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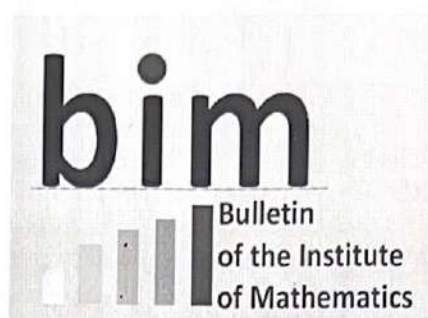
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QUADRATIC STOCHASTIC PROCESSES OF PERMUTATION MATRIX

Xudayarov S. S. ¹

Permutatsiya matritsasining kvadratik stoxastik jarayonlari
Ushbu maqolada $(\sigma|D)$ tipli kvadratik stoxastik jarayonlar
(KSJ)ga doir bir nechta misollari qurilgan, bu yerda σ - stoxastik
kubik matritsaning tipi, D esa kub matritsalar o'rtasidagi
ko'paytirishni bildiradi.

Kalit so'zlar: kvadratik stoxastik jarayon; kubik matritsa; vaqt;
Kolmogorov-Cherpan tenglamasi.

Квадратичные случайные процессы матрицы перестановок
В этой статье построены несколько примеров квадратичных
стохастических процессов (КСП) типа $(\sigma|D)$, где σ тип сто-
хастических кубических матриц, а D означает конкретное
умножение между кубическими матрицами.

Ключевые слова: квадратичная стохастическая матрица; ку-
бическая матрица; время; Уравнение Колмогорова-Чепмена.

MSC 2010: 35C15, 35L15.

Keywords: quadratic stochastic process; cubic matrix; time; Kolmogorov-Chapman equation

Introduction

In this paper following [1], [3], [4], [6] we define a type of quadratic stochastic processes (QSP) (also known as Markov processes of cubic matrices) in continuous time. These are dynamical systems given by σ -stochastic cubic matrices satisfying an analogue of Kolmogorov-Chapman equation (KCE) with respect to a fixed (in this paper called type D) multiplication between cubic matrices. In this setting the existence of a stochastic (at each time) solution to the KCE provides the existence of a QSP called a QSP of type $(\sigma|D)$. In this paper, our aim is to construct QSPs of type $(\sigma|D)$ and study their time dependent behavior.

Let us give necessary definitions (see [1]).

A square matrix $\mathcal{U} = (U_{ij})_{i,j=1}^m$ is called *right stochastic* if

$$U_{ij} \geq 0, \quad \forall i, j = 1, \dots, m; \quad \sum_{j=1}^m U_{ij} = 1, \quad \forall i = 1, \dots, m.$$

Similarly one can define a *left stochastic* matrix being a non-negative real square matrix, with each column summing to 1.

A family of stochastic matrices $\{\mathcal{U}^{[s,t]} : s, t \geq 0\}$ is called a *Markov process* if it satisfies the Kolmogorov-Chapman equation:

$$\mathcal{U}^{[s,t]} = \mathcal{U}^{[s,\tau]} \mathcal{U}^{[\tau,t]}, \quad \text{for all } 0 \leq s < \tau < t. \quad (1)$$

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Cubic matrices. Consider a cubic matrix $Q = (q_{ijk})_{i,j,k=1}^m$ as a m^3 -dimensional vector, i.e. an element of \mathbb{R}^{m^3} , which can be uniquely written as

$$Q = \sum_{i,j,k=1}^m q_{ijk} E_{ijk},$$

where E_{ijk} denotes the cubic unit (basis) matrix, i.e. E_{ijk} is a m^3 -cubic matrix whose (i, j, k) th entry is equal to 1 and all the other entries are equal to 0.

Denoting $Q_i = (q_{ijk})_{j,k=1}^m$ we can write the cubic matrix Q in the following form

$$Q = (Q_1 | Q_2 | \dots | Q_m).$$

Denote by \mathbf{C} the set of all cubic matrices over a field F . Then \mathbf{C} is an m^3 -dimensional vector space over F , i.e. for any matrices $A = (a_{ijk}), B = (b_{ijk}) \in \mathbf{C}, \lambda \in F$, we have

$$A + B = (a_{ijk} + b_{ijk}) \in \mathbf{C}, \quad \lambda A = (\lambda a_{ijk}) \in \mathbf{C}.$$

