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New branches of the essential spectrum of a 2×2 operator matrix

Dilmurodov E.B.

Abstract. We consider 2×2 operator matrix $\mathcal{A}_\mu$, $\mu > 0$ acting in the direct sum of the one- and two-particle subspaces of the bosonic Fock space. We describe the essential spectrum of spectrum of $\mathcal{A}_\mu$ via the spectrum of a family of generalized Friedrichs models and define its new branches. For the special cases the location of these branches are investigated.

Keywords: operator matrix, bosonic Fock space, annihilation and creation operators, the Faddeev equation, essential spectrum.

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1 Introduction

The well-known methods for the investigation of the location of essential spectra of Schrödinger operators are Weyl criterion for the one particle problem and the HWZ theorem for multiparticle problems, the modern proof of which is based on the Ruelle-Simon partition of unity. The theorem on the location of the branches of essential spectrum of multi-particle Hamiltonians was named the HWZ theorem in [3, 14] to the honor of Hunziker [6], van Winter [15] and Zhislin [16]. A lattice analogue of this theorem for the four-particle Schrödinger operator is proved in [1, 8].

The systems considered above have a fixed number of quasi-particles. In statistical physics [9], solid-state physics [10] and the theory of quantum fields [5], one considers systems, where the number of quasi-particles is bounded, but not fixed. Recall that the study of systems describing N particles in interaction, without conservation of the number of particles is reduced to the investigation of the spectral properties of self-adjoint operators, acting in the *cut subspace* $\mathcal{H}^{(N)}$ of Fock space, consisting of $n \leq N$ particles [9, 10, 5]. The modified version of HWZ theorem for the 4×4 operator matrix is proved in [11]. The bounds of a 2×2 block-operator matrix are estimated in [4].

In the present paper we consider 2×2 operator matrix $\mathcal{A}_\mu$, related with the Hamiltonian of a system describing three particles in interaction, without conservation of the number of particles. For the study of location of the essential spectrum of $\mathcal{A}_\mu$ and define its new branches we introduce the family of 2×2 operator matrices. We discuss the cases where the lower and upper bounds of the two- and three-particle branches of $\mathcal{A}_\mu$ are coincide. We remark that these results are important in the studying the finiteness or infiniteness of the number of discrete eigenvalues of $\mathcal{A}_\mu$, see [2].

2 2×2 operator matrices

Let $\mathbb{T}^3$ be the three-dimensional torus, the cube $(-\pi,\pi]^3$ with appropriately identified sides equipped with its Haar measure. Let $L_2(\mathbb{T}^3)$ be the Hilbert space of square integrable (complex) functions defined on $\mathbb{T}^3$ and $L_2^{\mathrm{s}}((\mathbb{T}^3)^2)$ be the Hilbert space of square integrable (complex) symmetric functions defined on $(\mathbb{T}^3)^2$. Denote by $\mathcal{H}$ the direct sum of spaces $\mathcal{H}_1 := L_2(\mathbb{T}^3)$ and $\mathcal{H}_2 := L_2^{\mathrm{s}}((\mathbb{T}^3)^2)$, that is, $\mathcal{H} := \mathcal{H}_1 \oplus \mathcal{H}_2$. The spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are called one- and two-particle subspaces of a bosonic Fock space $\mathcal{F}_{\mathrm{s}}(L_2(\mathbb{T}^3))$ over $L_2(\mathbb{T}^3)$, respectively, where

$$\mathcal{F}_{\mathrm{s}}(L_2(\mathbb{T}^3)) := \mathbb{C} \oplus L_2(\mathbb{T}^3) \oplus L_2^{\mathrm{s}}((\mathbb{T}^3)^2) \oplus \ldots \oplus L_2^{\mathrm{s}}((\mathbb{T}^3)^n) \oplus \ldots .$$

Let us consider the following 2×2 operator matrix $\mathcal{A}_\mu$ acting in the Hilbert space $\mathcal{H}$ as

$$\mathcal{A}_\mu := \begin{pmatrix} A_{11} & \mu A_{12} \\ \mu A_{12}^* & A_{22} \end{pmatrix}$$

with the entries

$$(A_{11}f_1)(p) = w_1(p)f_1(p), \quad (A_{12}f_2)(p) = \int_{\mathbb{T}^3} f_2(p,t)dt,$$
$$(A_{22}f_2)(p,q) = w_2(p,q)f_2(p,q), \quad f_i \in \mathcal{H}_i, \quad i=1,2.$$

