bo'lsa, u holda $[a,b]$ ning deyarli barcha yerida $({}^cD_{a+}^{\alpha}y)(x)$ va $({}^cD_{b-}^{\alpha}y)(x)$ Kaputo kasr hosilalar mavjud bo'ladi.

(a) Agar $\alpha \notin N_0$, $({}^cD_{a+}^{\alpha}y)(x)$ va $({}^cD_{b-}^{\alpha}y)(x)$ lar quyidagicha ifodalanadi

$$({}^cD_{a+}^{\alpha}y)(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{y^{(n)}(t)dt}{(x-t)^{\alpha-n+1}} =: (I_{\alpha+}^{n-\alpha}D^n y)(x) \qquad (17)$$

va

$$({}^cD_{b-}^{\alpha}y)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^b \frac{y^{(n)}(t)dt}{(t-x)^{\alpha-n+1}} =: (-1)^n\,(I_{b-}^{n-\alpha}D^n y)(x) \qquad (18)$$

Bu yerda $D = \frac{d}{dx}$ va $n = [Re(\alpha)] + 1$

$I_{a+}^{\alpha}y$ va $I_{b-}^{\alpha}y$ lar $\alpha \in C$ $(Re(\alpha) \geq 0)$ tartibli Riemann-Liouville kasr integrallari bo'lib, quyidagicha formula orqali aniqlanadi:

$$(I_{a+}^{\alpha}y)(x) := \frac{1}{\Gamma(\alpha)}\int_a^x \frac{y(t)dt}{(x-t)^{1-\alpha}} \quad (x > a; Re(\alpha) > 0)$$

va

$$(I_{b-}^{\alpha}y)(x) := \frac{1}{\Gamma(\alpha)}\int_x^b \frac{y(t)dt}{(t-x)^{1-\alpha}} \quad (x < b; Re(\alpha) > 0)$$

Odatda, $0 < Re(\alpha) < 1$ va y(x) $\in AC[a,b]$ bo'lganda

$$({}^cD_{a+}^{\alpha}y)(x) = \frac{1}{\Gamma(1-\alpha)}\int_a^x \frac{y'(t)dt}{(x-t)^{\alpha}} =: (I_{\alpha+}^{1-\alpha}Dy)(x) \qquad (19)$$

va

$$({}^cD_{b-}^{\alpha}y)(x) = -\frac{1}{\Gamma(1-\alpha)}\int_x^b \frac{y'(t)dt}{(t-x)^{\alpha}} =: -\,(I_{b-}^{1-\alpha}Dy)(x) \qquad (20)$$

(b) Agar $\alpha = n \in N_0$ bo'lsa, u holda $({}^{c}D_{a+}^{\alpha}y)(x)$ va $({}^{c}D_{b-}^{\alpha}y)(x)$ lar (16) kabi ifodalanadi. Odatda,

$$({}^{c}D_{a+}^{0}y)(x) = ({}^{c}D_{b-}^{0}y)(x) = y(x) \qquad (21)$$

Teorema 2. Faraz qilaylik, $Re(\alpha) \geq 0$ va n (5) tenglik orqali berilgan bo'lsin. Shuningdek,

$y(x) \in C^{n}[a,b]$. U holda , $({}^{c}D_{a+}^{\alpha}y)(x)$ va $({}^{c}D_{b-}^{\alpha}y)(x)$ Kaputo kasr hosilalar $[a,b]$da uzluksiz:

$({}^{c}D_{a+}^{\alpha}y)(x) \in C[a,b]$ va $({}^{c}D_{b-}^{\alpha}y)(x) \in C[a,b]$.

(a) Agar $\notin N_0$, $({}^{c}D_{a+}^{\alpha}y)(x)$ va $({}^{c}D_{b-}^{\alpha}y)(x)$ lar (17) va (18) orqali ifodalanadi. Bundan tashqari,

$$({}^{c}D_{a+}^{\alpha}y)(a) = ({}^{c}D_{b-}^{\alpha}y)(b) = 0 \qquad (22)$$

(b) Agar $\alpha = n \in N_0$ bo'lsa, u holda $({}^{c}D_{a+}^{\alpha}y)(x)$ va $({}^{c}D_{b-}^{\alpha}y)(x)$ kasr hosilalar (14) orqali ifodalanadi. Odatda (21)tenglik o'rinli.