Maksimov's multiplications. Following [5] (see also [3], [4], [7]) define the following multiplications for basis matrices E_{ijk} :

$$E_{ijk} *_0 E_{lmr} = \delta_{kl} \delta_{jn} E_{ijr}. \quad (2)$$

Then for any two cubic matrices $A = (a_{ijk}), B = (b_{ijk}) \in \mathbf{C}$ the matrix $A *_0 B = (c_{ijk})$ is defined by

$$c_{ijr} = \sum_{k=1}^m a_{ijk} b_{kjr}. \quad (3)$$

Let $I = \{1, 2, \dots, m\}$. Consider

$$E_{ijk} *_a E_{lmr} = \delta_{kl} E_{ia(j,n)r}, \quad (4)$$

where $a: I \times I \rightarrow I, (j, n) \mapsto a(j, n) \in I$, is an arbitrary associative binary operation and δ_{kl} is the Kronecker symbol. Note that (2) is not a particular case of (4).

Denote by O_m the set of all associative binary operations on I .

The general formula for the multiplication is the extension of (4) by bilinearity, i.e. for any two cubic matrices $A = (a_{ijk}), B = (b_{ijk}) \in \mathbf{C}$ the matrix $A *_a B = (c_{ijk})$ is defined by

$$c_{ijr} = \sum_{l,n: a(l,n)=j} \sum_k a_{ilk} b_{knr}.$$

Define several kinds of cubic stochastic matrices (see [5]): a cubic matrix $P = (p_{ijk})_{i,j,k=1}^m$ is called

(1,2)-stochastic if

$$p_{ijk} \geq 0, \quad \sum_{i,j=1}^m p_{ijk} = 1, \quad \text{for all } k.$$

(1,3)-stochastic if

$$p_{ijk} \geq 0, \quad \sum_{i,k=1}^m p_{ijk} = 1, \quad \text{for all } j.$$

(2,3)-stochastic if

$$p_{ijk} \geq 0, \quad \sum_{j,k=1}^m p_{ijk} = 1, \quad \text{for all } i.$$

3-stochastic if

$$p_{ijk} \geq 0, \quad \sum_{k=1}^m p_{ijk} = 1, \quad \text{for all } i, j.$$

The last one can be also given with respect to first and second index.

Define the following concrete multiplication basis matrices (called type (D)):

$$E_{ijk} \cdot E_{lmr} = \begin{cases} E_{ijk}, & \text{if } k = l, \\ 0, & \text{if otherwise.} \end{cases}$$

Extending this multiplication to arbitrary cubic matrices

$$A = (a_{ijk})_{i,j,k=1}^m, \quad B = (b_{ijk})_{i,j,k=1}^m, \quad C = (c_{ijk})_{i,j,k=1}^m,$$

we get that the entries of $C = AB$ can be written as

$$c_{ijr} = \sum_{k,n=1}^m a_{ijk} b_{knr}.$$

Let parameters $s \geq 0, t \geq 0$, are considered as time.

Denote $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ and by $M^{[s,t]} = (p_{ijk}^{[s,t]})_{i,j,k=1}^m$ a cubic matrix with two parameters.

Definition 1. A family $\{M^{[s,t]} : s, t \in \mathbb{R}_+\}$ is called a Markov process of cubic matrices (or a quadratic stochastic process (QSP)) of type $(\sigma|\mu)$ if for each time s and t the cubic matrix $M^{[s,t]}$ is σ -stochastic and satisfies the Kolmogorov-Chapman equation (for cubic matrices):

$$M^{[s,t]} = M^{[s,\tau]} *_{\mu} M^{[\tau,t]}, \quad \text{for all } 0 \leq s < \tau < t \tag{5}$$

with respect to the multiplication of type μ .

Theorem 1. (see [3]) Let $\{A^{[t]} = (a_{ij}^{[t]})_{i,j=1}^m, t \geq 0\}$ be a family of invertible (for all t), $m \times m$ square matrices and let $(A^{[t]})^{-1} = (b_{ij}^{[t]})$ denote the inverse of $A^{[t]}$. Let $B^{(s)} = (\beta_{ijk}^{(s)})$, be a cubic matrix, where $\beta_{ijk}^{(s)}, i, j, k = 1, \dots, m$, are arbitrary functions such that

$$\sum_{j=1}^m \beta_{ijk}^{(s)} = a_{ik}^{[s]}, \quad \text{for any } i, k \text{ and } s.$$

Then cubic matrix

$$M^{[s,t]} = \left(\sum_{k=1}^m \beta_{ijk}^{(s)} b_{kr}^{[t]} \right)_{i,j,r=1}^m \tag{6}$$

generates an flow of algebras (i.e. satisfies equation (5) of type (D)).