Here $\mu > 0$ is a coupling constant, the functions $w_1(\cdot)$ and $w_2(\cdot,\cdot)$ have the form

$$w_1(k) := \varepsilon(k) + \gamma, \quad w_2(k,p) := \varepsilon(k) + \varepsilon(\frac{1}{2}(k+p)) + \varepsilon(p)$$

with $\gamma \in \mathbb{R}$ and the dispersion function $\varepsilon(\cdot)$ is defined by

$$\varepsilon(k) := \sum_{i=1}^{3}(1-\cos k_i),\ k=(k_1,k_2,k_3) \in \mathbb{T}^3. \tag{2.1}$$

We denote by A_{12}^* the adjoint operator to A_{12}. Then

$$(A_{12}^* f_1)(p,q) = \frac{1}{2}(f_1(q) + f_1(p)), \quad f_1 \in \mathcal{H}_1.$$

Under these assumptions the operator $\mathcal{A}_\mu$ is bounded and self-adjoint in the Hilbert space $\mathcal{H}$.

We remark that the operators A_{12} and A_{12}^* are called annihilation and creation operators [5], respectively. In physics, an annihilation operator is an operator that lowers the number of particles in a given state by one, a creation operator is an operator that increases the number of particles in a given state by one, and it is the adjoint of the annihilation operator.

Main aim of the present paper is

(i) to investigate the spectrum of the corresponding family of generalized Friedrichs model;

(ii) to describe the essential spectrum of $\mathcal{A}_\mu$ and to define its new branches;

(iii) to study the cases where the essential spectrum of $\mathcal{A}_\mu$ consists one, two or three bounded closed intervals.

The next sections are devoted to the discussion of these problems.

Throughout the paper the spectrum, the essential spectrum, and the discrete spectrum of a bounded self-adjoint operator will be denote by $\sigma(\cdot)$, $\sigma_{\rm ess}(\cdot)$ and $\sigma_{\rm disc}(\cdot)$, respectively.

3 Family of generalized Friedrichs models and its spectrum

To study the spectral properties of the operator $\mathcal{A}_\mu$ we introduce a family of bounded self-adjoint operators (generalized Friedrichs models) $\mathcal{A}_\mu(k)$, $k \in \mathbb{T}^3$, which acts in $\mathcal{H}_0 \oplus \mathcal{H}_1$ ($\mathcal{H}_0 := \mathbb{C}$) as 2×2 operator matrices

$$\mathcal{A}_\mu(k) := \begin{pmatrix} A_{00}(k) & \frac{\mu}{\sqrt{2}} A_{01} \\ \frac{\mu}{\sqrt{2}} A_{01}^* & A_{11}(k) \end{pmatrix},$$

with matrix elements

$$A_{00}(k) f_0 = w_1(k) f_0, \ \ A_{01} f_1 = (f_1, 1),$$
$$(A_{11}(k) f_2)(p) = w_2(k,p) f_1(p), f_i \in \mathcal{H}_i, i = 0, 1.$$

Let the operator $\mathcal{A}_0(k)$, $k \in \mathbb{T}^3$ acts in $\mathcal{H}_0 \oplus \mathcal{H}_1$ as

$$\mathcal{A}_0(k) := \begin{pmatrix} 0 & 0 \\ 0 & A_{11}(k) \end{pmatrix}.$$

The perturbation $\mathcal{A}_\mu(k) - \mathcal{A}_0(k)$ of the operator $\mathcal{A}_0(k)$ is a self-adjoint operator of rank 2. Therefore in accordance with the invariance of the essential spectrum under the finite rank perturbations the essential spectrum $\sigma_{\rm ess}(\mathcal{A}_\mu(k))$ of $\mathcal{A}_\mu(k)$ fills the following interval on the real axis

$$\sigma_{\rm ess}(\mathcal{A}_\mu(k)) = [m(k), M(k)],$$

where the numbers $m(k)$ and $M(k)$ are defined by

$$m(k) := \min_{p \in \mathbb{T}^3} w_2(k,p), \quad M(k) := \max_{p \in \mathbb{T}^3} w_2(k,p). \tag{3.1}$$

For any $k \in \mathbb{T}^3$ we define an analytic function $I(k\,;\cdot)$ in $\mathbb{C} \setminus \sigma_{\mathrm{ess}}(\mathcal{A}_\mu(k))$ by

$$I(k\,;z) := \int_{\mathbb{T}^3} \frac{dt}{w_2(k,t)-z}.$$

The Fredholm determinant associated to the operator $\mathcal{A}_\mu(k)$ is defined by

$$\Delta_\mu(k\,;z) := w_1(k) - z - \frac{\mu^2}{2} I(k\,;z), \;\; z \in \mathbb{C} \setminus \sigma_{\mathrm{ess}}(\mathcal{A}_\mu(k)).$$

The following statement establishes connection between the eigenvalues of the operator $\mathcal{A}_\mu(k)$ and zeros of the function $\Delta_\mu(k\,;\cdot)$ [7, 12].

