O‘ZBEKISTON RESPUBLIKASI
FANLAR AKADEMIYASINING

MA’RUZALARI

4

2022

ДОКЛАДЫ

АКАДЕМИИ НАУК
РЕСПУБЛИКИ УЗБЕКИСТАН

МАТЕМАТИКА
ТЕХНИЧЕСКИЕ НАУКИ
ЕСТЕСТВОЗНАНИЕ

O‘ZBEKISTON RESPUBLIKASI FANLAR AKADEMIYASI
«FAN» NASHRIYOTI, TOSHKENT, 2022

УДК 517.938

S.S.Xudayarov

A QUADRATIC NON-STOCHASTIC OPERATOR ON 2D-SIMPLEX

(Submitted by Uz AS academician Sh.A.Ayupov)

Introduction. Non-linear dynamical systems arise in many problems of biology, physics and other sciences. In particular, quadratic dynamical systems describe the behavior of populations of different species (see [1]-[5] and the references therein).

In this paper, we consider a quadratic non-stochastic operator mapping the two-dimensional (2D) simplex to itself. We find all fixed points and invariant sets of the operator. Moreover, we study behavior of trajectories generated by the operator.

Let $E = \{1,2,\dots,m\}$. A distribution on the set E is a probability measure $x = (x_1,\dots,x_m)$, i.e., an element of the simplex:

$$S^{m-1} = \left\{x \in \mathbb{R}^m: \quad x_i \geq 0, \quad \sum_{i=1}^{m} x_i = 1\right\}.$$

In general, a quadratic operator V, $V: x \in \mathbb{R}^m \to x' = V(x) \in \mathbb{R}^m$ is defined by:

$$V: x'_k = \sum_{i,j=1}^{m} P_{ij,k} x_i x_j, \quad k = 1,\dots,m. \qquad (1)$$

The following theorem gives conditions for coefficients of V to preserve the simplex.

Theorem 1. [5] *For a quadratic operator V given by (1), to preserve a simplex S^{m-1} it is* sufficient *that*

i)
$$\sum_{k=1}^{m} P_{ij,k} = 1, \quad i,j = 1,\dots,m;$$

ii) $0 \leq P_{ii,k} \leq 1, \quad i,k = 1,\dots,m;$

iii) $-\frac{1}{m-1}\sqrt{P_{ii,k}P_{jj,k}} \leq P_{ij,k} \leq 1 + \sqrt{(1-P_{ii,k})(1-P_{jj,k})}.$

and necessary that the conditions i), ii) and

iii') $-\sqrt{P_{ii,k}P_{jj,k}} \leq P_{ij,k} \leq 1 + \sqrt{(1-P_{ii,k})(1-P_{jj,k})}$

are satisfied.

Definition 1 . [5] *A quadratic operator (1), preserving a simplex, is called non-stochastic (shartly QnSO) if at least one of its coefficients $P_{ij,k}$, $i \neq j$ is negative.*

In this paper, we consider (see Remark 2.2 in [5]) the following example of QnSO on the 2D-simplex S^2:

$$V_0: \begin{cases} x' = \frac{1}{2}(z-y)^2 + \frac{3}{2}x(y+z) \\ y' = \frac{1}{2}(x-z)^2 + \frac{3}{2}y(x+z) \\ z' = \frac{1}{2}(y-x)^2 + \frac{3}{2}z(x+y). \end{cases} \qquad (3)$$

Let s_3 be a permutation group of order 3. We define the action of s_3 on S^2 in the following way: if $g \in s_3$, $x \in S^2$ and $M \subseteq S^2$, then

$$g(x) = (x_{g(1)}, x_{g(2)}, x_{g(3)}),$$

$$g(M) = \{g(x): x \in M\}.$$

The action of s_3 on the operator V_0 is defined as follows:

$$(gV_0)(x) = g(V_0(x)).$$

Fixed points. The fixed points are solutions to the system

$$\begin{cases} x = \frac{1}{2}(z-y)^2 + \frac{3}{2}x(y+z) \\ y = \frac{1}{2}(x-z)^2 + \frac{3}{2}y(x+z) \\ z = \frac{1}{2}(y-x)^2 + \frac{3}{2}z(x+y). \end{cases}$$

Lemma 1. If x is a fixed point of the operator V_0, i.e., $V_0(x) = x$, then, for any $g \in s_3$, then point $g(x)$ is also a fixed point.

It is easy to find the following fixed points of the operator (3) :

$$a_1 = \left(0, \frac{1}{2}, \frac{1}{2}\right); a_2 = \left(\frac{1}{2}, 0, \frac{1}{2}\right), a_3 = \left(\frac{1}{2}, \frac{1}{2}, 0\right), a_4 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

Definition 2. [2] A fixed point x^* of the operator V is called hyperbolic if its Jacobian J at x^* has no eigenvalues on the unit circle.

Definition 3. [2] A hyperbolic fixed point x^* is called:

i) *attracting* if all the eigenvalues of the Jacobian $J(x^*)$ are less than 1 in absolute value;

ii) *repelling* if all the eigenvalues of the Jacobian $J(x^*)$ are greater than 1 in absolute value;

iii) *a saddle* otherwise.

