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A quadratic operator corresponding to a non-stochastic matrix on 2D-simplex

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Abstract. In this paper, we consider a quadratic non-stochastic operator mapping two-dimensional (2D) simplex to itself. We find all fixed points and invariant sets of the operator. Moreover, we study behavior of trajectories generated by the operator.

Keywords: simplex; quadratic (non)-stochastic operator; fixed point; invariant set, limit point.

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1 Definitions

Non-linear dynamical systems arise in many problems of biology, physics and other sciences. In particular, quadratic dynamical systems describe the behavior of populations of different species (see [1]-[7] and the references therein).

Let $E = \{1, 2, ..., m\}$. A distribution on the set E is a probability measure $x = (x_1, ..., x_m)$, i.e., an element of the simplex:

$$S^{m-1} = \left\{ x \in \mathbb{R}^m : x_i \geq 0, \sum_{i=1}^{m} x_i = 1 \right\}.$$

In general, a quadratic operator $V : x \in \mathbb{R}^m \to x' = V(x) \in \mathbb{R}^m$ is defined by:

$$V: \ x'_k = \sum_{i,j=1}^{m} P_{ij,k} x_i x_j, \ \ k = 1, ..., m. \tag{1}$$

The following theorem gives conditions for coefficients of V to preserve the simplex.

Theorem 1.1. *[7] For a quadratic operator V, given by (1), to preserve a simplex S^{m-1} it is* **sufficient** *that*

i) $\sum_{k=1}^{m} P_{ij,k} = 1, \ \ i, j = 1, \dots, m;$

ii) $0 \leq P_{ii,k} \leq 1, \ \ i, k = 1, \dots, m;$

iii) $-\frac{1}{m-1}\sqrt{P_{ii,k}P_{jj,k}} \leq P_{ij,k} \leq 1 + \sqrt{(1 - P_{ii,k})(1 - P_{jj,k})}.$

and **necessary** *that the conditions i), ii) and*

iii') $-\sqrt{P_{ii,k}P_{jj,k}} \leq P_{ij,k} \leq 1 + \sqrt{(1 - P_{ii,k})(1 - P_{jj,k})}$

are satisfied.

Definition 1.2. [7] A quadratic operator (1), preserving a simplex, is called non-stochastic (QnSO) if at least one of its coefficients $P_{ij,k}, \ i \neq j$ is negative.

In this paper, we study (see Remark 2.2 in [7]) the following example of QnSO on the 2D-simplex S^2:

$$V_0: \begin{cases} x' = \frac{1}{2}(z-y)^2 + \frac{3}{2}x(y+z) \\ y' = \frac{1}{2}(x-z)^2 + \frac{3}{2}y(x+z) \\ z' = \frac{1}{2}(y-x)^2 + \frac{3}{2}z(x+y). \end{cases} \tag{1.1}$$

Let s_3 be a permutation group of order 3. We define the action of s_3 on S^2 in the following way: if $g \in s_3$, $x \in S^2$ and $M \subseteq S^2$, then

$$g(x) = (x_{g(1)}, x_{g(2)}, x_{g(3)}), \quad g(M) = \{g(x) : x \in M\}.$$

The action of s_3 on the operator V_0 is defined as follows:

$$(gV_0)(x) = g(V_0(x)).$$

2 Fixed points

Lemma 2.1. *If x is a fixed point of the operator V_0, i.e., $V_0(x) = x$, then for any $g \in s_3$, the point $g(x)$ is also a fixed point.*

The fixed points are solutions of the system

$$\begin{cases} x = \frac{1}{2}(z-y)^2 + \frac{3}{2}x(y+z) \\ y = \frac{1}{2}(x-z)^2 + \frac{3}{2}y(x+z) \\ z = \frac{1}{2}(y-x)^2 + \frac{3}{2}z(x+y). \end{cases}$$

Subtracting from the first equation second one we get

$$(x-y)[1 - \frac{1}{2}(2z - y - x) - \frac{3}{2}z] = 0.$$

From this equality we get 1) $x = y$, 2) $z = \frac{1}{2}, x \neq y, x + y = \frac{1}{2}$.

1) Case: $x = y$. In this case we get $z = \frac{3}{2}z(1-z)$, consequently, $z = 0$ and $z = \frac{1}{3}$.