Lemma 3.1. *For any $\mu > 0$ and $k \in \mathbb{T}^3$ the operator $\mathcal{A}_\mu(k)$ has an eigenvalue $z_\mu(k) \in \mathbb{C} \setminus \sigma_{\mathrm{ess}}(\mathcal{A}_\mu(k))$ if and only if $\Delta_\mu(k\,;z_\mu(k)) = 0$.*

From Lemma 3.1 it follows that

$$\sigma_{\mathrm{disc}}(\mathcal{A}_\mu(k)) = \{z \in \mathbb{C} \setminus \sigma_{\mathrm{ess}}(\mathcal{A}_\mu(k)) :\ \Delta_\mu(k\,;z) = 0\}.$$

4 New branches of the essential spectrum

In this section we investigate the essential spectrum of $\mathcal{A}_\mu$.

It is easy to show that the function $w_2(\cdot,\cdot)$ has a unique non-degenerate minimum (resp. maximum) at the point $(\overline{0},\overline{0}) \in (\mathbb{T}^3)^2$ (resp. $(\overline{\pi},\overline{\pi}) \in (\mathbb{T}^3)^2$) and

$$\min_{k,p\in\mathbb{T}^3} w_2(k,p) = w_2(\overline{0},\overline{0}) = 0, \qquad \max_{k,p\in\mathbb{T}^3} w_2(k,p) = w_2(\overline{\pi},\overline{\pi}) = 18,$$

where $\overline{0} := (0,0,0)$, $\overline{\pi} := (\pi,\pi,\pi) \in \mathbb{T}^3$. Note that the function $w_1(\cdot)$ has also a unique non-degenerate minimum (resp. maximum) at the point $\overline{0} \in \mathbb{T}^3$ (resp. $\overline{\pi} \in \mathbb{T}^3$).

The following lemma describes the essential spectrum of $\mathcal{A}_\mu$.

Lemma 4.1. *For the essential spectrum of the $\mathcal{A}_\mu$ the following equality holds*

$$\sigma_{\mathrm{ess}}(\mathcal{A}_\mu) := \sigma_\mu \cup [0;18], \quad \sigma_\mu := \bigcup_{k\in\mathbb{T}^3} \sigma_{\mathrm{disc}}(\mathcal{A}_\mu(k)).$$

Lemma 4.1 can be proved in much same way as Theorem 1 in [7] and Theorem 6.1 in [12].

In the following we introduce the new subsets of $\sigma_{\mathrm{ess}}(\mathcal{A}_\mu)$.

Definition 4.2. The sets σ_μ and $[0;18]$ are called two- and three- particle branches of the essential spectrum of $\mathcal{A}_\mu$, respectively.

Using the extremal properties of the function $w_2(\cdot,\cdot)$, and the Lebesgue dominated convergence theorem one can show that the integral $I(\overline{0};0)$ is finite, see [13].

For the next investigations we introduce the following quantities

$$\mu_l^0(\gamma) := \sqrt{2\gamma}\,\big(I(\overline{0},0)\big)^{-1/2} \text{ for } \gamma > 0;$$
$$\mu_r^0(\gamma) := \sqrt{24-2\gamma}\,\big(I(\overline{0},0)\big)^{-1/2} \text{ for } \gamma < 12.$$

Since $\mathbb{T}^3$ is compact, and the functions $\Delta_\mu(\cdot;0)$ and $\Delta_\mu(\cdot;18)$ are continuous on $\mathbb{T}^3$, there exist points $k_0, k_1 \in \mathbb{T}^3$ such that the equalities

$$\max_{k\in\mathbb{T}^3} \Delta_\mu(k;0) = \Delta_\mu(k_0;0), \quad \min_{k\in\mathbb{T}^3} \Delta_\mu(k;18) = \Delta_\mu(k_1;18)$$

hold.