Xossa: Faraz qilaylik, $Re(\alpha) \geq 0$ va n (5) tenglik orqali berilgan bo'lsin. Shuningdek, $Re(\beta) > 0$ bo'lsin. U holda quyidagi munosabatlar o'rinli:

$$({}^{c}D_{a+}^{\alpha}(t-\alpha)^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(x-a)^{\beta-1} \qquad (Re(\beta) > n) \qquad (23)$$

$$({}^{c}D_{b-}^{\alpha}(b-t)^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(b-x)^{\beta-1} \qquad (Re(\beta) > n) \qquad (24)$$

va

$$({}^{c}D_{a+}^{\alpha}(t-\alpha)^{k})(x) = 0 \quad \text{va} \quad ({}^{c}D_{b-}^{\alpha}(b-t)^{k})(x) = 0 \qquad (k = 0,1,2,\dots,n-1) \qquad (25)$$

Odatda,

$$({}^{c}D_{a+}^{\alpha}1)(x) = 0 \text{ va } ({}^{c}D_{b-}^{\alpha}1)(x) = 0 \qquad (26)$$

Biz Kaputo kasr hosilasini chekli $[a,b]$ intervalda (3) va (4) formulalar orqali ta'rifladik, va Teorema 1 va Teorema 2 larga ko'ra y(x) $\in AC^{n}[a,b]$ va $y(x) \in C^{n}[a,b]$ bo'lganda (16) yoki (17) va (18) orqali ifodaladik. (17) va (18) formulalar yarim haqiqiy sonlar o'qi R^{+} va butun sonlar o'qi R da Kaputo kasr hosilasining ta'rifi sifatida ishlatilishi mumkin. Shuning uchun, yarim haqiqiy sonlar o'qi R^{+} va butun sonlar o'qi R da $\alpha \in C\ (Re(\alpha) > 0\ va\ \alpha \neq N)$ tartibli Kaputo kasr hosilasining mos formulalari quyidagichadir:

$$({}^{c}D_{0+}^{\alpha}y)(x) = \frac{1}{\Gamma(n-\alpha)}\int_{0}^{x}\frac{y^{(n)}(t)dt}{(x-t)^{\alpha-n+1}} \qquad x \in R^{+} \qquad (27)$$

$$({}^{c}D_{-}^{\alpha}y)(x) = \frac{(-1)^{n}}{\Gamma(n-\alpha)}\int_{x}^{\infty}\frac{y^{(n)}(t)dt}{(t-x)^{\alpha-n+1}} \qquad x \in R^{+} \qquad (28)$$

va

$$({}^{c}D_{+}^{\alpha}y)(x) = \frac{1}{\Gamma(n-\alpha)}\int_{-\infty}^{x}\frac{y^{(n)}(t)dt}{(x-t)^{\alpha-n+1}} \qquad x \in R \qquad (29)$$

$$({}^{c}D_{-}^{\alpha}y)(x) = \frac{(-1)^{n}}{\Gamma(n-\alpha)}\int_{x}^{\infty}\frac{y^{(n)}(t)dt}{(t-x)^{\alpha-n+1}} \qquad x \in R \qquad (30)$$

$0 < Re(\alpha) < 1$ bo'lganda (27)-(28) va (29)-(30) formulalar quyidagicha ko'rinishni oladi:

$$({}^{c}D_{0+}^{\alpha}y)(x) = \frac{1}{\Gamma(1-\alpha)}\int_{0}^{x}\frac{y'(t)dt}{(x-t)^{\alpha}} \qquad x \in R^{+} \qquad (31)$$

$$({}^{c}D_{-}^{\alpha}y)(x) = -\frac{1}{\Gamma(1-\alpha)}\int_{x}^{\infty}\frac{y'(t)dt}{(t-x)^{\alpha}} \qquad x \in R^{+} \qquad (32)$$

va

$$({}^{c}D_{+}^{\alpha}y)(x) = \frac{1}{\Gamma(1-\alpha)}\int_{-\infty}^{x}\frac{y'(t)dt}{(x-t)^{\alpha}} \qquad x \in R \qquad (33)$$

$$({}^{c}D_{b-}^{\alpha}y)(x) = -\frac{1}{\Gamma(1-\alpha)}\int_{x}^{\infty}\frac{y'(t)dt}{(t-x)^{\alpha}} \qquad x \in R \qquad (34)$$

Xossa 2. Agar $Re(\alpha) > 0\ va\ \lambda > 0$ bo'lsa, u holda

$$({}^{c}D_{+}^{\alpha}e^{\lambda t})(x) = \lambda^{\alpha}e^{\lambda x} \text{ va } ({}^{c}D_{-}^{\alpha}e^{-\lambda t})(x) = \lambda^{\alpha}e^{-\lambda x} \qquad (35)$$

$E_\alpha[\lambda(x-a)^\alpha]$ Mittag-Leffler funksiyasi ${}^{c}_{.}D^{\alpha}_{a+}$ Kaputo kasr hosilasida invariant, lekin ${}^{c}_{.}D^{\alpha}_{-}$ Kaputo kasr hosilasi uchun bu o'rinli emas.

