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QUADRATIC STOCHASTIC DYNAMICAL SYSTEMS OF THE TYPE $(\sigma \mid D)$

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Abstract. *The study of mathematical models in physics or biology comes down to the study of continuous or discrete-time quadratic random processes. In this case, an important issue is the construction of a family of cubic matrices that satisfy the Kolmogorov-Chapman equation.*

Maksimov introduced associative multiplication of cubic matrices, generalizing the usual matrix multiplication. He introduced analogues of cubic stochastic matrices and gave their probabilistic interpretation. The cubic stationarity of stochastic matrices is described and the statement that the cubic stochastic matrix approaches stationarity is proven. Generalizing the concept of a Markov process, Markov introduced the concept of an interaction process and showed that the concept of ergodicity of such a process is naturally related to the associative multiplication of cubic matrices.

Keywords: *quadratic random process with continuous time, quadratic random process with discrete time, Kolmogorov-Chapman equation, cubic matrix, stochastic matrix, associative multiplication of cubic matrices.*

КВАДРАТИЧНЫЕ СТОХАСТИЧЕСКИЕ ДИНАМИЧЕСКИЕ СИСТЕМЫ ТИПА $(\sigma \mid D)$

Аннотация. *Изучение математических моделей в физике или биологии сводится к изучению непрерывных или дискретных по времени квадратичных случайных процессов. В этом случае важным вопросом является построение семейства кубических матриц, удовлетворяющих уравнению Колмогорова-Чепмена.*

Максимов ввел ассоциативное умножение кубических матриц, обобщив обычное матричное умножение. Он ввел аналоги кубических стохастических матриц и дал их вероятностную интерпретацию. Описана кубическая стационарность стохастических матриц и доказано утверждение, что кубическая стохастическая матрица приближается к стационарности. Обобщая понятие марковского процесса, Марков ввел понятие процесса взаимодействия и показал, что понятие эргодичности такого процесса естественным образом связано с ассоциативным умножением кубических матриц.

Ключевые слова: *квадратичный случайный процесс с непрерывным временем, квадратичный случайный процесс с дискретным временем, уравнение Колмогорова-Чепмена, кубическая матрица, стохастическая матрица, ассоциативное умножение кубических матриц.*

$(\sigma \mid D)$ -TIPLI KVADRATIK STOXASTIK DINAMIK SISTEMALAR

Annotatsiya. *Fizika yoki biologiyadagi matematik modellarni o'rganish uzluksiz yoki diskret vaqtli kvadratik tasodifiy jarayonlarni o'rganishga keltiriladi. Bunday holda Kolmogorov-Chapman tenglamasini qanoatlantiradigan kubik matritsalar oilasini qurish muhim masala hisoblanadi.*

Maksimov tomonidan odatdagi matritsani ko'paytirishni umumlashtiruvchi kub matritsalarni assotsiativ ko'paytirish kiritildi. U kubik stoxastik matritsalarning analoglarini kiritdi va ularning ehtimollik talqinlarini berdi. Stoxastik matritsalarning kubik statsionarligi tavsiflandi va kubik stoxastik matritsaning statsionarga yaqinlashishi haqidagi tasdiqni isbotladi. Markov jarayoni tushunchasini umumlashtiruvchi Markov o'zaro ta'sir jarayoni tushunchasini kiritdi va bunday jarayonning ergodikligi tushunchasi tabiiy ravishda kub matritsalarni assotsiativ ko'paytirish bilan bog'liqligini ko'rsatdi.

Kalit so'zlar: *uzluksiz vaqtli kvadratik tasodifiy jarayon, diskret vaqtli kvadratik tasodifiy jarayon, Kolmogorov-Chapman tenglamasi, kubik matritsa, Stoxastik matritsa, kub matritsalarni assotsiativ ko'paytirish.*

Introduction and problem statement. In [1] by J.M.Casas, M.Ladra and U.A.Rozikov a Markov process of cubic stochastic matrices is introduced, which is also called quadratic stochastic process (QSP). QSP is a special case of a dynamic system with continuous time, the states of which are stochastic cubic matrices that satisfy an analog of the Kolmogorov-Chapman equation (KCE). Since there are several kinds of multiplication between cubic matrices, one has to first fix the multiplication and then consider the KCE with respect to the fixed multiplication. In the work [1], [2] J.M.Casas, M.Ladra, U.A.Rozikov to construct a QSP two specially chosen concepts of stochastic cubic matrices and two multiplications of such matrices (known as Maksimov multiplications) are considered.

