MINISTRY OF HIGHER EDUCATION, SCIENCE AND INNOVATIONS OF THE REPUBLIC OF UZBEKISTAN
V.I.ROMANOVSKIY INSTITUTE OF MATHEMATICS
UNIVERSITY OF EXACT AND SOCIAL SCIENCES
OXUS UNIVERSITY

INTERNATIONAL SCIENTIFIC CONFERENCE

MATHEMATICAL ANALYSIS AND DYNAMICAL SYSTEMS

CONFERENCE PROGRAMME

MAY 20 - 21, 2025

TASHKENT, UZBEKISTAN

On the fixed point and invariant set QnSO

Khudayarov S.S.[1], **Dilmurodov E.B.**[2]
Bukhara State University, Bukhara, Uzbekistan;
Bukhara Branch Institute of Mathematics named after V.I. Romanovsky of the Academy of Sciences of the Republic of Uzbekistan;
[1]s.s.xudayarov@buxdu.uz
[2] e.b.dilmurodov@buxdu.uz

Let $I = \{1, 2, \dots, m\}$. A distribution of the set I is a probability measure $x = (x_1, \dots, x_m)$, i.e. an element of the simplex:

$$S^{m-1} = \left\{ x \in \mathbb{R}^m : x_i \geq 0, \ \sum_{i=1}^{m} x_i = 1 \right\}. \quad (1)$$

The quadratic stochastic operator (QSO) is a mapping of the simplex S^{m-1} into itself, of the form

$$V : x'_k = \sum_{i,j=1}^{m} P_{ij,k} x_i x_j, \ \ k = 1, \dots, m. \quad (2)$$

where $P_{ij,k}$ are coefficients of heredity and

$$P_{ij,k} \geq 0, \ \ P_{ij,k} = P_{ji,k}, \ \ \sum_{k=1}^{m} P_{ij,k} = 1, \ \ i, j, k = 1, \dots, m. \quad (3)$$

Thus, each quadratic stochastic operator V can be uniquely defined by a cubic matrix $(P_{ij,k})_{i,j,k=1}^{m}$ with conditions (3).

For a given $x^{(0)} \in S^{m-1}$ the trajectory (orbit) $\{x^{(n)}\}$ of $x^{(0)}$ under the action of QSO (2) is defined by

$$x^{(n+1)} = V(x^{(n)}), \ \ n = 0, 1, 2... \quad (4)$$

The main problem: is to know what ultimately happens with the sequence (4). Does the limit $\lim_{n\to\infty} x_n =$? This is very difficult problem, in general (see [1],[2]).

Definition 1.[1] A quadratic operator (2) preserving a simplex, is called non-stochastic (QnSO) if at least one of its coefficients $P_{ij,k}, \ i \neq j$ is negative.

Theorem 1. [3] For a quadratic operator V (given by (2)), to preserve a simplex S^{m-1} it is **sufficient** that

i) $\sum\limits_{k=1}^{m} P_{ij,k} = 1, \ \ i, j = 1, \dots, m;$

ii) $0 \leq P_{ii,k} \leq 1, \ \ i, k = 1, \dots, m;$

iii) $-\frac{1}{m-1}\sqrt{P_{ii,k}P_{jj,k}} \leq P_{ij,k} \leq 1 + \sqrt{(1 - P_{ii,k})(1 - P_{jj,k})}$

and **necessary** that the conditions (i), (ii) and

iii′) $-\sqrt{P_{ii,k}P_{jj,k}} \leq P_{ij,k} \leq 1 + \sqrt{(1 - P_{ii,k})(1 - P_{jj,k})}$

are satisfied.

Let's consider the following quadratic stochastic operator S^2:

$$W : \begin{cases} x' = x^2 + y^2 + z^2 - axy - axz - ayz \\ y' = (2 + a)(x + z)y \\ z' = (2 + a)zx. \end{cases} \quad (5)$$

where

$$a \in [-2, 2] \tag{6}$$

If this operator W takes a value in the interval $a \in [0, 2]$ for the parameter a, then W is a quadratic non-stochastic operator (QnSO) according to Theorem 3.

$$W_0 : \begin{cases} x' = x^2 + y^2 + z^2 - axy - axz - ayz \\ y' = (2 + a)(x + z)y \\ z' = (2 + a)zx. \end{cases} \tag{7}$$

where

$$a \in [0, 2] \tag{8}$$

Fixed points. The fixed points of this operator are solutions to the system $W_0(x, y, z) = (x, y, z)$. The following lemma gives all fixed points of operator W_0.

Lemma. The set of all fixed points of operator (7) is

$$\text{Fix}(W) = \left\{ A_1 = (1, 0, 0)\,, A_2 = \left(\frac{1}{2+a}, \frac{1+a}{2+a}, 0\right), A_3 = \left(\frac{1}{2+a}, 0, \frac{1+a}{2+a}\right) \right\}$$

Definition 2. A fixed point x^* of a mapping W is called hyperbolic point if its Jacobian J_W at x^* has no eigenvalues on the unit circle; **attracting** point if all the eigenvalues of the Jacobi matrix $J_W(x^*)$ are less than 1 in absolute value; **repelling** point if all the eigenvalues of the Jacobi matrix $J_W(x^*)$ are greater than 1 in absolute value; a **saddle** point otherwise. The following theorem gives type (see [1] for definitions) of each fixed point of the operator W_2:

Theorem 2. For the operator W_2, the points e_1, e_2, e_3 are fixed points. Moreover,

1) the point e_2 is non-hyperbolic;

2) if $a \in [0, 1)$, then the point e_1 is a repeller, point e_3 is a saddle;

3) if $a = 1$, then e_1 and e_3 are non-hyperbolic;

4) if $a \in (1, 2]$, then e_1 is a saddle and point e_3 is a repeller.

On invariant sets. Recall that a set W is called invariant with respect to an operator V if $V(W) \subset W$.

Denote

$$\mathcal{I}_1 = \{(x, y, z) \in S^2 : y = 0\}, \;\; \mathcal{I}_2 = \{(x, y, z) \in S^2 : z = 0\},$$

It is easy to see that $\mathcal{I}_1$ and $\mathcal{I}_2$ is an edge of the simplex and it is invariant: $y = 0 \;\; \Rightarrow y' = 0$, $z = 0 \;\; \Rightarrow z' = 0$.

References

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