Let us introduce the following notations:

$$\gamma_0 := \left(12\frac{I(k_0;0)}{I(\overline{0};0)} - \varepsilon(k_0)\right)\left(1+\frac{I(k_0;0)}{I(\overline{0};0)}\right)^{-1};$$
$$\gamma_1 := (18-\varepsilon(k_1))\left(1-\frac{I(k_1;18)}{I(\overline{0};0)}\right).$$

Denote

$$E_\mu^{(1)} := \min\{\sigma_\mu \cap (-\infty;0]\}\,; E_\mu^{(2)} := \max\{\sigma_\mu \cap (-\infty;0]\}\,;$$
$$E_\mu^{(3)} := \min\{\sigma_\mu \cap [18;\infty)\}\,; E_\mu^{(4)} := \max\{\sigma_\mu \cap [18;\infty)\}\,.$$

We formulate the first main result of the paper. It is precisely describe the structure of the essential spectrum of $\mathcal{A}_\mu$. The structure of the essential spectrum depends on the location of the parameters $\mu > 0$ and $\gamma \in \mathbb{R}$.

Theorem 4.3. *Let $\mu = \mu_r^0(\gamma)$, with $\gamma < 12$. The following equality holds*

$$\sigma_{\text{ess}}(\mathcal{A}_\mu) = \begin{cases} [E_\mu^{(1)}; E_\mu^{(2)}]\bigcup[0;18], & \textit{if} \quad \gamma < \gamma_0; \\ [E_\mu^{(1)}; 18], & \textit{if} \quad \gamma_0 \le \gamma < 6; \\ [0;18], & \textit{if} \quad 6 \le \gamma < 12. \end{cases}$$

Proof. Suppose that $\mu = \mu_r^0(\gamma)$, with $\gamma < 12$. It is easy to see that $\lim\limits_{z\to+\infty} \Delta_\mu(k;z) = -\infty$ and $\Delta_\mu(k;18) \le 0$, for any $k \in \mathbb{T}^3$. Then Lemma 3.1 implies that for any $k \in \mathbb{T}^3$ the operator $\mathcal{A}_\mu(k)$ has no eigenvalues, bigger than 18.

We recall the following properties of $\Delta_\mu(k;z)$: for any $k \in \mathbb{T}^3$, the function $\Delta_\mu(k;\cdot)$ is strictly decreasing in the interval $(-\infty;0)$ and the equality $\lim\limits_{z\to-\infty} \Delta_\mu(k;z) = +\infty$ holds.

First we consider the case $\gamma < \gamma_0$. Similar discussions shows that under this assumption $\Delta_\mu(k;0) < 0$, for any $k \in \mathbb{T}^3$.

Then using the continuity of the function $\Delta_\mu(k;\cdot)$ in the interval $(-\infty;0)$, we easily obtain than, for any $k \in \mathbb{T}^3$ there exists a point $z_\mu(k) \in (-\infty;0)$ such that $\Delta_\mu(k;z_\mu(k)) = 0$. Then by Lemma 3.1 the number $z_\mu(k)$ is an eigenvalue of operator $\mathcal{A}_\mu(k)$. So, we define the mapping $z_\mu : k \in \mathbb{T}^3 \to z_\mu(k)$. From the analyticity of the function $\varepsilon(\cdot)$ we have that the function $z_\mu(\cdot)$ is continuous on the compact set $\mathbb{T}^3$, and its range is a closed subset of $(-\infty;0)$, i.e., $\mathrm{I}mz_\mu(k) = [E_\mu^{(1)};E_\mu^{(2)}]$, with $E_\mu^{(2)} < 0$.

Summarizing above mentioned facts by Lemma 4.1 we conclude that the equality $\sigma_{\mathrm{ess}}(\mathcal{A}_\mu) = [E_\mu^{(1)};E_\mu^{(2)}] \cup [0;18]$ holds.

We proceed the proof considering the case $\gamma_0 \leq \gamma < 6$. Simple calculations show that $\Delta_\mu(\overline{0};0) < 0$ and $\Delta_\mu(k_0;0) > 0$.

Introduce the notation $G_\mu :\equiv \{k \in \mathbb{T}^3 : \Delta_\mu(k;0) < 0\}$. By the construction the set $G_\mu \subset \mathbb{T}^3$ is an open. Since $\overline{0} \in G_\mu$, it is a non-empty. From the fact $\Delta_\mu(k_0;0) > 0$, that is, $k_0 \notin G_\mu$ we obtain that $G_\mu \neq \mathbb{T}^3$.

By Lemma 3.1 for any $k \in G_\mu$ operator $\mathcal{A}_\mu(k)$ has an unique eigenvalue $z_\mu(k) < 0$. Since the function $\varepsilon(\cdot)$ is an analytic on its domainone can see that $z_\mu : k \in G_\mu \to z_\mu(k)$ is a continuous mapping on G_μ.

