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A QUADRATIC NON-STOCHASTIC OPERATOR ON 2D-SIMPLEX

(Submitted by Uz AS academician Sh.A.Ayupov)

Introduction. Non-linear dynamical systems arise in many problems of biology, physics and other sciences. In particular, quadratic dynamical systems describe the behavior of populations of different species (see [1]-[5] and the references therein).

In this paper, we consider a quadratic non-stochastic operator mapping the two-dimensional (2D) simplex to itself. We find all fixed points and invariant sets of the operator. Moreover, we study behavior of trajectories generated by the operator.

Let $E = \{1, 2, \dots, m\}$. A distribution on the set E is a probability measure $x = (x_1, \dots, x_m)$, i.e., an element of the simplex:

$$S^{m-1} = \left\{ x \in \mathbb{R}^m : x_i \geq 0, \sum_{i=1}^m x_i = 1 \right\}.$$

In general, a quadratic operator $V, V: x \in \mathbb{R}^m \rightarrow x' = V(x) \in \mathbb{R}^m$ is defined by:

$$V: x'_k = \sum_{i,j=1}^m P_{ij,k} x_i x_j, \quad k = 1, \dots, m. \quad (1)$$

The following theorem gives conditions for coefficients of V to preserve the simplex.

Theorem 1. [5] For a quadratic operator V given by (1), to preserve a simplex S^{m-1} it is sufficient that

i)

$$\sum_{k=1}^m P_{ij,k} = 1, \quad i, j = 1, \dots, m;$$

ii)

$$0 \leq P_{ii,k} \leq 1, \quad i, k = 1, \dots, m;$$

iii)

$$-\frac{1}{m-1} \sqrt{P_{ii,k} P_{jj,k}} \leq P_{ij,k} \leq 1 + \sqrt{(1 - P_{ii,k})(1 - P_{jj,k})}.$$

and necessary that the conditions i), ii) and

$$iii') -\sqrt{P_{ii,k} P_{jj,k}} \leq P_{ij,k} \leq 1 + \sqrt{(1 - P_{ii,k})(1 - P_{jj,k})}$$

are satisfied.

Definition 1. [5] A quadratic operator (1), preserving a simplex, is called non-stochastic (shartly QnSO) if at least one of its coefficients $P_{ij,k}, i \neq j$ is negative.

In this paper, we consider (see Remark 2.2 in [5]) the following example of QnSO on the 2D-simplex S^2 :

$$V_0: \begin{cases} x' = \frac{1}{2}(z-y)^2 + \frac{3}{2}x(y+z) \\ y' = \frac{1}{2}(x-z)^2 + \frac{3}{2}y(x+z) \\ z' = \frac{1}{2}(y-x)^2 + \frac{3}{2}z(x+y). \end{cases} \quad (3)$$

Let s_3 be a permutation group of order 3. We define the action of s_3 on S^2 in the following way: if $g \in s_3, x \in S^2$ and $M \subseteq S^2$, then

$$g(x) = (x_{g(1)}, x_{g(2)}, x_{g(3)}),$$

$$g(M) = \{g(x) : x \in M\}.$$

The action of s_3 on the operator V_0 is defined as follows:

$$(gV_0)(x) = g(V_0(x)).$$

Fixed points. The fixed points are solutions to the system

$$\begin{cases} x = \frac{1}{2}(z-y)^2 + \frac{3}{2}x(y+z) \\ y = \frac{1}{2}(x-z)^2 + \frac{3}{2}y(x+z) \\ z = \frac{1}{2}(y-x)^2 + \frac{3}{2}z(x+y). \end{cases}$$

Lemma 1. If x is a fixed point of the operator V_0 , i.e., $V_0(x) = x$, then, for any $g \in s_3$, then point $g(x)$ is also a fixed point.

It is easy to find the following fixed points of the operator (3):

$$\alpha_1 = \left(0, \frac{1}{2}, \frac{1}{2}\right), \alpha_2 = \left(\frac{1}{2}, 0, \frac{1}{2}\right), \alpha_3 = \left(\frac{1}{2}, \frac{1}{2}, 0\right), \alpha_4 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

Definition 2. [2] A fixed point x^* of the operator V is called hyperbolic if its Jacobian J at x^* has no eigenvalues on the unit circle.

Definition 3. [2] A hyperbolic fixed point x^* is called:

- i) *attracting* if all the eigenvalues of the Jacobian $J(x^*)$ are less than 1 in absolute value;
- ii) *repelling* if all the eigenvalues of the Jacobian $J(x^*)$ are greater than 1 in absolute value;
- iii) *a saddle* otherwise.