In general, the matrix (6) does not generate a QSP. Here our aim is to find conditions on matrices $A^{[t]}$ and $B^{(s)}$ (mentioned in Theorem 1) ensuring that the matrix (6) generates a QSP.

QSP of type $(\sigma|D)$

Lemma 1. Let $A^{[t]} = (a_{ij}^{[t]})_{i,j=1}^m$ be a right stochastic and invertible matrix and let $B^{[t]} = (b_{ij}^{[t]})_{i,j=1}^m$ be its inverse matrix. If $B^{[t]}$ is not a stochastic matrix, then it has a least one negative element.

Proof. Let $A^{[t]}$ be a right stochastic matrix, then for any $i, j \in I, a_{ij}^{[t]} \geq 0$ and for each $i \in I, \sum_{j=1}^m a_{ij}^{[t]} = 1$.

Since the matrix $A^{[t]}$ and $B^{[t]}$ are inversely, we have $A^{[t]}B^{[t]} = B^{[t]}A^{[t]} = E$, that is:

$$\sum_{j=1}^m b_{ij}^{[t]} a_{jl}^{[t]} = e_{il} = \begin{cases} 1 & \text{if } i = l, \\ 0 & \text{if } i \neq l. \end{cases}$$

Then for each $i \in I$ we have

$$\begin{aligned} \sum_{j=1}^m b_{ij}^{[t]} &= \sum_{j=1}^m (b_{ij}^{[t]} \cdot 1) = \sum_{j=1}^m \left(b_{ij}^{[t]} \sum_{l=1}^m a_{jl}^{[t]} \right) \\ &= \sum_{j=1}^m \sum_{l=1}^m b_{ij}^{[t]} a_{jl}^{[t]} = \sum_{l=1}^m \sum_{j=1}^m b_{ij}^{[t]} a_{jl}^{[t]} = \sum_{l=1}^m e_{il} = 1. \end{aligned}$$

Since the matrix $B^{[t]}$ is not a stochastic, it has at least one negative element. □

Proposition 1. If $A^{[t]}$ and $B^{(s)} = (\beta_{ijk}^{(s)})_{i,j,k=1}^m$ in Theorem 1 satisfies the following conditions:

- $(A^{[t]})^{-1}$ is a left stochastic;
- $B^{(s)}$ is a (1,2)-stochastic

then the matrix $M^{[s,t]}$ (given by (6)) generates a QSP of type (12|D).

Proof. By the conditions of the proposition it is easy to see that all elements of the matrix $M^{[s,t]}$ are non-negative. We calculate for the following sum

$$\sum_{i,j=1}^m \sum_{k=1}^m \beta_{ijk}^{(s)} b_{kr}^{[t]} = \sum_{k=1}^m \left(\sum_{i,j=1}^m \beta_{ijk}^{(s)} b_{kr}^{[t]} \right) = \sum_{k=1}^m b_{kr}^{[t]} \sum_{i,j=1}^m \beta_{ijk}^{(s)}$$

Since $B^{(s)}$ is a (1,2)-stochastic we have $\sum_{i,j=1}^m \beta_{ijk}^{(s)} = 1$. Then the right hand side of the last equality becomes

$$\sum_{k=1}^m b_{kr}^{[t]} \sum_{i,j=1}^m \beta_{ijk}^{(s)} = \sum_{k=1}^m b_{kr}^{[t]}$$

Since $(A^{[t]})^{-1}$ is a left stochastic matrix we obtain

$$\sum_{k=1}^m b_{kr}^{[t]} = 1.$$

Consequently, $M^{[s,t]}$ is a (1,2)-stochastic matrix. This completes the proof. □

Recall that a permutation matrix is a square binary matrix that has exactly one entry of 1 in each row and each column and 0s elsewhere. Note that a permutation matrix is (right and left) stochastic.

Example 1. Let

$$A^{[t]} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$B^{(s)} = \left(\begin{array}{cc|cc} f(s) & 0 & 0 & g(s) \\ 1-f(s) & 0 & 0 & 1-g(s) \end{array} \right)$$

Here $0 \leq f(s), g(s) \leq 1$. Then $A^{[t]}$ and $B^{[t]}$ satisfy all conditions of Theorem 1.

In this case the QSP is defined by the matrix

$$M^{[s]} = \left(\begin{array}{cc|cc} f(s) & 0 & 0 & g(s) \\ 1-f(s) & 0 & 0 & 1-g(s) \end{array} \right)$$

Theorem 2. If in Theorem 1 both $A^{[t]}$ and $(A^{[t]})^{-1}$ are stochastic for some $t > 0$ and $(\beta_{ijk}^{(s)})_{i,j,k=1}^m$ is 3-stochastic for some $s < t$, then $M^{[s,t]}$ is not 3-stochastic.