2) Case $z = \frac{1}{2}, x \neq y, x + y = \frac{1}{2}$. In this case we have $y = \frac{1}{2}(\frac{1}{2} - y - \frac{1}{2})^2 + \frac{3}{2}y(1-y)$. Consequently, we obtain $x = 0$ and $x = \frac{1}{2}$.

Therefore, by Lemma 2.1 we get the following fixed points:

$$a_1 = (0, \frac{1}{2}, \frac{1}{2}), \ a_2 = (\frac{1}{2}, 0, \frac{1}{2}), \ a_3 = (\frac{1}{2}, \frac{1}{2}, 0), \ a_4 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}).$$

Definition 2.2. [2]. A fixed point x^* of the operator V is called hyperbolic if its Jacobian J at x^* has no eigenvalues on the unit circle.

Definition 2.3. [2]. A hyperbolic fixed point x^* is called:

i) attracting if all the eigenvalues of the Jacobian $J(x^*)$ are less than 1 in absolute value;

ii) repelling if all the eigenvalues of the Jacobian $J(x^*)$ are greater than 1 in absolute value;

iii) a saddle otherwise.

To study the type of each fixed point rewrite operator (1.1) (using $z = 1 - y - x$) as

$$W : \begin{cases} x' = \frac{1}{2}(1 - x - 2y)^2 + \frac{3}{2}x(1 - x) \\ y' = \frac{1}{2}(2x - 1 + y)^2 + \frac{3}{2}y(1 - y). \end{cases} \tag{2.1}$$

Note that W maps the set $K = \{(x, y) \in [0, 1]^2 : 0 \le x + y \le 1\}$ to itself.

The Jacobian of W at point (x, y) is

$$J_W(x, y) = \begin{pmatrix} 2y - 2x + \frac{1}{2} & 2x + 4y - 2 \\ 4x + 2y - 2 & 2x - 2y + \frac{1}{2} \end{pmatrix}.$$

For each fixed point we have the following eigenvalues:

Case a_1: $\lambda_1 = \frac{3}{2}$, $\lambda_2 = -\frac{1}{2}$.

Case a_2: $\lambda_1 = -\frac{1}{2}$, $\lambda_2 = \frac{3}{2}$.

Case a_3: $\lambda_1 = -\frac{1}{2}$, $\lambda_2 = \frac{3}{2}$.

Case a_4: $\lambda_{1,2} = \frac{1}{2}$.

Thus a_1, a_2 and a_3 are saddle points, but a_4 is an attracting fixed point.

3 Invariant sets

Let s_3 be a permutation group of order 3.

Lemma 3.1. *If M is an invariant set of the operator V_0, defined $V_0(M) \subseteq M$, then, for any $g \in s_n$, the set $g(M)$ is also an invariant set for V_0.*

Introduce the following sets (see Fig. 2):
$M_1 = \{(x, y, z) \in S^2 : x > y > z > 1/6\}$, $M_2 = \{(x, y, z) \in S^2 : x > z > y > 1/6\}$, $M_3 = \{(x, y, z) \in S^2 : y > x > z > 1/6\}$, $M_4 = \{(x, y, z) \in S^2 : y > z > x > 1/6\}$, $M_5 = \{(x, y, z) \in S^2 : z > x > y > 1/6\}$, $M_6 = \{(x, y, z) \in S^2 : z > y > x > 1/6\}$, $H = \{(x, y, z) \in S^2 : z > x > y, x < 1/6\}$, $I = \{(x, y, z) \in S^2 : x < z < 1/6\}$, $J = \{(x, y, z) \in S^2 : x < \frac{1}{6}, z > y > \frac{1}{6}\}$, $K = \{(x, y, z) \in S^2 : x < \frac{1}{6}, y > z > \frac{1}{6}\}$, $L = \{(x, y, z) \in S^2 : x < z < 1/6\}$, $N = \{(x, y, z) \in S^2 : 1/6 > x > z\}$, $P = \{(x, y, z) \in S^2 : y > x > \frac{1}{6}, z < \frac{1}{6}\}$, $Q = \{(x, y, z) \in S^2 : x > y > \frac{1}{6}, z < \frac{1}{6}\}$, $D = \{(x, y, z) \in S^2 : 1/6 > y > z\}$, $E = \{(x, y, z) \in S^2 : y < z < 1/6\}$, $F = \{(x, y, z) \in S^2 : x > z > \frac{1}{6}, y < \frac{1}{6}\}$, $G = \{(x, y, z) \in S^2 : z > x > \frac{1}{6}, y < \frac{1}{6}\}$, $l_1 := \{(x, y, z) \in S^2 : x = y, x + y + z = 1\}$, $l_2 := \{(x, y, z) \in S^2 : x = z, x + y + z = 1\}$, $l_3 := \{(x, y, z) \in S^2 : y = z,\ x + y + z = 1\}$.

