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Quadratic stochastic processes of permutation matrix

Xudayarov S. S. [1]

Permutatsiya matritsasining kvadratik stoxastik jarayonlari
Ushbu maqolada $(\sigma|D)$ tipli kvadratik stokastik jarayonlar (KSJ)ga doir bir nechta misollari qurilgan, bu yerda σ - stoxastik kubik matritsaning tipi, D esa kub matritsalar o'rtasidagi ko'paytirishni bildiradi.
Kalit so'zlar: kvadratik stoxastik jarayon; kubik matritsa; vaqt; Kolmogorov-Chepman tenglamasi.

Квадратичные случайные процессы матрицы перестановок
В этой статье построены несколько примеров квадратичных стохастических процессов (КСП)типа $(\sigma|D)$, где σ тип стохастических кубических матриц, а D означает конкретное умножение между кубическими матрицами.
Ключевые слова: квадратичная стохастическая матрица; кубическая матрица; время; Уравнение Колмогорова-Чепмена.

MSC 2010: 35C15, 35L15.
Keywords: quadratic stochastic process; cubic matrix; time; Kolmogorov-Chapman equation

Introduction

In this paper following [1], [3], [4], [6] we define a type of quadratic stochastic processes (QSP) (also known as Markov processes of cubic matrices) in continuous time. These are dynamical systems given by σ-stochastic cubic matrices satisfying an analogue of Kolmogorov-Chapman equation (KCE) with respect to a fixed (in this paper called type D) multiplication between cubic matrices. In this setting the existence of a stochastic (at each time) solution to the KCE provides the existence of a QSP called a QSP of type $(\sigma|D)$. In this paper, our aim is to construct QSPs of type $(\sigma|D)$ and study their time dependent behavior.

Let us give necessary definitions (see [1]).

A square matrix $\mathcal{U} = (U_{ij})_{i,j=1}^{m}$ is called *right stochastic* if

$$U_{ij} \geq 0, \quad \forall i, j = 1, \dots, m; \qquad \sum_{j=1}^{m} U_{ij} = 1, \ \ \forall i = 1, \dots, m.$$

Similarly one can define a *left stochastic* matrix being a non-negative real square matrix, with each column summing to 1.

A family of stochastic matrices $\{\mathcal{U}^{[s,t]} : s, t \geq 0\}$ is called a *Markov process* if it satisfies the Kolmogorov-Chapman equation:

$$\mathcal{U}^{[s,t]} = \mathcal{U}^{[s,\tau]}\mathcal{U}^{[\tau,t]}, \qquad \text{for all } \ 0 \leq s < \tau < t. \tag{1}$$

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Cubic matrices. Consider a cubic matrix $Q = (q_{ijk})_{i,j,k=1}^{m}$ as a m^3-dimensional vector, i.e. an element of $\mathbb{R}^{m^3}$, which can be uniquely written as

$$Q = \sum_{i,j,k=1}^{m} q_{ijk} E_{ijk},$$

where E_{ijk} denotes the cubic unit (basis) matrix, i.e. E_{ijk} is a m^3- cubic matrix whose (i, j, k)th entry is equal to 1 and all the other entries are equal to 0.

Denoting $Q_i = (q_{ijk})_{j,k=1}^{m}$ we can write the cubic matrix Q in the following form

$$Q = (Q_1|Q_2|\dots|Q_m).$$

Denote by $\mathbf{C}$ the set of all cubic matrices over a field F. Then $\mathbf{C}$ is an m^3-dimensional vector space over F, i.e. for any matrices $A = (a_{ijk})$, $B = (b_{ijk}) \in \mathbf{C}$, $\lambda \in F$, we have

$$A + B = (a_{ijk} + b_{ijk}) \in \mathbf{C}, \qquad \lambda A = (\lambda a_{ijk}) \in \mathbf{C}.$$

Maksimov's multiplications. Following [5] (see also [3], [4], [7]) define the following multiplications for basis matrices E_{ijk}:

$$E_{ijk} *_0 E_{lnr} = \delta_{kl}\delta_{jn} E_{ijr}. \tag{2}$$

Then for any two cubic matrices $A = (a_{ijk}), B = (b_{ijk}) \in \mathbf{C}$ the matrix $A *_0 B = (c_{ijk})$ is defined by

$$c_{ijr} = \sum_{k=1}^{m} a_{ijk} b_{kjr}. \tag{3}$$

Let $I = \{1, 2, \dots, m\}$. Consider

$$E_{ijk} *_a E_{lnr} = \delta_{kl} E_{ia(j,n)r}, \tag{4}$$

where $a \colon I \times I \to I$, $(j, n) \mapsto a(j, n) \in I$, is an arbitrary associative binary operation and δ_{kl} is the Kronecker symbol. Note that (2) is *not* a particular case of (4).