Proof. If $A^{[t]}$ and $(A^{[t]})^{-1}$ are stochastic matrices, then $A^{[t]}$ is a permutation of unitary matrix (see [2]). More precisely, there are $k_1 \neq k_2 \neq k_3 \neq \dots \neq k_m$, such that for the elements of matrix $A^{[t]}$ we have the following

$$a_{1k_1}^{[t]} = 1, a_{2k_2}^{[t]} = 1, \dots, a_{mk_m}^{[t]} = 1,$$

and all other elements are 0. Indeed, let \mathcal{P} be the set of stochastic matrices:

$$\mathcal{P} = \{P = (p_{ij})_{i,j=1}^n : p_{ij} \geq 0, \sum_{j=1}^n p_{ij} = 1, \forall i\}$$

and \mathcal{P}_{per} be the set of permutation matrices. For $P = (p_{ij})_{i,j=1}^n$ we denote $p_i := (p_{i1}, p_{i2}, \dots, p_{in})$, $supp(p_i) = \{k : p_{ik} > 0\}$.

First we show that if $P \in \mathcal{P}_{per}$ then $P^{-1} \in \mathcal{P}_{per}$:

If P is a permutation matrix then there exists a rearrangement π of coordinates such that for all $x = (x_1, x_2, \dots, x_n)$ one has $Px = \pi \circ x$. This implies that $P^{-1}x = \pi^{-1} \circ x$ and so P^{-1} also rearranges coordinates. Therefore P^{-1} is a permutation matrix as well.

Second we prove that if $P \in \mathcal{P} \setminus \mathcal{P}_{per}$ such that P is invertible (i.e. P^{-1} exists) then $P^{-1} \notin \mathcal{P}$:

Let us assume that $T := P^{-1} = (q_{ij})_{i,j=1}^n \in \mathcal{P}$ is stochastic. Since $P \in \mathcal{P} \setminus \mathcal{P}_{per}$ then there exists i in $\{1, 2, \dots, n\}$ such that $|supp(p_i)| > 1$. This implies p_{ij} has two nonzero elements. Now, since $PT = I$, then $\sum_{j=1}^n p_{ij}q_{ji} = \sum_{j \in supp(p_i)} p_{ij}q_{ji} = 1$. However $0 \leq q_{ji} \leq 1$ implies that $q_{ji} = 1$ for all $j \in supp(p_i)$. Otherwise, one would have $\sum_{j \in supp(p_i)} p_{ij}q_{ji} < \sum_{j \in supp(p_i)} p_{ij} = 1$. Considering $TP = I$ we get $(TP)_{ij} = \sum_{k=1}^n q_{jk}p_{kj} = \sum_{k \in supp(q_j)} q_{jk}p_{kj} = 1$. However, since T is stochastic one has $q_{jm} = 0$ for all $m \neq i$ ($q_{ji} = 1$). Therefore $(TP)_{ij} = q_{ji}p_{ij} = 1$ implies $p_{ij} = 1$. This however, contradict to $|supp(p_i)| > 1$.

Since (by Theorem 1

$$\sum_{j=1}^m \beta_{ijk}^{(s)} = a_{ik}^{[s]}$$

we have

$$\begin{cases} \sum_{j=1}^m \beta_{1jk_1}^{(s)} = 1, & \sum_{j=1}^m \beta_{1jk}^{(s)} = 0, k \neq k_1 \Rightarrow \beta_{1jk}^{(s)} = 0, k \neq k_1 \\ \sum_{j=1}^m \beta_{2jk_2}^{(s)} = 1, & \sum_{j=1}^m \beta_{2jk}^{(s)} = 0, k \neq k_2 \Rightarrow \beta_{2jk}^{(s)} = 0, k \neq k_2 \\ \dots & \dots \\ \sum_{j=1}^m \beta_{mjk_m}^{(s)} = 1, & \sum_{j=1}^m \beta_{mj k}^{(s)} = 0, k \neq k_m \Rightarrow \beta_{mj k}^{(s)} = 0, k \neq k_m \end{cases} \quad (7)$$

Since $(\beta_{ijk}^{(s)})_{i,j,k=1}^m$ is 3-stochastic we have

$$\beta_{1j1}^{(s)} + \beta_{1j2}^{(s)} + \dots + \beta_{1jk_1}^{(s)} + \dots + \beta_{1jm}^{(s)} = 1$$

$$\beta_{1jk_1}^{(s)} = 1, j = 1, \dots, m$$

But it contradicts to $\sum_{k=1}^m \beta_{1jk}^{(s)} = 1$. Theorem 2 is proved. \square

Remark. It follows from the theorem that in order for $M^{[s,t]}$ to represent a quadratic stochastic process, the matrix $A^{[t]}$ must be a permutation matrix.