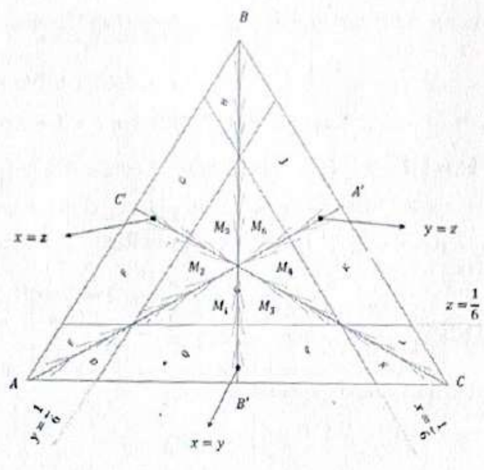

Рис. 2: The 2D-simplex and invariant sets of V_0.

Theorem 3.2. *The sets $M_i, i = 1,2,3,4,5,6,$ $l_j,\ j = 1,2,3$ are invariant with respect to the operator V_0.*

Proof. We show that M_5 is invariant. Let $(x,y,z) \in M_5$, i.e., $z > x > y > 1/6$. From formula of V_0 we get

a) $z' - x' = \frac{1}{2}(z-x)(6y-1) > 0 \Rightarrow z' > x'$
b) $z' - y' = \frac{1}{2}(z-y)(6x-1) > 0 \Rightarrow z' > y'$
c) $x' - y' = \frac{1}{2}(x-y)(6z-1) > 0 \Rightarrow x' > y'$
From a), b), c) we get $z' > x' > y'$.
Now we shall show that $y' > \frac{1}{6}$. This follows from

$$\min_{(x,y,z)\in D}\left(\frac{1}{2}(x-z)^2 + \frac{3}{2}y(x+z)\right) = \frac{5}{24},$$

where $D = \{(x,y,z) \in \mathbb{R}^3 : x+y+z = 1\ z \geq x \geq y \geq \frac{1}{6}\}$. Thus M_5 is an invariant. By Lemma 3.1 we get that M_1, M_2, M_3, M_4, M_6 are invariant sets too. From a), b), c) it follows that if $x = y$ then $x' = y'$, therefore each median $l_j,\ j = 1,2,3$ is invariant. $\square$ It is easy to check that sets H-G, introduced above, are not invariant. Indeed, for example, we show that H is not invariant. By b) we have

$$z' - y' = \frac{1}{2}(z-y)(6x-1) < 0 \Rightarrow z' < y'.$$

Thus H is not invariant. Similarly for G by a), b), c) we get $x' > z' > y'$, i.e., G is not an invariant.

4 Trajectories

For $v^{(0)} \in S^2$ define its trajectory $\{v^{(n)}\}_{n=0}^{\infty}$, with respect to operator V_0, as

$$v^{(n+1)} = V_0(v^{(n)}), \quad n = 0, 1, 2, \dots$$

Let $\varphi(x,y,z) = |x-y||y-z||z-x|$, $v = (x,y,z) \in S^2$.

Lemma 4.1. *For any $v^{(0)} \in S^2$ the following holds*

$$\lim_{n\to\infty} \varphi(v^{(n)}) = 0.$$

Proof. We have

$$|x^{(n+1)} - y^{(n+1)}| = \frac{1}{2}|x^{(n)} - y^{(n)}||6z^{(n)} - 1|,$$

$$|y^{(n+1)} - z^{(n+1)}| = \frac{1}{2}|y^{(n)} - z^{(n)}||6x^{(n)} - 1|,$$

$$|z^{(n+1)} - x^{(n+1)}| = \frac{1}{2}|z^{(n)} - x^{(n)}||6y^{(n)} - 1|.$$

Therefore,

$$\varphi(x^{(n+1)}, y^{(n+1)}, z^{(n+1)}) = \frac{1}{8}\varphi(x^{(n)}, y^{(n)}, z^{(n)}) \cdot |f(x^{(n)}, y^{(n)}, z^{(n)})|,$$