Denote by O_m the set of all associative binary operations on I.

The general formula for the multiplication is the extension of (4) by bilinearity, i.e. for any two cubic matrices $A = (a_{ijk}), B = (b_{ijk}) \in \mathbf{C}$ the matrix $A *_a B = (c_{ijk})$ is defined by

$$c_{ijr} = \sum_{l,n:\, a(l,n)=j} \sum_{k} a_{ilk} b_{knr}.$$

Define several kinds of cubic stochastic matrices (see [5]): a cubic matrix $P = (p_{ijk})_{i,j,k=1}^{m}$ is called

$(1, 2)$-*stochastic* if

$$p_{ijk} \geq 0, \qquad \sum_{i,j=1}^{m} p_{ijk} = 1, \text{ for all } k.$$

$(1, 3)$-*stochastic* if

$$p_{ijk} \geq 0, \qquad \sum_{i,k=1}^{m} p_{ijk} = 1, \text{ for all } j.$$

$(2, 3)$-*stochastic* if

$$p_{ijk} \geq 0, \qquad \sum_{j,k=1}^{m} p_{ijk} = 1, \text{ for all } i.$$

3-*stochastic* if

$$p_{ijk} \geq 0, \qquad \sum_{k=1}^{m} p_{ijk} = 1, \text{ for all } i, j.$$

The last one can be also given with respect to first and second index.

Define the following concrete multiplication basis matrices (called type (D)):

$$E_{ijk}\cdot E_{lnr}=\begin{cases}E_{ijk}, & \text{if } k=l,\\ 0, & \text{if otherwise.}\end{cases}$$

Extending this multiplication to arbitrary cubic matrices

$$A=(a_{ijk})_{i,j,k=1}^m,\quad B=(b_{ijk})_{i,j,k=1}^m,\quad C=(c_{ijk})_{i,j,k=1}^m,$$

we get that the entries of $C=AB$ can be written as

$$c_{ijr}=\sum_{k,n=1}^m a_{ijk}b_{knr}.$$

Let parameters $s\geq 0$, $t\geq 0$, are considered as time.

Denote $\mathbb{R}_+=\{x\in\mathbb{R}: x\geq 0\}$ and by $M^{[s,t]}=\left(P_{ijk}^{[s,t]}\right)_{i,j,k=1}^m$ a cubic matrix with two parameters.

Definition 1. A family $\{M^{[s,t]}:\ s,t\in\mathbb{R}_+\}$ is called a Markov process of cubic matrices (or a quadratic stochastic process (QSP)) of type $(\sigma|\mu)$ if for each time s and t the cubic matrix $M^{[s,t]}$ is σ-stochastic and satisfies the Kolmogorov-Chapman equation (for cubic matrices):

$$M^{[s,t]}=M^{[s,\tau]}*_\mu M^{[\tau,t]},\qquad \text{for all } 0\leq s<\tau<t \tag{5}$$

with respect to the multiplication of type μ.

Theorem 1. (see [3]) *Let $\{A^{[t]}=(a_{ij}^{[t]}, t\geq 0)\}$ be a family of invertible (for all t), $m\times m$ square matrices and let $\left(A^{[t]}\right)^{-1}=(b_{ij}^{[t]})$ denote the inverse of $A^{[t]}$. Let $B^{(s)}=\left(\beta_{ijk}^{(s)}\right)$, be a cubic matrix, where $\beta_{ijk}^{(s)}$, $i,j,k=1,\dots,m$, are arbitrary functions such that*

$$\sum_{j=1}^m \beta_{ijk}^{(s)}=a_{ik}^{[s]},\quad \textit{for any } i,k \textit{ and } s.$$

Then cubic matrix

$$M^{[s,t]}=\left(\sum_{k=1}^m \beta_{ijk}^{(s)}b_{kr}^{[t]}\right)_{i,j,r=1}^m \tag{6}$$

generates an flow of algebras (i.e. satisfies equation (5) of type (D).

In general, the matrix (6) does not generate a QSP. Here our aim is to find conditions on matrices $A^{[t]}$ and $B^{(s)}$ (mentioned in Theorem 1) ensuring that the matrix (6) generates a QSP.