In this paper we have several examples of quadratic stochastic processes (QSP) of type $(\sigma \mid D)$, where the type of σ stochastic cubic matrices, and D stands for a specific multiplication between cubic matrices.

Markov process of square matrices. One of well studied time evolution is Markov process, which is defined by linear mappings as follows.

Definition 1. *A family of stochastic matrices* $F_{s,t} = \{\mathrm{U}^{[s,t]} : s,t \geq 0\}$ *is called a Markov process if it satisfies the Kolmogorov-Chapman equation* [*e.g.*7]

$$\mathrm{U}^{[s,t]} = \mathrm{U}^{[s,\tau]}\mathrm{U}^{[\tau,t]}, \quad \textit{for all} \quad 0 \leq s < \tau < t \tag{1}$$

Let $x^{(0)} = (x_1^{(0)},\ldots,x_m^{(0)}) \in S^{m-1}$ be an initial distribution on I. Denote by $x^{(t)} = (x_1^{(t)},\ldots,x_m^{(t)}) \in S^{m-1}$ the distribution of the system at the moment t. For arbitrary moments of time s and t with $s < t$ the matrix $\mathrm{U}^{[s,t]} = \left(U_{ij}^{[s,t]}\right)$ gives the transition probabilities from the distribution $x^{(s)}$ to the distribution $x^{(t)}$. Moreover, $x^{(t)}$ depends linearly from $x^{(s)}$:(see [7])

$$x_k^{(t)} = \sum_{i=1}^{m} U_{ik}^{[s,t]} x_i^{(s)}, \qquad k = 1,\ldots,m.$$

Maksimov's cubic stochastic matrices. Denote $I = \{1,2,\ldots,m\}$. Let $\mathfrak{C}$ be the set of all m^3-dimensional cubic matrices over the field of real numbers. Denote by E_{ijk}, $i,j,k \in I$ the basis cubic matrices in $\mathfrak{C}$.(see [7])

Following [4] define the following multiplications for basis matrices E_{ijk}:

$$E_{ijk} *_0 E_{lnr} = \delta_{kl}\delta_{jn}E_{ijr}, \tag{2}$$

where δ_{kl} is the Kronecker symbol, i.e.

$$\delta_{kl} = \begin{cases} 0, & k \neq l \\ 1, & k = l \end{cases}$$

Then for any two cubic matrices $A = (a_{ijk}), B = (b_{ijk}) \in \mathfrak{C}$ the matrix $A *_0 B = (c_{ijk})$ is defined by

$$c_{ijr} = \sum_{k=1}^{m} a_{ijk} b_{kjr}. \tag{3}$$

The following results of this section are proven in [4] (see also [7] for detailed proofs)

Proposition 1. *The algebra of cubic matrices* $(\mathfrak{C}, *_0)$ *is a direct sum of algebras of square matrices.*

Define multiplication:

$$E_{ijk} *_a E_{lnr} = \delta_{kl} E_{ia(j,n)r}, \tag{4}$$

where $a : I \times I \to I$, $(j,n) \mapsto a(j,n) \in I$, is an arbitrary associative binary operation.

Note that (4) is *not* a particular case of (4).

Denote by O_m the set of all associative binary operations on I.

The general formula for the multiplication is the extension of (4) by bilinearity, i.e. for any two cubic matrices $A = (a_{ijk}), B = (b_{ijk}) \in \mathfrak{C}$ the matrix $A *_a B = (c_{ijk})$ is defined by

$$c_{ijr}=\sum_{l,n:a(l,n)=j}\sum_k a_{ilk}b_{knr}.$$

Note that $c_{ijr}=0$ for j such that $\{l,n:a(l,n)=j\}=\varnothing$.

If the equation $a(x,u)=v$ (resp. $a(u,x)=v$) is uniquely solvable for any $u,v\in I$ then the operation a on I has right (resp. left) unique solvability.