where $f(x,y,z) = (6x-1)(6y-1)(6z-1)$. It is easy to see that the function satisfies

$$|f(x,y,z)| \le 5, \quad \forall (x,y,z) \in S^2.$$

Consequently, $\varphi(x^{(n+1)}, y^{(n+1)}, z^{(n+1)}) \le \frac{5}{8}\varphi(x^{(n)}, y^{(n)}, z^{(n)})$, where

$$v^{(n)} = (x^{(n)}, y^{(n)}, z^{(n)}).$$

Thus

$$0 \le \varphi(v^{(n)}) \le (\frac{5}{8})^{n-1}\varphi(v^{(0)}).$$

This completes the proof. □

From this lemma it follows that any trajectory of the operator V_0 will be mainly close to medians of the simplex.

Lemma 4.2. *For each $i = 1,2,3$, $k \in N$ the equation $V_0^{(k)}(v) = a_i$ does not have solution except solutions $v = e_i$ and a_i.*

Proof. By symmetry of the operator it suffices to prove for $i = 1$. Take first $k = 1$, then $V_0(v) = a_1$ is reduced to

$$\begin{cases} x' - y' = \frac{1}{2}(x-y)(6z-1) \\ y' - z' = \frac{1}{2}(y-z)(6x-1) \\ x' - z' = \frac{1}{2}(x-z)(6y-1) \end{cases} \Rightarrow \begin{cases} 0 - \frac{1}{2} = \frac{1}{2}(x-y)(5-6x-6y) \\ \frac{1}{2} - \frac{1}{2} = \frac{1}{2}(y-z)(6x-1) \\ 0 - \frac{1}{2} = \frac{1}{2}(2x-1+y)(6y-1) \end{cases} \Rightarrow$$

$$\Rightarrow \frac{1}{2} - \frac{1}{2} = \frac{1}{2}(y-z)(6x-1) \Rightarrow x = \frac{1}{6} \text{ or } x = 1-2y.$$

Consider $x = \frac{1}{6}$ and $x = 1-2y$ separately:

1) Case: $x = \frac{1}{6}$. We have

$$(y - \frac{1}{6})(4-6y) = 1 \Rightarrow 36y^2 - 30y + 10 = 0 \Rightarrow \mathcal{D} < 0.$$

$$(1-6y)(\frac{1}{3} - 1 + y) = 1 \Rightarrow 18y^2 - 15y + 5 = 0 \Rightarrow \mathcal{D} < 0.$$

Thus the system has no solution in case $x = \frac{1}{6}$.

2) Case: $x = 1-2y$. In this case we have

$$(3y-1)(6y-1) = 1 \Rightarrow y_1 = 0,\ y_2 = \frac{1}{2} \Rightarrow x_1 = 1,\ x_2 = 0.$$

$$(1-6y)(2(1-3y)) = 1 \Rightarrow y_1 = 0,\ y_2 = \frac{1}{2} \Rightarrow x_1 = 1,\ x_2 = 0.$$

Thus, for $k = 1$, the system has solutions $e_1 = (1,0,0)$ and $a_1 = (0, \frac{1}{2}, \frac{1}{2})$ only. We note that

$$V_0(e_1) = a_1,\ \ V_0(v) \neq e_1, \ \text{ for all } \ v \in S^2.$$

Assume $k \geq 2$ and that the $V_0^{(k)}(v) = a_1$ has some solution different from e_1 and a_1 then

$$V_0^{(k)}(v) = V_0(V_0^{(k-1)}(v)) = V_0(\overline{v}) = a_1. \tag{4.1}$$

Since $v \neq e_1, a_1$ we have

$$\overline{v} = V_0^{(k-1)}(v) \neq e_1, a_1.$$

This by (4.1) contradicts to the result of $k = 1$. □

Lemma 4.3. *Let $M = \bigcup_{i=1}^{6} M_i$. For any initial point $v^{(0)} \in M$, the trajectory has the following limit*

$$\lim_{n\to\infty} V_0^{(n)}(v^{(0)}) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}).$$