The following propositions can be proved similar to Proposition 1. So the proofs will be skipped.

Proposition 2. If $A^{[t]}$ and $B^{(s)} = (\beta_{ijk}^{(s)})_{j,k=1}^m$ in Theorem 1 satisfies the following conditions:

- $(A^{[t]})^{-1}$ is a right stochastic;
- $B^{(s)}$ is a $(1,3)$ -stochastic.

then the matrix $M^{[s,t]}$ (given by (6)) generates a QSP of type $(13|D)$

Example 2. Let

$$A^{[t]} = (A^{[t]})^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and

$$B^{(s)} = \left(\begin{array}{ccc|ccc} f_1(s) & 0 & 0 & 0 & 0 & f_2(s) & 0 & f_3(s) & 0 \\ \varphi_1(s) & 0 & 0 & 0 & 0 & \varphi_2(s) & 0 & \varphi_3(s) & 0 \\ 1 - F_1(s) & 0 & 0 & 0 & 0 & 1 - F_2(s) & 0 & 1 - F_3(s) & 0 \end{array} \right).$$

Then $A^{[t]}$ and $B^{[t]}$ satisfy all conditions of Theorem 1. In this case the QSP is defined by the matrix

$$M^{[s]} = \left(\begin{array}{ccc|ccc} f_1(s) & 0 & 0 & 0 & f_2(s) & 0 & 0 & 0 & f_3(s) \\ \varphi_1(s) & 0 & 0 & 0 & \varphi_2(s) & 0 & 0 & 0 & \varphi_3(s) \\ 1 - F_1(s) & 0 & 0 & 0 & 1 - F_2(s) & 0 & 0 & 0 & 1 - F_3(s) \end{array} \right).$$

Where $F_i(s) = f_i(s) + \varphi_i(s)$, $i = 1, 2, 3$ and functions $f_i(s)$ and $\varphi_i(s)$ satisfy the following conditions

$$\begin{cases} \sum_{i=1}^3 f_i(s) = 1 \\ \sum_{i=1}^3 \varphi_i(s) = 1 \\ f_i(s) \geq 0 \\ \varphi_i(s) \geq 0 \\ f_i(s) + \varphi_i(s) \leq 1 \end{cases} \quad (8)$$

Proposition 3. If $A^{[l]}$ and $B^{(s)} = (\beta_{ijk}^{(s)})_{j,k=1}^m$ in Theorem 1 satisfies the following conditions:

– $(A^{[l]})^{-1}$ is a right stochastic;

– $B^{(s)}$ is a (2,3)-stochastic.

then the matrix $M^{[s,l]}$ (given by (6)) generates a QSP of type (23|D)

Example 3. Let

$$A^{[l]} = (A^{[l]})^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and

$$B^{(s)} = \left(\begin{array}{ccc|ccc} 0 & 0 & f_1(s) & 0 & f_2(s) & 0 & f_3(s) & 0 & 0 \\ 0 & 0 & \varphi_1(s) & 0 & \varphi_2(s) & 0 & \varphi_3(s) & 0 & 0 \\ 0 & 0 & 1 - F_1(s) & 0 & 1 - F_2(s) & 0 & 1 - F_3(s) & 0 & 0 \end{array} \right),$$

here $F_i(s) = f_i(s) + \varphi_i(s)$, $i = 1, 2, 3$. Then $A^{[l]}$ and $B^{[l]}$ satisfy all conditions of Theorem 1. In this case the QSP is defined by the matrix

$$M^{(s)} = \left(\begin{array}{ccc|ccc} 0 & 0 & f_1(s) & 0 & f_2(s) & 0 & f_3(s) & 0 & 0 \\ 0 & 0 & \varphi_1(s) & 0 & \varphi_2(s) & 0 & \varphi_3(s) & 0 & 0 \\ 0 & 0 & 1 - F_1(s) & 0 & 1 - F_2(s) & 0 & 1 - F_3(s) & 0 & 0 \end{array} \right).$$

The Functions $f_i(s)$ and $\varphi_i(s)$ satisfy the following conditions.

$$\begin{cases} f_i(s) \geq 0 \\ \varphi_i(s) \geq 0 \\ f_i(s) + \varphi_i(s) \leq 1, \quad i = 1, 2, 3. \end{cases} \quad (9)$$

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