To study the type of each fixed point, rewrite operator (3) (using $x = 1 - y - z$) as

$$W: \begin{cases} x' = \frac{1}{2}\ (1 - x - 2y)^2 + \frac{3}{2}x(1-x) \\ y' = \frac{1}{2}\ (2x - 1 + 2y)^2 + \frac{3}{2}y(1-y) . \end{cases}$$

Note that W maps the set $K = \{(x,y) \in [0,1]^2: 0 \le x + y \le 1\}$ to itself.

For eigenvalues of the Jacobian at fixed points we have

Case a_1: $\lambda_1 = \frac{3}{2},\ \lambda_2 = -\frac{1}{2}$.

Case a_2: $\lambda_1 = -\frac{1}{2},\ \lambda_2 = \frac{3}{2}$.

Case a_3: $\lambda_1 = -\frac{1}{2},\ \lambda_2 = \frac{3}{2}$.

Case a_4: $\lambda_{1,2} = \frac{1}{2}$.

Invariant sets. Lemma 2. For any $g \in s_3$ and $x \in S^2$, the following equality is true:

$$(gV_0)(x) = g(V_0(x)).$$

Lemma 3. If M is an invariant set of the operator V_0, i.e., $V_0(M) \subseteq M$, then, for any $g \in s_3$, the set $g(M)$ is also an invariant set for V_0.

Denine

$$M_1 = \{(x,y,z) \in S^2: \ x > y > z > \frac{1}{6}\},$$

$$M_2 = \{(x,y,z) \in S^2: x > z > y > \frac{1}{6}\},$$

$$M_3 = \{(x,y,z) \in S^2: \ y > x > z > \frac{1}{6}\},$$

$$M_4 = \{(x,y,z) \in S^2: \ y > z > x > \frac{1}{6}\},$$

$$M_5 = \{(x, y, z) \in S^2:\ z > x > y > \frac{1}{6}\},$$

$$M_6 = \{(x, y, z) \in S^2:\ z > y > x > \frac{1}{6}\}.$$

Proposition 1. The sets M_i, $i = 1, 2, 3, 4, 5, 6$ are invariant with respect to the operator V_0. Moreover, each median of the simplex S^2 is an invariant.

Trajectories. For any $v^0 = (x^{(0)}, y^{(0)}, z^{(0)}) \in S^2$ its trajectory is defined by

$$v^{(n+1)} = (x^{(n+1)}, y^{(n+1)}, z^{(n+1)}) = V_0(x^{(n)}, y^{(n)}, z^{(n)}), n \geq 0.$$

Theorem 2. For the operator V_0 the following hold

1) There are invariant curves $\gamma_i, i = 1,2,3$ such that $a_i \in \gamma_i$ and if $v^{(0)} \in \gamma_i$ then $\lim_{n\to\infty} v^{(n)} = a_i$.

2) For any $v^{(0)} \in S^2 \setminus \cup_{i=1}^3 \gamma_i$ the following holds

$$\lim_{n\to\infty} v^{(n)} = \lim_{n\to\infty} (x^{(n)}, y^{(n)}, z^{(n)}) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

Conclusions. The results have the following biological interpretations:

Let $v = (x, y, z)$ be an initial state (the probability distribution on the set $E = \{1,2,3\}$ of genotypes). Theorem 2 says that, almost surely, the state of the system tends to the equilibrium state $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ with the passage of time, i.e. the future of the system is stable: all genotypes 1, 2 and 3 are survived always with equal probability.

REFERENCES

1. J.M.Casas, M.Ladra, U.A.Rozikov, Markov processes of cubic stochastic matrices: quadratic stochastic processes. Linear Algebra Appl. 2019, Vol. 575, pp.273-298.

2. B.J.Mamurov, U.A.Rozikov, S.S.Xudayarov, Quadratic stochastic processes of type $(\sigma|\mu)$. Markov Processes Related Fields. 2020, Vol. 26, No. 5, pp. 915-933.

3. Yu.I.Lyubich, Mathematical structures in population genetics, Springer-Verlag, 1992.

4. U.A.Rozikov, Population dynamics: algebraic and probabilistic approach. World Sci. Publ. Singapore. 2020, 460 pp.

5. U.A.Rozikov, S.S.Xudayarov, Quadratic non-stochastic operators: examples of splitted chaos. Ann. Funct. Anal. 2022, Vol.13, No. 1, 17 p.

С.С.Худаяров

2D- симплексдаги квадратик ностохастик оператор

Ушбу мақолада икки ўлчовли 2-D симплексни ўзини-ўзига ўтказувчи квадратик стокастик бўлмаган оператор қаралган. Берилган оператор учун қўзғалмас нуқталари, инвариант тўпламлари ва динамик система траекториясининг лимит нуқталари ўрганилган.

С.С.Худаяров

Квадратический нестохастический оператор на 2D-симплекс

В статье рассматривается квадратичный нестохастический оператор, переводящий двумерный (2D) симплекс в себя. Найдены все неподвижные точки и инвариантные множества оператора. Кроме того, мы изучаем поведение траекторий, порожденных оператором.

S.S.Xudayarov

A quadratic non-stochastic operator on 2D-simplex

In this paper, we consider a quadratic non-stochastic operator mapping the two-dimensional (2D) simplex to itself. We find all fixed points and invariant sets of the operator. Moreover, we study behavior of trajectories generated by the operator.

Institute of mathematics named after V.I.Romanovskiy
Uzbekistan Academy of Sciences

Received 13.06.2022

CONTENTS