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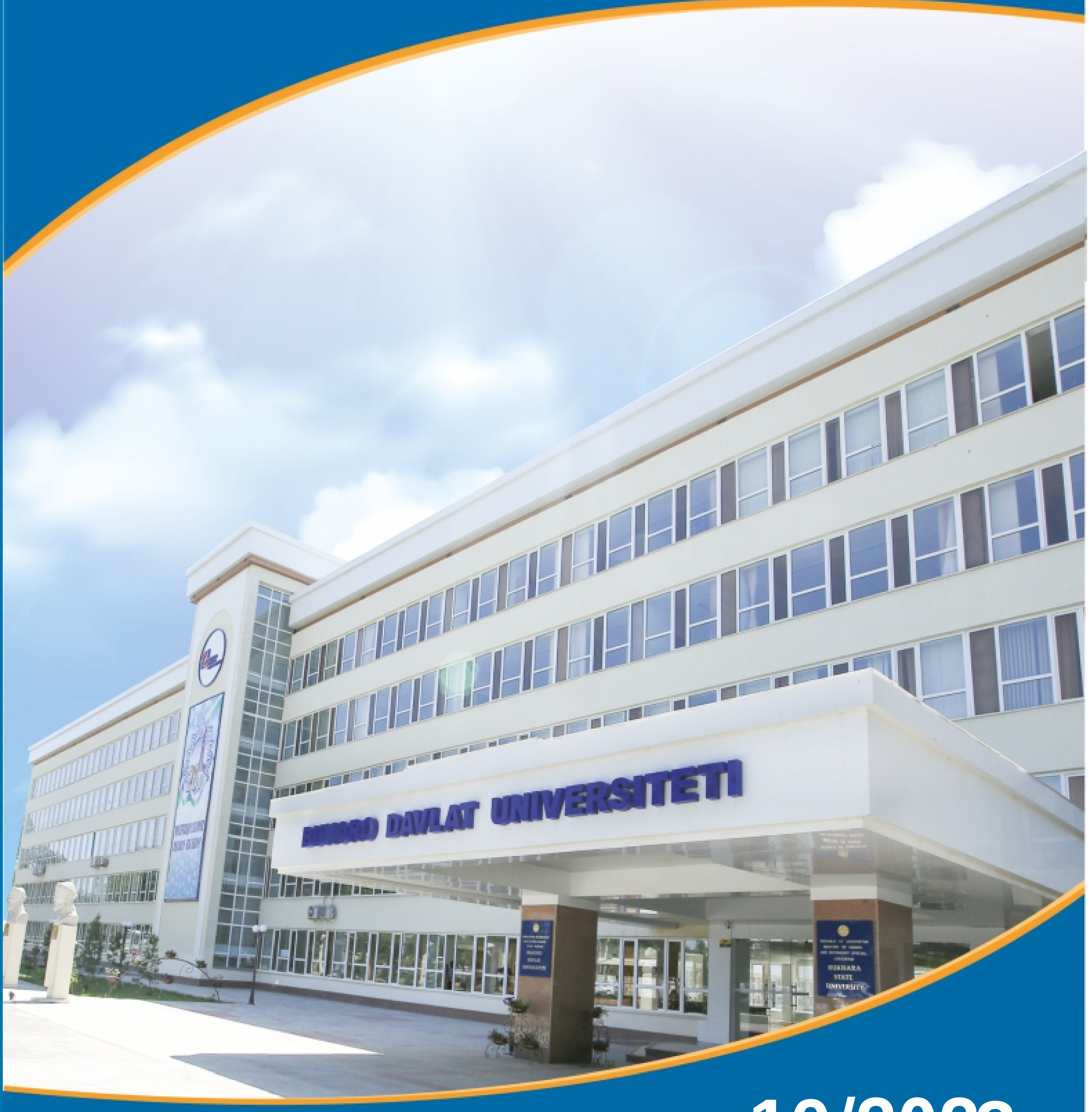