To study the type of each fixed point, rewrite operator (3) (using $x = 1 - y - z$) as

$$W: \begin{cases} x' = \frac{1}{2}(1-x-2y)^2 + \frac{3}{2}x(1-x) \\ y' = \frac{1}{2}(2x-1+2y)^2 + \frac{3}{2}y(1-y) \end{cases}$$

Note that W maps the set $K = \{(x, y) \in [0, 1]^2 : 0 \leq x + y \leq 1\}$ to itself.

For eigenvalues of the Jacobian at fixed points we have

Case α_1 : $\lambda_1 = \frac{3}{2}, \lambda_2 = -\frac{1}{2}$.

Case α_2 : $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{3}{2}$.

Case α_3 : $\lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{3}{2}$.

Case α_4 : $\lambda_{1,2} = \frac{1}{2}$.

Invariant sets. Lemma 2. For any $g \in s_3$ and $x \in S^2$, the following equality is true:

$$(gV_0)(x) = g(V_0(x)).$$

Lemma 3. If M is an invariant set of the operator V_0 , i.e., $V_0(M) \subseteq M$, then, for any $g \in s_3$, the set $g(M)$ is also an invariant set for V_0 .

Denine

$$M_1 = \{(x, y, z) \in S^2 : x > y > z > \frac{1}{6}\},$$

$$M_2 = \{(x, y, z) \in S^2 : x > z > y > \frac{1}{6}\},$$

$$M_3 = \{(x, y, z) \in S^2 : y > x > z > \frac{1}{6}\},$$

$$M_4 = \{(x, y, z) \in S^2 : y > z > x > \frac{1}{6}\}.$$

$$M_5 = \{(x, y, z) \in S^2: z > x > y > \frac{1}{6}\},$$

$$M_6 = \{(x, y, z) \in S^2: z > y > x > \frac{1}{6}\}.$$

Proposition 1. The sets M_i , $i = 1, 2, 3, 4, 5, 6$ are invariant with respect to the operator V_0 . Moreover, each median of the simplex S^2 is an invariant.

Trajectories. For any $v^0 = (x^{(0)}, y^{(0)}, z^{(0)}) \in S^2$ its trajectory is defined by

$$v^{(n+1)} = (x^{(n+1)}, y^{(n+1)}, z^{(n+1)}) = V_0^n(x^{(0)}, y^{(0)}, z^{(0)}), n \geq 0.$$

Theorem 2. For the operator V_0 the following hold

- 1) There are invariant curves γ_i , $i = 1, 2, 3$ such that $a_i \in \gamma_i$ and if $v^{(0)} \in \gamma_i$ then $\lim_{n \rightarrow \infty} v^{(n)} = a_i$.
- 2) For any $v^{(0)} \in S^2 \setminus \bigcup_{i=1}^3 \gamma_i$ the following holds

$$\lim_{n \rightarrow \infty} v^{(n)} = \lim_{n \rightarrow \infty} (x^{(n)}, y^{(n)}, z^{(n)}) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

Conclusions. The results have the following biological interpretations:

Let $v = (x, y, z)$ be an initial state (the probability distribution on the set $E = \{1, 2, 3\}$ of genotypes). Theorem 2 says that, almost surely, the state of the system tends to the equilibrium state $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ with the passage of time, i.e. the future of the system is stable: all genotypes 1, 2 and 3 are survived always with equal probability.

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С.С.Худаяров

2D- симплексадаги квадратик ностохастик оператор

Ушбу мақолада икки ўлчовли 2-D симплексни ўзини-ўзига ўтказувчи квадратик стохастик бўлмаган оператор қаралган. Берилган оператор учун қўзғалмас нукталари, инвариант тўпламлари ва динамик система траекториясининг лимит нукталари ўрганилган.

С.С.Худаяров

Квадратический нестохастический оператор на 2D-симплексе

В статье рассматривается квадратичный нестохастический оператор, переводящий двумерный (2D) симплекс в себя. Найдены все неподвижные точки и инвариантные множества оператора. Кроме того, мы изучаем поведение траекторий, порожденных оператором.

S.S.Xudayarov

A quadratic non-stochastic operator on 2D-simplex

In this paper, we consider a quadratic non-stochastic operator mapping the two-dimensional (2D) simplex to itself. We find all fixed points and invariant sets of the operator. Moreover, we study behavior of trajectories generated by the operator.

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