QSP of type $(\sigma|D)$

Lemma 1. *Let $A^{[t]}=\left(a_{ij}^{[t]}\right)_{i,j=1}^m$ be a right stochastic and invertible matrix and let $B^{[t]}=\left(b_{ij}^{[t]}\right)_{i,j=1}^m$ be its inverse matrix. If $B^{[t]}$ is not a stochastic matrix,then it has a least one negative element.*

Proof. Let $A^{[t]}$ be a right stochastic matrix, then for any $i,j\in I$, $a_{ij}^{[t]}\geq 0$ and for each $i\in I$, $\sum\limits_{j=1}^m a_{ij}^{[t]}=1$. Since the matrixs $A^{[t]}$ and $B^{[t]}$ are inversely, we have $A^{[t]}B^{[t]}=B^{[t]}A^{[t]}=E$, that is:

$$\sum_{j=1}^m b_{ij}^{[t]}a_{jl}^{[t]}=e_{il}=\begin{cases}1 & \text{if } i=l,\\ 0 & \text{if } i\neq l.\end{cases}$$

Then for each $i\in I$ we have

$$\sum_{j=1}^m b_{ij}^{[t]}=\sum_{j=1}^m\left(b_{ij}^{[t]}\cdot 1\right)=\sum_{j=1}^m\left(b_{ij}^{[t]}\sum_{l=1}^m a_{jl}^{[t]}\right)$$

$$=\sum_{j=1}^m\sum_{l=1}^m b_{ij}^{[t]}a_{jl}^{[t]}=\sum_{l=1}^m\sum_{j=1}^m b_{ij}^{[t]}a_{jl}^{[t]}=\sum_{l=1}^m e_{il}=1.$$

Since the matrix $B^{[t]}$ in not a stochastic, it is has a least one negative element. □

Prorosition 1. If $A^{[t]}$ and $B^{(s)} = (\beta_{ijk}^{(s)})_{i,j,k=1}^m$ in Theorem 1 satisfies the following conditions:

– $(A^{[t]})^{-1}$ is a left stochastic;

– $B^{(s)}$ is a $(1,2)$-stochastic

then the matrix $M^{[s,t]}$ (given by (6)) generates a QSP of type $(12|D)$.

Proof. By the conditions of the proposition it is easy to see that all elements of the matrix $M^{[s,t]}$ are non-negative. We calculate for the following sum

$$\sum_{i,j=1}^m \sum_{k=1}^m \beta_{ijk}^{(s)} b_{kr}^{[t]} = \sum_{k=1}^m \left(\sum_{i,j=1}^m \beta_{ijk}^{(s)} b_{kr}^{[t]} \right) = \sum_{k=1}^m b_{kr}^{[t]} \sum_{i,j=1}^m \beta_{ijk}^{(s)}$$

Since $B^{(s)}$ is a (1,2)-stochastic we have $\sum\limits_{i,j=1}^m \beta_{ijk}^{(s)} = 1$. Then the right hand side of the last equality becomes

$$\sum_{k=1}^m b_{kr}^{[t]} \sum_{i,j=1}^m \beta_{ijk}^{(s)} = \sum_{k=1}^m b_{kr}^{[t]}$$

Since $\left(A^{[t]}\right)^{-1}$ is a left stochastic matrix we obtain

$$\sum_{k=1}^m b_{kr}^{[t]} = 1.$$

Consequently, $M^{[s,t]}$ is a (1,2)-stochastic matrix. This completes the proof. □

Recall that a permutation matrix is a square binary matrix that has exactly one entry of 1 in each row and each column and 0s elsewhere. Note that a permutation matrix is (right and left) stochastic.

Example 1. Let

$$A^{[t]} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$B^{(s)} = \left(\begin{array}{cc|cc} f(s) & 0 & 0 & g(s) \\ 1-f(s) & 0 & 0 & 1-g(s) \end{array} \right)$$

Here $0 \le f(s), g(s) \le 1$. Then $A^{[t]}$ and $B^{[t]}$ satisfy all conditions of Theorem 1.

In this case the QSP is defined by the matrix

$$M^{[s]} = \left(\begin{array}{cc|cc} f(s) & 0 & 0 & g(s) \\ 1-f(s) & 0 & 0 & 1-g(s) \end{array} \right).$$

Theorem 2. *If in Theorem 1 both $A^{[t]}$ and $A^{[t]})^{-1}$ are stochastic for some $t > 0$ and $(\beta_{ijk}^{(s)})_{i,j,k=1}^m$ is 3-stochastic for some $s < t$, then $M^{[s,t]}$ is not 3-stochastic.*