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DYNAMICS OF A COOPERATIVE SYSTEM WITH ORDER ONE IN THE PLANE

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Abstract. *In this paper we investigate the dynamical systems generated by cooperative discrete system with order one in the plane. For the cooperative discrete system, we find all fixed points and their types. Moreover, we study limit points of the trajectories of the corresponding dynamical system.*
Keywords: *Cooperative system, fixed point, trajectories, limit point.*

ДИНАМИКА КООПЕРАТИВНОЙ СИСТЕМЫ ПЕРВОГО ПОРЯДКА НА ПЛОСКОСТИ

Аннотация. *В данной статье мы исследуем динамические системы, порожденные кооперативной дискретной системой первого порядка на плоскости. Для кооперативной дискретной системы найдены все неподвижные точки и их типы. Кроме того, мы изучаем предельные точки траекторий соответствующей динамической системы.*
Ключевые слова: *Кооперативная система, неподвижная точка, траектории, предельная точка.*

TEKISLIKDA BIRINCHI TARTIBLI KOOPERATIV SISTEMANING DINAMIKASI

Annotatsiya. *Ushbu maqolada biz tekislikda birinchi tartibli kooperativ diskret sistema tomonidan hosil qilingan dinamik sistemalarni o'rganamiz. Kooperativ diskret sistema uchun barcha qo'zg'almas nuqtalarni va ularning turlarini topamiz. Bundan tashqari, qaralayotgan dinamik sistema trayektoriyalari limit nuqtalarini o'rganamiz.*
Kalit so'zlar: *kooperativ sistema, qo'zg'almas nuqta, trayektoriyalar, limit nuqta.*

In this piece of work, we consider the cooperative discrete system

$$V: \begin{cases} x' = ax + \dfrac{by^k}{1+y^k} \\ y' = dy + \dfrac{cx^k}{1+x^k} \end{cases} \tag{1}$$

where all parameters a, b, c, d are positive real numbers and the initial conditions x_0, y_0 are nonnegative real numbers. In this system k is called the order of cooperative discrete system (1). We describe the dynamics of this system when $k = 1$.

The main problem for a given operator V and for arbitrarily initial point $s^{(0)} \in \mathbb{R}_+^2$ is to describe the limit points of the trajectory $\{s^{(m)}\}_{m=0}^{\infty}$, where

$$s^{(m)} = V^m(s^{(0)}) = \underbrace{V(V(\dots V}_{m}(s^{(0)}))\dots)$$

Definition 1. A point s is called fixed point of the operator V if $V(s) = s$. The set of all fixed points is denoted by $FixV$.

For the fixed points of the given nonlinear operator V when $k = 1$ we have the following theorem.

Theorem 1. *If* $a \geq 1$ *or* $d \geq 1$ *or* $a, d \in (0,1)$ *and* $bc \leq (1-a)(1-d)$ *holds then the nonlinear operator* V *has a unique fixed point which is*

$$s_1 = (0; 0).$$

If $a, d \in (0,1)$ *and* $bc > (1-a)(1-d)$ *holds then the nonlinear operator* V *has two fixed points which are*

$$s_1 = (0; 0) \quad and \quad s_2 = (x_0;\ y_0).$$

where

$$x_0 = \frac{bc-(1-a)(1-d)}{(1-a)(1-d+c)}, \qquad y_0 = \frac{bc-(1-a)(1-d)}{(1-d)(1-a+b)}.$$

Proof. In order to find fixed points of the given operator V we have to solve the following system of equations for (x,y):

$$\begin{cases}(1-a)x = \dfrac{by}{1+y}\\ (1-d)y = \dfrac{cx}{1+x}\end{cases} \qquad (2)$$

It is obviously that if either $a \geq 1$ or $d \geq 1$ holds then system of equations (1) has a unique solution

$$(x,y) = (0,0).$$

Let $a,d \in (0,1)$ then from the first equation of the system (1) we find $x = \frac{by}{(1-a)(1+y)}$. Substituting this value to the second equation of the system (1) we get following equation:

$$(1-d)y = \frac{bcy}{by+(1-a)(1+y)}$$

which under the condition $bc > (1-a)(1-d)$ gives two fixed points

$$s_1 = (0;0) \quad and \quad s_2 = (x_0;\, y_0)$$

where

$$x_0 = \frac{bc-(1-a)(1-d)}{(1-a)(1-d+c)}, \qquad y_0 = \frac{bc-(1-a)(1-d)}{(1-d)(1-a+b)}.$$

Theorem is proved.