Since for any $k \in \mathbb{T}^3$ the operator $\mathcal{A}_\mu(k)$ is bounded and $\mathbb{T}^3$ is compact, there exists a positive number C such that $\sup\limits_{k\in\mathbb{T}^3} ||\mathcal{A}_\mu(k)|| \leq C$, and for any $k \in \mathbb{T}^3$ we have

$$\sigma(\mathcal{A}_\mu(k)) \subset [-C;C]. \tag{4.1}$$

For any $p \in \partial G_\mu = \{k \in \mathbb{T}^3 : \Delta_\mu(k;0) = 0\}$, there exist a sequence $\{k_n\} \subset G_\mu$ such that $k_n \to p$ as $n \to \infty$. Set $z_\mu^{(n)} := z_\mu(k_n)$. Then $z_\mu(k_n) < 0$ for any $k_n \in G_\mu$ and from (4.1), we see that $\{z_\mu^{(n)}\} \subset [-C;0]$. Then there exist a subsequence $\{z_\mu^{(n_m)}\} \subset \{z_\mu^{(n)}\}$ such that $z_\mu^{(n_m)} \to z_\mu^{(0)}$ $(z_\mu^{(n_m)} = z_\mu(k_{n_m}), k_{n_m} \in \{k_n\})$ as $m \to \infty$ for some $z_\mu^{(0)} \in [-C;0]$.

The continuity of the function $\Delta_\mu(\cdot;\cdot)$ in $\mathbb{T}^3 \times (-\infty;0]$ and the relations $k_{n_m} \to p$ and $z_\mu^{(n_m)} \to z_\mu^{(0)}$ as $m \to \infty$ imply

$$0 = \lim_{m\to\infty} \Delta_\mu(k_{n_m};z_\mu^{(n_m)}) = \Delta_\mu(p;z_\mu^{(0)}).$$

Since for any $k \in \mathbb{T}^3$ the function $\Delta_\mu(k;\cdot)$ is monotonically decreasing in $(-\infty;0]$ and $p \in \partial G_\mu$, it follows that $\Delta_\mu(p;z_\mu^{(0)}) = 0$ if and only if $z_\mu^{(0)} = 0$.

For any $p \in \partial G_\mu$, we set

$$z_\mu(p) := \lim_{k\to p, p\in\partial G_\mu} z_\mu(k) = 0.$$

The function $z_\mu(\cdot)$ is a continuous on the compact set $G_\mu \cup \partial G_\mu$ and $z_\mu(p) = 0$ for any $p \in \partial G_\mu$; moreover, we have

$$\mathrm{Im} z_\mu(\cdot) = [E_\mu^{(1)}; 0], \quad E_\mu^{(1)} < 0.$$

Hence the set $\{\lambda \in \sigma_\mu : \lambda \leq 0\}$ coincides with the set $[E_\mu^{(1)}; 0]$.

Then by Lemma 4.1 we get $\sigma_{\mathrm{ess}}(\mathcal{A}_\mu) := [E_\mu^{(1)}; 18]$.

Let $6 \leq \gamma < 12$. In this case we can see that for any $k \in \mathbb{T}^3$ the following relations $\Delta_\mu(k; 0) \geq 0$ and $\Delta_\mu(k; 18) \leq 0$ hold. Then by Lemma 3.1 the operator $\mathcal{A}_\mu(k)$ has no eigenvalues in $(-\infty; 0) \bigcup (18; +\infty)$, i.e. $\sigma_\mu \backslash [0; 18] = \varnothing$. So, by Lemma 4.1 we obtain $\sigma_{\mathrm{ess}}(\mathcal{A}_\mu) = [0; 18]$. □

The following Theorem is a second main result of the paper and it can be proved in much the same way as Theorem 4.3.

Theorem 4.4. *Let $\mu = \mu_l^0(\gamma)$, with $\gamma > 0$.*
The following equality holds

$$\sigma_{\mathrm{ess}}(\mathcal{A}_\mu) = \begin{cases} [0; 18], & \text{if} \quad 0 < \gamma \leq 6; \\ [0; E_\mu^{(4)}], & \text{if} \quad 6 < \gamma \leq \gamma_1; \\ [0; 18] \bigcup [E_\mu^{(3)}; E_\mu^{(4)}], & \text{if} \quad \gamma > \gamma_1. \end{cases}$$

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Dilmurodov E.B.
Bukhara State University, M.Ikbol 11, Bukhara 705018, Uzbekistan.
e-mail: eloy.dilmurodov@mail.ru