O‘ZBEKISTON RESPUBLIKASI
OLIY TA’LIM, FAN VA INNOVATSIYALAR VAZIRLIGI

SHAROF RASHIDOV NOMIDAGI

SAMARQAND DAVLAT UNIVERSITETI

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MATEMATIKA INSTITUTI

SAMARQAND BO‘LINMASI

ZAMONAVIY ANALIZ VA MATEMATIK FIZIKA MASALALARI

Respublika ilmiy-amaliy anjumani materiallari to‘plami

2024-yil 16-17-sentabr

SAMARQAND – 2024

ABOUT FIXED POINT OF A QnSO

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Let $I = \{1, 2, \dots, m\}$. A distribution of the set I is a probability measure $x = (x_1, \dots, x_m)$, i.e. an element of the simplex:

$$S^{m-1} = \left\{ x \in \mathbb{R}^m : x_i \geq 0, \ \sum_{i=1}^{m} x_i = 1 \right\}. \tag{1}$$

The quadratic stochastic operator (QSO) is a mapping of the simplex S^{m-1} into itself, of the form

$$V : x'_k = \sum_{i,j=1}^{m} P_{ij,k} x_i x_j, \ \ k = 1, \dots, m. \tag{2}$$

where $P_{ij,k}$ are coefficients of heredity and

$$P_{ij,k} \geq 0, \ \ P_{ij,k} = P_{ji,k}, \ \ \sum_{k=1}^{m} P_{ij,k} = 1, \ \ i, j, k = 1, \dots, m. \tag{3}$$

Thus, each quadratic stochastic operator V can be uniquely defined by a cubic matrix $(P_{ij,k})_{i,j,k=1}^m$ with conditions (3).

For a given $x^{(0)} \in S^{m-1}$ the trajectory (orbit) $\{x^{(n)}\}$ of $x^{(0)}$ under the action of QSO (2) is defined by

$$x^{(n+1)} = V(x^{(n)}), \ \ n = 0, 1, 2...$$

In general, a quadratic operator V, $V : x \in \mathbb{R}^m \to x' = V(x) \in \mathbb{R}^m$ corresponding to a cubic matrix $(P_{ij,k})_{i,j,k=1}^m$ is defined by (2):[1],[2]

Definition A quadratic operator (2) preserving a simplex, is called non-stochastic (QnSO) if at least one of its coefficients $P_{ij,k}, \ i \neq j$ is negative.

Quadratic dynamical systems describe the behavior of populations of different species with population models [3],[4],[5].

Theorem[6]. For a quadratic operator V (given by (2)), to preserve a simplex S^{m-1} it is **sufficient** that

i) $\sum\limits_{k=1}^{m} P_{ij,k} = 1, \ \ i, j = 1, \dots, m;$
ii) $0 \leq P_{ii,k} \leq 1, \ \ i, k = 1, \dots, m;$
iii) $-\frac{1}{m-1}\sqrt{P_{ii,k}P_{jj,k}} \leq P_{ij,k} \leq 1 + \sqrt{(1 - P_{ii,k})(1 - P_{jj,k})}$
and **necessary** that the conditions (i), (ii) and
iii′) $-\sqrt{P_{ii,k}P_{jj,k}} \leq P_{ij,k} \leq 1 + \sqrt{(1 - P_{ii,k})(1 - P_{jj,k})}$
are satisfied.

Consider the following QnSO on S^2:

$$W_0 : \begin{cases} x' = ax^2 + 2bxy + cy^2 \\ y' = (1-a)x^2 + 2(1-b)xy + (1-c)y^2 \\ z' = z(2-z). \end{cases} \tag{4}$$

where

$$a, c \in [0, 1], \ \ \ b \in [-\sqrt{ac}, \ \ 1 + \sqrt{(1-a)(1-c)}]. \tag{5}$$

Denote

$$\mathcal{Z} = \{(x, y, z) \in S^2 : z = 0\}.$$

It is easy to see that $\mathcal{Z}$ is an edge of the simplex and it is invariant: $z = 0 \quad \Rightarrow z' = 0$.

The fixed points are solutions to the system (4)

$$\begin{cases} x = ax^2 + 2bxy + cy^2 \\ y = (1-a)x^2 + 2(1-b)xy + (1-c)y^2 \\ z = z(2-z) \end{cases} \tag{6}$$

From the third equation of the system we find $z = 0$ and $z = 1$.

1) Case: If $z = 1$, then according to the definition of a simplex, i.e. $x + y + z = 1$, it follows that $x = y = 0$. Thus, the point $e_1 = (0, 0, 1)$ will be a fixed point of the operator W_0. That is, there is only one fixed point outside $\mathcal{Z}$.