Proof. If $A^{[t]}$ and $(A^{[t]})^{-1}$ are stochastic matrices, then $A^{[t]}$ is a permutation of unitary matrix (see [2]). More precisely, there are $k_1 \neq k_2 \neq k_3 \neq ... \neq k_m$, such that for the elements of matrix $A^{[t]}$ we have the following

$$a_{1k_1}^{[t]} = 1, \; a_{2k_2}^{[t]} = 1, ..., a_{mk_m}^{[t]} = 1,$$

and all other elements are 0. Indeed, let $\mathcal{P}$ be the set of stochastic matrices:

$$\mathcal{P} = \{P = (p_{ij})_{i,j=1}^n : p_{ij} \ge 0, \sum_{j=1}^n p_{ij} = 1, \forall i\}$$

and $\mathcal{P}_{per}$ be the set of permutation matrices. For $P = (p_{ij})_{i,j=1}^n$ we denote $p_i := (p_{i1}, p_{i2}, ..., p_{in}), \;\; supp(p_i) = \{k : p_{ik} > 0\}$.

First we show that if $P \in \mathcal{P}_{per}$ then $P^{-1} \in \mathcal{P}_{per}$:

If P is a permutation matrix then there exists a rearrangement π of coordinates such that for all $x = (x_1, x_2, ..., x_n)$ one has $Px = \pi \circ x$. This implies that $P^{-1}x = \pi^{-1} \circ x$ and so P^{-1} also rearranges coordinates. Therefor P^{-1} is a permutation matrix as well.

Second we prove that if $P \in \mathcal{P}\backslash\mathcal{P}_{per}$ such that P is invertible (i.e.P^{-1} exists) then $P^{-1} \notin \mathcal{P}$:

Let us assume that $T := P^{-1} = (q_{ij})_{i,j=1}^{n} \in \mathcal{P}$ is stochastic. Since $P \in \mathcal{P}\backslash\mathcal{P}_{per}$ then there exists i in $\{1,2,...,n\}$ such that $|supp(p_i)| > 1$. This implies p_{ij} has two nonzero elements. Now, since $PT = I$, then $\sum\limits_{j=1}^{n} p_{ij}q_{ji} = \sum\limits_{j\in supp(p_i)} p_{ij}q_{ji} = 1$. However $0 \le q_{ji} \le 1$ implies that $q_{ij} = 1$ for all $j \in supp(p_i)$. Otherwise, one would have $\sum\limits_{j\in supp(p_i)} p_{ij}q_{ji} < \sum\limits_{j\in supp(p_i)} p_{ij} = 1$. Considering $TP = I$ we get $(TP)_{ij} = \sum\limits_{k=1}^{n} q_{jk}p_{kj} = \sum\limits_{k\in supp(q_j)} q_{jk}p_{kj} = 1$. However, since T is stochastic one has $q_{jm} = 0$ for all $m \neq i(q_{ji} = 1)$. Therefore $(TP)_{ij} = q_{ji}p_{ij} = 1$ implies $p_{ij} = 1$. This however, contradict to $|supp(p_i)| > 1$.

Since (by Theorem 1

$$\sum_{j=1}^{m} \beta_{ijk}^{(s)} = a_{ik}^{[s]}$$

we have

$$\begin{cases} \sum\limits_{j=1}^{m} \beta_{1jk_1}^{(s)} = 1, & \sum\limits_{j=1}^{m} \beta_{1jk}^{(s)} = 0, k \neq k_1 \Rightarrow \beta_{1jk}^{(s)} = 0, k \neq k_1 \\ \sum\limits_{j=1}^{m} \beta_{2jk_2}^{(s)} = 1, & \sum\limits_{j=1}^{m} \beta_{2jk}^{(s)} = 0, k \neq k_2 \Rightarrow \beta_{2jk}^{(s)} = 0, k \neq k_2 \\ \dots & \dots \\ \sum\limits_{j=1}^{m} \beta_{mjk_m}^{(s)} = 1, & \sum\limits_{j=1}^{m} \beta_{mjk}^{(s)} = 0, k \neq k_m \Rightarrow \beta_{mjk}^{(s)} = 0, k \neq k_m \end{cases} \tag{7}$$

Since $(\beta_{ijk}^{(s)})_{i,j,k=1}^{m}$ is 3-stochastic we have

$$\beta_{1j1}^{(s)} + \beta_{1j2}^{(s)} + \ldots + \beta_{1jk_1}^{(s)} + \ldots + \beta_{1jm}^{(s)} = 1$$

$$\beta_{1jk_1}^{(s)} = 1, \;\; j = 1, \ldots, m$$

But it contradicts to $\sum\limits_{k=1}^{m} \beta_{1jk}^{(s)} = 1$. Theorem 2 is proved. □

Remark. It follows from the theorem that in order for $M^{[s,t]}$ to represent a quadratic stochastic process, the matrix $A^{[t]}$ must be a permutation matrix.