Proof. It suffices to prove this lemma for M_1. Take $(x,y,z) \in M_1$, since M_1 is invariant we have

$$x^{(n)} > y^{(n)} > z^{(n)} > \frac{1}{6}.$$

Thus $z^{(n)}$ is minimal coordinate of the vector $v^{(n)}$ for each $n \geq 1$. Moreover, $z^{(n)} \leq \frac{1}{3}$. Because if $z^{(n)} > \frac{1}{3}$ then we have $x^{(n)} > \frac{1}{3}$, $y^{(n)} > \frac{1}{3}$ and so $v^{(n)} \notin S^2$. Now we shall show that $z^{(n)}$ is an increasing sequence:

$$z^{(n+1)} = \frac{1}{2}(y^{(n)} - x^{(n)})^2 + \frac{3}{2}z^{(n)}(1 - z^{(n)}) \geq \frac{3}{2}z^{(n)}(1 - z^{(n)}) > z^{(n)}.$$

Here we used that the function $f(z) = \frac{3}{2}z(1-z)$ satisfies $f(z) \geq z$ for any $0 \leq z \leq \frac{1}{3}$. Thus, $z^{(n)}$ is monotone increasing and its limit is $\frac{1}{3}$ as a_4 is an attracting fixed point:

$$\lim_{n\to\infty} z^{(n)} = \frac{1}{3}.$$

Now we show that $x^{(n)}$ has a limit too. Assume it has not limit, then there are sub sequences $\{n_k\} \subset \mathbb{N}$ and $\{\overline{n}_m\} \subset \mathbb{N}$ such that

$$\lim_{k\to\infty} x^{(n_k)} = \alpha, \quad \lim_{m\to\infty} x^{(\overline{n}_m)} = \beta,$$

where $\alpha \neq \beta$.

By the third equation of the operator (1.1) we get

$$z^{(n_k+1)} = \frac{1}{2}(2x^{(n_k)} + z^{(n_k)} - 1)^2 + \frac{3}{2}z^{(n_k)}(1 - z^{(n_k)}),$$

taking limit for $k \to \infty$ we obtain

$$\frac{1}{3} = \frac{1}{2}(\frac{2}{3} - 2\alpha)^2 + \frac{1}{3} \Rightarrow \alpha = \frac{1}{3}$$

similarly for $\overline{n}_m$ we have

$$z^{(\overline{n}_m+1)} = \frac{1}{2}(2x^{(\overline{n}_m)} + z^{(\overline{n}_m)} - 1)^2 + \frac{3}{2}z^{(\overline{n}_m)}(1 - z^{(\overline{n}_m)}),$$

and taking limit for $m \to \infty$:

$$\frac{1}{3} = \frac{1}{2}(\frac{2}{3} - 2\beta)^2 + \frac{1}{3} \Rightarrow \beta = \frac{1}{3}.$$

Thus, $\alpha = \beta = \frac{1}{3}$ which a contradiction to $\alpha \neq \beta$. Consequently,

$$\lim_{n\to\infty} z^{(n)} = \lim_{n\to\infty} x^{(n)} = \frac{1}{3}.$$

Now by $x^{(n)} + y^{(n)} + z^{(n)} = 1$ we get

$$\lim_{n\to\infty} y^{(n)} = \frac{1}{3}.$$

□ Denote

$$Q_a = \{(x, y, z) \in Q : z \in [\frac{3 - \sqrt{9 - 4a}}{6}, \frac{a}{6}]\}, \quad a \in [0, 1]\}.$$

Lemma 4.4. *If $(x, y, z) \in Q_1$ then $V_0(x, y, z) \in M_3$.*

Proof. Let $(x, y, z) \in Q$, i.e., $x > y > 1/6, z < 1/6$. From formula of V_0 we get $y' > x' > z'$. Now under conditions of lemma we show that $z' > 1/6$. Indeed,

$$z' = \frac{1}{2}(y - x)^2 + \frac{3}{2}z(x + y) > \frac{1}{6} \quad \Leftrightarrow$$

$$3(y - x)^2 + 9(1 - x - y)(x + y) > 1 \quad \Leftrightarrow$$

$$y^2 - 2xy + x^2 + 3x + 3y - 3(x^2 + 2xy + y^2) > \frac{1}{3}.$$

Then
$$2(x+y)^2+4xy-3(x+y)+\frac{1}{3}<0.$$
$$2(x+y)^2+4xy-3(x+y)+\frac{1}{3}\le 3(x+y)^2-3(x+y)+\frac{1}{3}. \quad \Leftrightarrow$$
Therefore it is sufficient to show that
$$3(x+y)^2-3(x+y)+\frac{1}{3}\le 0.$$