2) Case: If $z = 0$, then the restriction of operator (4) to $\mathcal{Z}$ is an arbitrary one-dimensional QnSO in S^1:

$$\begin{cases} x' = ax^2 + 2bxy + cy^2 \\ y' = (1-a)x^2 + 2(1-b)xy + (1-c)y^2, \end{cases} \tag{7}$$

Since $x + y = 1$ the fixed point equation is reduced to one-dimensional form:

$$x = f(x) := ax^2 + 2bx(1-x) + c(1-x)^2. \tag{8}$$

From this we get

$$(a - 2b + c)x^2 + (2b - 2c - 1)x + c = 0,$$

$$D = (2b - 2c - 1)^2 - 4c(a - 2b + c) = (2b - 1)^2 + 4c(1 - a),$$

$$x_{1,2} = \frac{1 + 2c - 2b \pm \sqrt{(2b-1)^2 + 4c(1-a)}}{2 \cdot (a - 2b + c)}.$$

These fixed points are possible if the parameters a, b, c satisfying (5) are in the following sets:

$$A_1 = \{(a, b, c) : 0 \le x_1 = \frac{1 + 2c - 2b + \sqrt{(2b-1)^2 + 4c(1-a)}}{2 \cdot (a - 2b + c)} \le 1\}$$

or

$$A_2 = \{(a, b, c) : 0 \le x_2 = \frac{1 + 2c - 2b - \sqrt{(2b-1)^2 + 4c(1-a)}}{2 \cdot (a - 2b + c)} \le 1\}$$

Since $(2b - 1)^2 + 4c(1 - a) > 0$ and $f : [0, 1] \to [0, 1]$, we note that f has always at least one fixed point.

Adabiyotlar

1. R.N. Ganikhodzhaev, F.M. Mukhamedov, U.A. Rozikov, *Quadratic stochastic operators and processes: results and open problems*, Inf. Dim. Anal. Quant. Prob. Rel. Fields. **14**(2) (2011), 279–335.

2. Ganikhodzhaev R.N. *Thesis Doctor of Sciences,* Tashkent. 1995, (Russian).

3. H. Kesten, *Quadratic transformations: a model for population growth.* I, II, Adv. Appl.Probab, **2** (1970). 1–82; 179–228.

4. Yu.I. Lyubich, *Mathematical structures in population genetics*, Springer-Verlag, 1992.

5. U.A. Rozikov, *Population dynamics: algebraic and probabilistic approach. World Sci. Publl.*

Singapore. 2020.
6. B.J. Mamurov, U.A. Rozikov, S.S. Xudayarov. Quadratic stochastic processes of type $(\sigma|\mu)$. Markov Processes Related Fields. **26**(5) (2020), 915–933.

SHTOLTS-CHEZARO TEOREMASI VA UNING TADBIQLARI

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Annotatsiya. Ushbu tezisda Shtolts teoremasidan foydalanib ketma-ketliklarning yaqinlashuvchiligini isbotlash va limitini topish usullari keltirilgan.

Kalit so‘zlar: ketma-ketlik va uning limiti, Shtolts teoremasi, Teylor formulasi, Sterling formulasi.

Sonli ketma-ketlikning limitini topishning turli usullari mavjud bo‘lib[1]-[3], shular orasida Shtolts-Chezaro teoremasi[4] alohida o‘rin tutadi. Quyidagi dastlab Shtolts-Chezaro teoremasini keltiramiz, so‘ngra ushbu teorema yordamida ba’zi ketma-ketliklarning limitini hisoblaymiz.

Teorema(Shtolts-Chezaro teoremasi). Aytaylik, $\{a_n\}_{n\geq 1}$ va $\{b_n\}_{n\geq 1}$ haqiqiy ketma-ketliklar berilgan bo‘lsin. Agar $\{b_n\}$ ketma-ketlik qat’iy monoton va $\lim_{n\to\infty} b_n = \infty$ shartlar o‘rinli bo‘lib, ushbu

$$\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = l < \infty$$

limit mavjud bo‘lsa, u holda $\lim_{n\to\infty} \frac{a_n}{b_n} = l$ tenglik o‘rinli bo‘ladi.

1-misol. Quyidagi limitni hisoblang:

$$\lim_{n\to\infty} \frac{1}{\left(\sqrt[n]{(2n-1)!}\right)^2} \sum_{k=1}^{n} \left[\left(\sqrt[2k]{k!} + \sqrt[2(k+1)]{(k+1)!}\right)^2\right].$$

Bu yerda $[x]-x$ sonning butun qismini bildiradi.

Yechimi. Shtolts-Chezaro teoremasining barcha shartlari qanoatlantirilganini e’tiborga olgan holda, limitni hisoblaymiz:

$$\lim_{n\to\infty} \frac{1}{\left(\sqrt[n]{(2n-1)!}\right)^2} \sum_{k}^{n} \left[\left(\sqrt[2k]{k!} + \sqrt[2(k+1)]{(k+1)!}\right)^2\right] =$$

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n+1} \left[\left(\sqrt[2k]{k!} + \sqrt[2(k+1)]{(k+1)!}\right)^2\right] - \sum_{k=1}^{n} \left[\left(\sqrt[2k]{k!} + \sqrt[2(k+1)]{(k+1)!}\right)^2\right]}{\left(\sqrt[n+1]{(2n+1)!}\right)^2 - \left(\sqrt[n]{(2n-1)!}\right)^2} =$$

$$= \lim_{n\to\infty} \frac{\left[\left(\sqrt[2(n+1)]{(n+1)!} + \sqrt[2(n+2)]{(n+2)!}\right)^2\right]}{\left(\sqrt[n+1]{(2n+1)!}\right)^2 - \left(\sqrt[n]{(2n-1)!}\right)^2}.$$

So‘nggi limitni hisoblash uchun quyidagi Sterling formulasidan foydalanamiz:

$$n! \sim \left(\frac{n}{e}\right)^n.$$