Lemma 1. *If the operation a on I has right or left unique solvability, then*

$$\sum_{d\in I}\sum_{j,m:a(j,m)=d}\gamma_{j,m}=\sum_{j\in I}\sum_{m\in I}\gamma_{j,m}.$$

Stochasticity. Define several kinds of cubic stochastic matrices (see [4, 6]): a cubic matrix $P=(p_{ijk})_{i,j,k=1}^m$ is called

- $(1,2)$-*stochastic* if

$$p_{ijk}\geq 0,\qquad \sum_{i,j=1}^m p_{ijk}=1,\ \textit{for all}\ k.$$

- $(1,3)$-*stochastic* if

$$p_{ijk}\geq 0,\qquad \sum_{i,k=1}^m p_{ijk}=1,\ \textit{for all}\ j.$$

- $(2,3)$-*stochastic* if

$$p_{ijk}\geq 0,\qquad \sum_{j,k=1}^m p_{ijk}=1,\ \textit{for all}\ i.$$

- 3-*stochastic* if

$$p_{ijk}\geq 0,\qquad \sum_{k=1}^m p_{ijk}=1,\ \textit{for all}\ i,j.$$

The last one can be also given with respect to first and second index.

Maksimov [4] also defined a twice stochastic matrix: a (2,3)-stochastic cubic matrix is called *twice stochastic* if

$$\sum_{i=1}^m p_{ijk}=\frac{1}{m},\ \textit{for all}\ j,k.$$

Proposition 2. $(2,3)$-*stochastic (and twice) stochastic cubic matrices form a convex semigroup with respect to multiplication* (4).

Remark 1. *One also can show that* $(1,2)$-*stochastic cubic matrices form a convex semigroup. But the collection of* $(1,3)$-*stochastic matrices does not form a semigroup with respect to multiplication* (4).

Discrete-time quadratic dynamical systems. Denote by S the set of all possible kinds of stochasticity and denote by M the set of all possible multiplication rules of cubic matrices.

Let parameters $s\geq 0$, $t\geq 0$, are considered as time.

Denote by $M^{[s,t]}=\left(P_{ijk}^{[s,t]}\right)_{i,j,k=1}^m$ a cubic matrix with two parameters.

Definition 2. [3]. *A family* $\{M^{[s,t]}:s,t\in R_+\}$ *is called a Markov process of cubic matrices (or a quadratic stochastic process (QSP)) of type* $(\sigma\,|\,\mu)$ *if for each time* s *and* t *the cubic matrix* $M^{[s,t]}$ *is stochastic in sense* $\sigma\in\mathrm{S}$ *and satisfies the Kolmogorov-Chapman equation (for cubic matrices):*

$$M^{[s,t]}=M^{[s,\tau]}*_{\mu}M^{[\tau,t]},\ \textit{for all}\ 0\leq s<\tau<t \tag{5}$$

with respect to the multiplication $\mu\in\mathrm{M}$.

Theorem 1. [3]. *Let* $\{A^{[t]}=(a_{ij}^{[t]}, t\geq 0)\}$ *be a family of invertible (for all t),* $m\times m$ *square matrices and let* $\left(A^{[t]}\right)^{-1}=(b_{ij}^{[t]})$ *denote the inverse of* $A^{[t]}$. *Let* $\mathrm{B}^{(s)}=\left(\beta_{ijk}^{(s)}\right)$, *be a cubic matrix, where* $\beta_{ijk}^{(s)}, i, j, k=1,\ldots,m,$ *are arbitrary functions such that*

$$\sum_{j=1}^{m}\beta_{ijk}^{(s)}=a_{ik}^{[s]},\ \textit{for any } i,k \textit{ and } s.$$

Then cubic matrix

$$M^{[s,t]}=\left(\sum_{k=1}^{m}\beta_{ijk}^{(s)}b_{kr}^{[t]}\right)_{i,j,r=1}^{m} \tag{6}$$

generates an flow of algebras (i.e. satisfies equation (5) *of type* (D)).

In general, the matrix (6) does not generate a QSP. Here our aim is to find conditions on matrices $A^{[t]}$ and $B^{(s)}$ (mentioned in Theorem 6) ensuring that the matrix (6) generates a QSP.

QSP of type $(\sigma\,|\,D)$.