Definition 2. A point s of the operator V is called hyperbolic if its Jacobian J at s has no eigenvalues on the unit circle.

Definition 3. A hyperbolic fixed point s is called:

i) attracting if all the eigenvalues of the Jacobi matrix $J(s)$ are less than 1 in absolute value,

ii) repelling if all the eigenvalues of the Jacobi matrix $J(s)$ are greater than 1 in absolute value,

iii) a saddle otherwise.

In order to find the type of the fixed points of the operator (1) when $k = 1$ we consider the Jacobi matrix

$$J(s) = J_V = \begin{pmatrix} a & \dfrac{b}{(1+y)^2} \\ \dfrac{c}{(1+x)^2} & d \end{pmatrix}$$

and the corresponding characteristic equation $det(J(s) - \lambda I) = 0$. The characteristic equation has the form

$$\lambda^2 - (a+d)\lambda + ad - \frac{bc}{(1+x)^2(1+y)^2} = 0,$$

Teorema 2. *When $k = 1$ for the type of fixed points of the operator V the following statements are hold:*

$$s_1 \ is \ a/an$$

$$\begin{cases} attractor, & if \ a,d \in (0,1) \ and \ bc < (1-a)(1-d) \\ saddle, & if \ a,d \in (0,1) \ and \ (1-a)(1-d) < bc < (1+a)(1+d) \\ repelling, & if \ a \geq 1 \ or \ d \geq 1 \ or \ a,d \in (0,1) \ and \ bc > (1+a)(1+d) \\ non-hyperbolic, & if \ a,d \in (0,1) \ and \ bc = (1-a)(1-d) \ or \ bc = (1+a)(1+d) \end{cases}$$

$$s_2 \ is \ a/an \ \ attractor, \ \ if \ a,d \in (0,1) \ and \ bc > (1-a)(1-d).$$

The proof of the theorem 2 based on finding λ from the characteristic equation at the find points and using definitions 2 and 3. Thus the proof of the theorem 2 remains for the readier.

For the limit points of the given operator V when $k = 1$ we have:

Teorema 3. *If $a,d \in (0,1)$, and $bc \leq (1-a)(1-d)$ hold then for any initial point $(x,y) \in \Delta = \{(x,y) \in \mathbb{R}^2 \colon\ x \geq 0, y \geq 0\}$ for the trajectories of the operator V the following holds*

$$\lim_{n\to\infty} V^n(x,y) = s_1 = (0,0).$$

Proof. For any initial point $(x,y) \in \Delta = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ we obtain:

$$\begin{cases} x' = ax + \dfrac{by}{1+y} \\ y' = dy + \dfrac{cx}{1+x} \end{cases} \Rightarrow \begin{cases} (x'-x) + (1-a)x = \dfrac{by}{1+y} \\ (y'-y) + (1-d)y = \dfrac{cx}{1+x} \end{cases} \Rightarrow$$

$$\Rightarrow [(x'-x) + (1-a)x][(y'-y) + (1-d)y] \leq (1-a)(1-d)xy$$

$$\Rightarrow (x'-x)(y'-y) + (1-d)y(x'-x) + (1-a)x(y'-y) \leq 0$$

$$\Rightarrow either\ x'-x \leq 0\ or\ y'-y \leq 0$$

$$\Rightarrow either\ \{x^{(n)}\}\ is\ monotone\ decreasing\ or\ \{y^{(n)}\}\ is\ monotone\ decreasing$$

$$and\ both\ of\ them\ are\ bounded\ from\ below.$$

We know that if the operator is a continuous map and the trajectories of the operator are convergent then the set of limit points of the operator should consist of the fixed points. Thus

$$\begin{cases} x^{(n)} = \dfrac{y^{(n+1)} - dy^{(n)}}{c - y^{(n+1)} + dy^{(n)}} \\ y^{(n)} = \dfrac{x^{(n+1)} - ax^{(n)}}{b - x^{(n+1)} + ax^{(n)}} \end{cases}$$

and

$$\lim_{n\to\infty} V^n(x,y) = \lim_{n\to\infty} \left(x^{(n)}, y^{(n)}\right) = s_1 = (0,0).$$

This gives the proof of the theorem.