Lemma. Agar $\alpha > 0, \alpha \in R \; va \; \lambda \in C$ bo'lsa , u holda

$$({}^{c}D^{\alpha}_{a+} E_\alpha[\lambda(t-a)^\alpha])(x) = \lambda E_\alpha[\lambda(x-a)^\alpha] \qquad (36)$$

va

$$({}^{c}D^{\alpha}_{-} t^{\alpha-1} E_\alpha[\lambda t^{-\alpha}])(x) = \frac{1}{x} E_{\alpha,1-\alpha}[\lambda x^{-\alpha}] \qquad (37)$$

Odatda, $\alpha = n \in N$ bo'lganda

$$D^n E_n[\lambda(x-a)^n] = E_n[\lambda(x-a)^n] \qquad (38)$$

va

$$D^n\left[t^{n-1} E_n[\lambda t^{-n}]\right](x) = \frac{1}{x} E_{n,1-n}[\lambda x^{-n}] = \frac{\lambda}{x^{n+1}} E_n[\lambda x^{-n}] \qquad (39)$$

Quyidagilar orqali $({}^{c}D^{\alpha}_{0+} y)(x)$ Kaputo kasr hosilasining Laplas almashtirishini keltiramiz:

Lemma. Faraz qilaylik, $\alpha > 0, n-1 < \alpha \leq n \;\; (n \epsilon N)$, $\forall b > 0$ uchun $y(x) \in C^n(R^+)$, $y^{(n)}(x) \in L_1(0,b)$, $y^{(n)}(x)$funksiya exponensial funksiya va $(Ly)(p)$ va $L[D^n \mathrm{y(t)}]$ Laplas almashtirishlari mavjud va $\lim\limits_{x\to\infty} D^k \mathrm{y(x)} = 0$ barcha $k = 0,1,\dots,n-1.$ U holda quyidagi tengliklar o'rinli:

$$(L\; {}^{c}D^{\alpha}_{0+} y)(s) = s^\alpha (Ly)(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} (D^k \mathrm{y})(0) \qquad (40)$$

Odatda, agar $0 < \alpha \leq 1$ bo'lsa, u holda

$$(L\; {}^{c}D^{\alpha}_{0+} y)(s) = s^\alpha (Ly)(s) - s^{\alpha-1} y(0) \qquad (41)$$

Mustaqil bajarish uchun mashqlar:

Quyidagi funksiyalarning (${}^{c}D_{0+}^{\alpha}y)(x)$ Kaputo kasr hosilasini toping.

1. $e^{\lambda x}$ 2. sin(λx) 3. cos(λx)

2. $\alpha > 0$ son uchun quyidagini isbotlang:

$$E_{\alpha}(-x) = E_{2\alpha}(x^2) - xE_{2\alpha,\alpha+1}(x^2)$$

3. $x > 0\ va\ \alpha,\beta,\gamma > 0$ sonlar uchun quyidagini isbotlang:

$$\frac{1}{\Gamma(\gamma)}\int_0^x (x-\xi)^{\gamma-1}\,\xi^{\beta-1}E_{\alpha,\beta}(\xi^{\alpha})d\xi = x^{\beta+\gamma-1}E_{\alpha,\beta+\gamma}(x^{\alpha})$$

4. Isbotlang:

$$E_{\alpha,\beta}(-x) = \frac{1}{\alpha\Gamma(\beta-\alpha)}\int_0^1 \left(1-\xi^{\frac{1}{\alpha}}\right)^{\beta-\alpha-1} E_{\alpha,\alpha}(-x\xi)d\xi$$

5. Isbotlang:

$$\frac{d^k}{dx^k}\left[x^{n-1}E_{k,n}(\lambda x^k)\right] = \lambda x^{n-1}E_{k,n}(\lambda x^k)$$

6. Isbotlang:

$$\left(x\frac{d}{dx}+p\right)E_{\alpha,\beta}^{p}(x) = pE_{\alpha,\beta}^{p+1}(x)$$

7. Isbotlang:

$$E_{1,2}(x) = 1 + xE_{1,3}(x)$$

8. Isbotlang:

$$xE_{2,2}(-x)^2 = sinx$$

9. $\rho = 2, \alpha = 1\ va\ \beta = 2$ uchun $E_{\alpha,\beta}^{\rho}$ ni hisoblang.

10. $\frac{d^n}{dx^n}E_n(x^n) = E_n(x^n)$ ni n=2 uchun isbotlang.

11. $f(x) = e^{-x^2}$, $x\epsilon\mathbb{R}$ funksiyaning Furye almashtirishini hisoblang.