Recall that a permutation matrix is a square binary matrix that has exactly one entry of 1 in each row and each column and 0 s elsewhere. Note that a permutation matrix is (right and left) stochastic.

Proposition 3. *If* $A^{[t]}$ *and* $\mathrm{B}(s)=(\beta_{ijk}^{(s)})_{j,j,k=1}^{m}$ *in Theorem 1 satisfies the following conditions:*

– $(A^{[t]})^{-1}$ *is a left stochastic;*

– $\mathrm{B}^{(s)}$ is a 2-*stochastic*.

Then the matrix $M^{[s,t]}$ (*given by* (1)) *generates a QSP of type* $(2\,|\,D)$

Proof. By the conditions of the proposition it is easy to see that all elements of the matrix $M^{[s,t]}$ are non-negative. We calculate for the following sum

$$\sum_{j=1}^{m}\sum_{k=1}^{m}\beta_{ijk}^{(s)}b_{kr}^{[t]}=\sum_{k=1}^{m}\sum_{j=1}^{m}\beta_{ijk}^{(s)}b_{kr}^{[t]}=\sum_{k=1}^{m}\left[\left(\sum_{j=1}^{m}\beta_{ijk}^{(s)}\right)b_{kr}^{[t]}\right]$$

Since $\mathrm{B}^{(s)}$ is a 2-stochastic we have $\sum_{j=1}^{m}\beta_{ijk}^{(s)}=1$. Then the right hand side of the last equality becomes

$$\sum_{k=1}^{m}b_{kr}^{[t]}\sum_{i,j=1}^{m}\beta_{ijk}^{(s)}=\sum_{k=1}^{m}b_{kr}^{[t]}$$

Since $\left(A^{[t]}\right)^{-1}$ is a left stochastic matrix we obtain

$$\sum_{k=1}^{m}b_{kr}^{[t]}=1.$$

Consequently, $M^{[s,t]}$ is a 2-stochastic matrix. This completes the proof.

Example 1. *Let*

$$A^{[t]}=\left(A^{[t]}\right)^{-1}=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$$

and

$$B^{(s)} = \left(\begin{array}{ccc|ccc|ccc} f_1(s) & f_2(s) & f_3(s) & f_4(s) & f_5(s) & f_6(s) & f_7(s) & f_8(s) & f_9(s) \\ \varphi_1(s) & \varphi_2(s) & \varphi_3(s) & \varphi_4(s) & \varphi_5(s) & \varphi_6(s) & \varphi_7(s) & \varphi_8(s) & \varphi_9(s) \\ 1-F_1(s) & 1-F_2(s) & 1-F_3(s) & 1-F_4(s) & 1-F_5(s) & 1-F_6(s) & 1-F_7(s) & 1-F_8(s) & 1-F_9(s) \end{array}\right)$$

Then $A^{[t]}$ and $(A^{[t]})^{-1}$ satisfy all conditions of Theorem 1. In this case the QSP is defined by the matrix

$$M^{(s)} = \left(\begin{array}{ccc|ccc|ccc} f_1(s) & f_3(s) & f_2(s) & f_4(s) & f_6(s) & f_5(s) & f_7(s) & f_9(s) & f_8(s) \\ \varphi_1(s) & \varphi_3(s) & \varphi_2(s) & \varphi_4(s) & \varphi_6(s) & \varphi_5(s) & \varphi_7(s) & \varphi_9(s) & \varphi_8(s) \\ 1-F_1(s) & 1-F_3(s) & 1-F_2(s) & 1-F_4(s) & 1-F_6(s) & 1-F_5(s) & 1-F_7(s) & 1-F_9(s) & 1-F_8(s) \end{array}\right)$$

Where $F_i(s) = f_i(s) + \varphi_i(s), i = 1,2,3,4,5,6,7,8,9$ and functions $f_i(s)$ and $\varphi_i(s)$ satisfy the following conditions

$$\begin{cases} f_i(s) \geq 0 \\ \varphi_i(s) \geq 0 \\ f_i(s) + \varphi_i(s) \leq 1 \end{cases}$$

Conclusion

In this paper we have several examples of quadratic stochastic processes (QSP) of type $(\sigma \mid D)$, where the type of σ stochastic cubic matrices, and D stands for a specific multiplication between cubic matrices.

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