The following propositions can be proved similar to Proposition 1. So the proofs will be skipped.

Proposition 2. If $A^{[t]}$ and $B^{(s)} = (\beta_{ijk}^{(s)})_{i,j,k=1}^{m}$ in Theorem 1 satisfies the following conditions:

– $(A^{[t]})^{-1}$ is a right stochastic;

– $B^{(s)}$ is a (1,3)-stochastic.

then the matrix $M^{[s,t]}$ (given by (6)) generates a QSP of type $(13|D)$

Example 2. Let

$$A^{[t]} = \left(A^{[t]}\right)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and

$$B^{(s)} = \left(\begin{array}{ccc|ccc|ccc} f_1(s) & 0 & 0 & 0 & 0 & f_2(s) & 0 & f_3(s) & 0 \\ \varphi_1(s) & 0 & 0 & 0 & 0 & \varphi_2(s) & 0 & \varphi_3(s) & \\ 1-F_1(s) & 0 & 0 & 0 & 0 & 1-F_2(s) & 0 & 1-F_3(s) & 0 \end{array} \right).$$

Then $A^{[t]}$ and $B^{[t]}$ satisfy all conditions of Theorem 1. In this case the QSP is defined by the matrix

$$M^{[s]} = \left(\begin{array}{ccc|ccc|ccc} f_1(s) & 0 & 0 & 0 & f_2(s) & 0 & 0 & 0 & f_3(s) \\ \varphi_1(s) & 0 & 0 & 0 & \varphi_2(s) & 0 & 0 & 0 & \varphi_3(s) \\ 1-F_1(s) & 0 & 0 & 0 & 1-F_2(s) & 0 & 0 & 0 & 1-F_3(s) \end{array} \right).$$

Where $F_i(s) = f_i(s) + \varphi_i(s), i = 1, 2, 3$ and functions $f_i(s)$ and $\varphi_i(s)$ satisfy the following conditions

$$\begin{cases} \sum\limits_{i=1}^{3} f_i(s) = 1 \\ \sum\limits_{i=1}^{3} \varphi_i(s) = 1 \\ f_i(s) \geq 0 \\ \varphi_i(s) \geq 0 \\ f_i(s) + \varphi_i(s) \leq 1 \end{cases} \tag{8}$$

Proposition 3. If $A^{[t]}$ and $B^{(s)} = (\beta_{ijk}^{(s)})_{i,j,k=1}^{m}$ in Theorem 1 satisfies the following conditions:
– $(A^{[t]})^{-1}$ is a right stochastic;
– $B^{(s)}$ is a (2,3)-stochastic.
then the matrix $M^{[s,t]}$ (given by (6)) generates a QSP of type $(23|D)$
Example 3. Let

$$A^{[t]} = \left(A^{[t]}\right)^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and

$$B^{(s)} = \left(\begin{array}{ccc|ccc|ccc} 0 & 0 & f_1(s) & 0 & f_2(s) & 0 & f_3(s) & 0 & 0 \\ 0 & 0 & \varphi_1(s) & 0 & \varphi_2(s) & 0 & \varphi_3(s) & 0 & 0 \\ 0 & 0 & 1 - F_1(s) & 0 & 1 - F_2(s) & 0 & 1 - F_3(s) & 0 & 0 \end{array} \right),$$

here $F_i(s) = f_i(s) + \varphi_i(s), i = 1, 2, 3$. Then $A^{[t]}$ and $B^{[t]}$ satisfy all conditions of Theorem 1. In this case the QSP is defined by the matrix

$$M^{(s)} = \left(\begin{array}{ccc|ccc|ccc} 0 & 0 & f_1(s) & 0 & f_2(s) & 0 & f_3(s) & 0 & 0 \\ 0 & 0 & \varphi_1(s) & 0 & \varphi_2(s) & 0 & \varphi_3(s) & 0 & 0 \\ 0 & 0 & 1 - F_1(s) & 0 & 1 - F_2(s) & 0 & 1 - F_3(s) & 0 & 0 \end{array} \right).$$

The Functions $f_i(s)$ and $\varphi_i(s)$ satisfy the following conditions.

$$\begin{cases} f_i(s) \geq 0 \\ \varphi_i(s) \geq 0 \\ f_i(s) + \varphi_i(s) \leq 1, \ \ i = 1, 2, 3. \end{cases} \tag{9}$$

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