12. $f(t) = e^{2t}cost$ funksiyaning Laplas almashtirishini aniqlang.

13. $f(t) = 4tcos^2t$ funksiyaning Laplas almashtirishini aniqlang.

14. $E_{\alpha,\beta}^{1}(x) = E_{\alpha,\beta}(x)$ ni isbotlang.

Xulosa.

Kasr tartibli differensiallash butun sonli tartibli differensiallashning umumlashmasidir. Kasr tartibli hosilalarni hisoblash klassik butun tartibli hosilalarga qaraganda ancha murakkabroq. Shu bilan birga, kasr tartibli integral va hosilalarga asoslangan matematik modellar ko'pgina hodisalarning xossalarini ilgari qo'llanilgan butun tartibli modellarga qaraganda aniqroq tasvirlashi ko'rsatilgan. Buning sababi, tizimlarning odatda mukammal emasligi va, masalan, tashqi kuchlar tomonidan ta’sir etishi mumkin. Shuning uchun, butun tartibli hosilalar holat o'zgaruvchilari traektoriyalarini tushunish uchun mos kelmasligi mumkin. Kasr hosilalari bilan bizning ixtiyorimizda cheksiz miqdordagi hosila tartiblari mavjud, shuning uchun biz qaysi kasrli differentsial tenglama modelning dinamikasini yaxshiroq tavsiflashini aniqlashimiz mumkin. Eksperimental ma'lumotlar va ba'zi real hodisalar uchun algoritmlar kasr tartib hosilalari yechim egri chizig'ini yanada samarali modellashtirishni ta'minlaydi.

Foydalanilgan adabiyotlar

1. T.M. Atanackovi´c, S. Pilipovi´c, B. Stankovi´c and D. Zorica (2014): Fractional Calculus with Applications in Mechanics, 2 Vol l., Wiley, New York (2014).
2. Yu.I. Babenko (1986): Heat and Mass Transfer, Chimia, Leningrad 1986 [in Russian].
3. R.T. Baillie and M.L. King (Editors)(1996): Fractional Differencing and Long Memory Processes, Journal of Econometrics 73(1)(1996), 1-324.
4. A.V. Balakrishnan (1959): Operational calculus for infinitesimal generators of semi-groups, Trans. Amer. Math. Soc. 91(2)(1959), 330-353.
5. A.V. Balakrishnan (1960): Fractional powers of closed operators and the semi-groups generated by them, Pacific J. Math. 10(2)(1960). 419-437.
6. D. Baleanu, K. Diethelm, E. Scalas and J.J. Trujillo (2012): Fractional Calculus: Models and Numerical Methods. World Scientific, Singapore (2012).
7. R. Balescu (2007): V-Langevin equations, continuous time random walks and fractional diffusion, Chaos, Solitons and Fractals 34(2007), 62-80.
8. J.H. Barret, (1954): Differential equations of non-integer order, Canad. J. Math., 6(1954), 529-541.
9. C.M. Bender and S.A. Orszag (1987): Advanced Mathematical Methods for Scientists and Engineers, McGraw-Hill, Singapore 1987, Ch 3.
10. L. Blank (1996): Numerical treatment of differential equations of fractional order, MCCM Numerical Analysis Report No. 287, The University of Manchester 1996.
11. S. Bochner (1949): Diffusion equation and stochastic processes, Proc. Nat. Acad. Sciences, USA, 35(1949), 368-370.
12. P. Butzer and U. Westphal (2000): Introduction to fractional calculus, in: H. Hilfer (Editor), Fractional Calculus, Applications in Physics, World Scientific, Singapore 2000, pp. 1-85.
13. R. Gorenflo (1997): Fractional calculus: some numerical methods, in: A. Carpinteri and F. Mainardi (Editors), Fractals and Fractional Calculus in Continuum Mechanics, Springer Verlag, Wien, 1997, pp. 277-290.

14. A.A. Kilbas, H.M. Srivastava and J.J. Trujillo (2006): Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam 2006 [North-Holland Mathematics Studies No 204].
15. A.V. Letnikov (1868): On historical development of differentiation theory with an arbitrary index, Mat. Sb., 3(1868), 85-112 [In Russian].
16. K.S. Miller and B. Ross (1993): An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York (1993).
17. I. Podlubny (1999): Fractional Differential Equations, Academic Press, San Diego (1999).
18. B. Ross (Editor)(1975): Fractional Calculus and its Applications, Springer Verlag, Berlin (1975), [Lecture Notes in Mathematics # 457].
19. B. Rubin (1996): Fractional Integrals and Potentials, Addison Wesley and Longman, Harlow 1996, [Pitman Monographs and Surveys in Pure and Applied Mathematics # 82].

MUNDARIJA