Denoting $x+y=t=1-z$ from the last equality we get $z\in[\frac{3-\sqrt{5}}{6},\frac{1}{6}]$. □

Lemma 4.5. *For each $(x,y,z)\in Q$ there exists a natural number n such that $V_0^{(n)}(x,y,z)\in M_3$.*

Proof. For the subset $Q_1\subset Q$ mentioned in the previous lemma we have $n=1$. Denote $g(x):=3-\sqrt{9-4x}$. Consider trajectory
$$b_{n+1}=g(b_n),\quad b_0=1,\quad n\ge 1.$$
Now we take $(x,y,z)\in Q$ and $b_1:=g(1)\in[0,1)$ with $z<\frac{b_1}{6}$. By formula of V_0 we get $y'>x'>z'$. We show that $z'>\frac{b_1}{6}$:
$$z'=\frac{1}{2}(y-x)^2+\frac{3}{2}z(x+y)>\frac{b_1}{6}\quad\Leftrightarrow$$
$$(y-x)^2+3(1-x-y)(x+y)>\frac{b_1}{3}\quad\Leftrightarrow$$
$$y^2-2xy+x^2+3x+3y-3(x^2+2xy+y^2)>\frac{b_1}{3}.$$
Then
$$2(x+y)^2+4xy-3(x+y)+\frac{b_1}{3}<0$$
$$2(x+y)^2+4xy-3(x+y)+\frac{b_1}{3}\le 3(x+y)^2-3(x+y)+\frac{b_1}{3}\quad\Leftrightarrow$$
Therefore it is sufficient to show that
$$3(x+y)^2-3(x+y)+\frac{b_1}{3}\le 0.$$
Since $x+y=t=1-z$ we get
$$3(1-z)^2-3(1-z)+\frac{b_1}{3}\le 0$$
$$3z^2-3z+\frac{b_1}{3}\le 0\quad\Leftrightarrow$$
$z\in\left[\frac{b_2}{6},\frac{3+\sqrt{9-4b_1}}{6}\right]$, where $b_2=g(b_1)$.

Thus if $z \in [\frac{b_2}{6}, \frac{b_1}{6}]$ then $z' > \frac{b_1}{6}$. Consequently, any point $(x, y, z) \in Q_{b_1}$ after second iteration of V_0 goes to M_3, i.e., $n = 2$. Using induction one can show that if $(x, y, z) \in Q_{b_n}$ then $z' > \frac{b_n}{6}$, i.e., $V_0(x, y, z) \in Q_{b_{n-1}}$. Therefore, we conclude that $V_0^{(n)}(x, y, z) \in Q_1$ and $V_0^{(n+1)}(x, y, z) \in M_3$. □

Denote

$$\Delta_1 = \{(x, y, z) \in S^2 : x > \frac{3+\sqrt{5}}{6}\},$$

$$\Delta_2 = \{(x, y, z) \in S^2 : y > \frac{3+\sqrt{5}}{6}\},$$

$$\Delta_3 = \{(x, y, z) \in S^2 : z > \frac{3+\sqrt{5}}{6}\}.$$

Lemma 4.6. *If $(x, y, z) \in \bigcup_{i=1}^{3} \Delta_i$ then*

$$V_0(x, y, z) \notin \bigcup_{i=1}^{3} \Delta_i .$$

Proof. Take $x > \frac{3+\sqrt{5}}{6}$. Since $\Delta_1 \subset E \cup D$ we get $y < \frac{3-\sqrt{5}}{6}$ and $z < \frac{3-\sqrt{5}}{6}$. Therefore,

$$x' = \frac{1}{2}(z-y)^2 + \frac{3}{2}x(y+z) = \frac{1}{2}(z^2 - 2zy + y^2) + \frac{3}{2}(x - x^2) \le \frac{1}{2}(z^2 + y^2) + \frac{3}{2}(x - x^2) <$$