Now let us make the notations i.e. we consider the subsets of the set $\Delta = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$:

$$\Delta_1 = \{(x,y) \in \mathbb{R}^2:\ 0 \leq x \leq x_0,\quad 0 \leq y \leq y_0\}$$

$$\Delta_2 = \{(x,y) \in \mathbb{R}^2:\ 0 \leq x \leq x_0,\quad y \geq y_0\}$$

$$\Delta_3 = \{(x,y) \in \mathbb{R}^2:\ x \geq x_0,\quad y \geq y_0\}$$

$$\Delta_4 = \{(x,y) \in \mathbb{R}^2:\ x \geq x_0,\quad 0 \leq y \leq y_0\}$$

where

$$x_0 = \frac{bc - (1-a)(1-d)}{(1-a)(1-d+c)}, \qquad y_0 = \frac{bc - (1-a)(1-d)}{(1-d)(1-a+b)}.$$

For the limit points of the given operator V when $k = 1$ we have:

Teorema 4. *If* $a, d \in (0,1)$, $and\ bc > (1-a)(1-d)$ $hold\ then\ for\ any\ initial\ point\ (x,y) \in \Delta_2 \cup \Delta_4$ *for the trajectories of the operator* V *the following holds*

$$\lim_{n\to\infty} V^n(x,y) = s_2 = (x_0;\ y_0).$$

Moreover we have

$$V(\Delta_1) \subset \Delta_1\ \ and\ \ V(\Delta_3) \subset \Delta_3,$$

i.e. Δ_1 and Δ_3 are invariant sets respect to the operator V.

Proof. Let $(x,y) \in \Delta_2$. If $a, d \in (0,1)$, $and\ bc > (1-a)(1-d)$ hold then the fixed point $s_2 = (x_0;\ y_0)$ exists and

$$x' = ax + \frac{by}{1+y} \geq ax + \frac{by_0}{1+y_0} = a(x - x_0) + x_0 = x + (1-a)(x_0 - x) \geq x$$

Moreover,

$$x' = ax + \frac{by}{1+y} \leq ax_0 + b$$

Thus $\{x^{(n)}\}$ is convergent. Since

$$y^{(n)} = \frac{x^{(n+1)} - ax^{(n)}}{b - x^{(n+1)} + ax^{(n)}} \quad and \quad x_0 < \frac{b}{1-a}$$

Then $lim_{n\to\infty} V^n(x,y)$ exists and it should be fixed point.

Let $(x,y) \in \Delta_4$. Then

$$y' = dy + \frac{cx}{1+x} \geq dy + \frac{cx_0}{1+x_0} = d(y - y_0) + y_0 = y + (1-d)(y_0 - y) \geq y$$

Moreover,

$$y' = dy + \frac{cx}{1+x} \leq dy_0 + c$$

Thus $\{y^{(n)}\}$ is convergent. Since

$$x^{(n)} = \frac{y^{(n+1)} - dy^{(n)}}{c - y^{(n+1)} + dy^{(n)}} \quad and \quad y_0 < \frac{c}{1-d}$$

Then $lim_{n\to\infty} V^n(x,y)$ exists and it should be fixed point. Thus for any initial point $(x,y) \in \Delta_2 \cup \Delta_4$ for the trajectories of the operator V the following holds

$$\lim_{n\to\infty} V^n(x,y) = s_2 = (x_0;\ y_0).$$

Now let $(x,y) \in \Delta_1$. Then

$$0 \leq x' = ax + \frac{by}{1+y} \leq ax_0 + \frac{by_0}{1+y_0} = x_0$$
$$0 \leq y' = dy + \frac{cx}{1+x} \leq dy_0 + \frac{cx_0}{1+x_0} = y_0$$

That means $V(\Delta_1) \subset \Delta_1$. Similarly one can show that $V(\Delta_3) \subset \Delta_3$. The proof of the theorem is completed.

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