$$< \frac{1}{2}\Big((\frac{3-\sqrt{5}}{6})^2 + (\frac{3-\sqrt{5}}{6})^2\Big) + \frac{3}{2}(x - x^2) = \frac{14-6\sqrt{5}}{36} + \frac{3}{2}(x - x^2) \le$$

$$\le \frac{14-6\sqrt{5}}{36} + \frac{3}{2} \cdot \frac{3+\sqrt{5}}{6}(1 - \frac{3+\sqrt{5}}{6}) = \frac{14-6\sqrt{5}}{36} + \frac{3}{2} \cdot \frac{(3+\sqrt{5})(3-\sqrt{5})}{36} =$$

$$= \frac{14-6\sqrt{5}}{36} + \frac{3}{2} \cdot \frac{4}{36} = \frac{20-6\sqrt{5}}{36} < \frac{3+\sqrt{5}}{6}.$$

Now

$$y' = \frac{1}{2}(x-z)^2 + \frac{3}{2}y(x+z) = \frac{1}{2}(x-z)^2 + \frac{3}{2}(1-x-z)(x+z) =$$

$$= \frac{1}{2}(x-z)^2 + \frac{3}{2}(x+z) - \frac{3}{2}(x+z)^2 =$$

$$= \frac{1}{2}[x^2 - 2xz + z^2 + 3x + 3z - 3x^2 - 6xz - 3z^2] = \frac{1}{2}[-2x^2 - 2z^2 - 8xz + 3x + 3z] =$$

$$= \frac{1}{2}[-2(x+z)^2 + 3(x+z) - 4xz] \le \frac{1}{2}[-2(x+z)^2 + 3(x+z)] = \frac{1}{2}[-2t^2 + 3t] <$$

$$< \frac{1}{2}[-2 \cdot (\frac{3}{4})^2 + 3 \cdot \frac{3}{4}) = \frac{1}{2}(-\frac{9}{8} + \frac{9}{4}) = \frac{1}{2} \cdot \frac{9}{8} = \frac{9}{16} < \frac{3+\sqrt{5}}{6}.$$

Similarly one can show that

$$z' < \frac{3+\sqrt{5}}{6}.$$

Thus

$$\begin{cases} x' < \frac{3+\sqrt{5}}{6} \\ y' < \frac{3+\sqrt{5}}{6} \\ z' < \frac{3+\sqrt{5}}{6} \end{cases} \Rightarrow (x',y',z') \notin \Delta_1 \cup \Delta_2 \cup \Delta_3 .$$

□

Lemma 4.7. *If $(x,y,z) \in E \cup D$ with $x \in [\frac{1}{6}, \frac{3+\sqrt{5}}{6}]$ then, $V_0(x,y,z) \in M_4$.*

Proof. This is similar to the proof of Lemma 4.4. □ Summarizing from above proved lemmas we obtain

Theorem 4.8. *For the operator V_0, for any $v^{(0)} \in S^2 \backslash \{a_i, e_i : i = 1,2,3\}$ the following holds*

$$\lim_{n\to\infty} V_0^{(n)}(v^{(0)}) = \lim_{n\to\infty} v^{(n)} = \lim_{n\to\infty} (x^{(n)}, y^{(n)}, z^{(n)}) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}). \qquad (4.2)$$

Proof.

1) If $v^{(0)} \in \cup_{i=1}^{6} M_i$ then (4.2) follows from Lemma 4.3;

2) If $v^{(0)} \in Q \cup P \cup F \cup G \cup J \cup K$ then (4.2) follows from Lemma 4.5 and Lemma 4.3.

3) The case

$$v^{(0)} \in \{(x,y,z) \in E \cup D : x \leq \frac{3+\sqrt{5}}{6}\} \cup \{(x,y,z) \in H \cup I : z \leq \frac{3+\sqrt{5}}{6}\} \cup$$

$$\cup \{(x,y,z) \in L \cup N : y \leq \frac{3+\sqrt{5}}{6}\}$$

follows from Lemma 4.7.

4) If $v^{(0)} \in \cup_{i=1}^{n} \Delta_i$ then $V_0(v^{(0)}) \in S^2 \cup \backslash \{\partial S^2 \cup_{i=1}^{n} \Delta_i\})$.

5) If $v^{(0)} \in \partial S^2 \backslash \{a_i, e_i : i = 1,2,3\}$, then $V_0(v^{(0)}) . \in \mathrm{int} S^2$. Therefore (4.2) holds.

□

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