



**ALI QUSHCHINING 620 YILLIK TAVALLUDIGA  
BAG‘ISHLANGAN “ALI QUSHCHI – MIRZO ULUG‘BEK  
ILMIY MAKTABINING BUYUK ELCHISI”  
MAVZUSIDAGI XALQARO ILMIY ANJUMAN  
MATERIALLARI**

2023-yil 21-22 sentyabr  
Samarqand, O‘zbekiston



**“ALI QUSHJI – AN OUTSTANDING AMBASSADOR OF  
THE SCIENTIFIC SCHOOL OF ULUGH BEG”** CELEBRATING THE  
620<sup>TH</sup> ANNIVERSARY OF ALI QUSHJI’S BIRTH  
INTERNATIONAL CONFERENCE

September 21-22, 2023  
Samarkand, Uzbekistan

**Ali Qushchi tavalludining 620 yilligi hamda O‘zbekiston respublikasi fanlar akademiyasining 80 yilligiga bag‘ishlangan “Ali Qushchi – Mirzo Ulug‘bek ilmiy maktabining buyuk elchisi” mavzusidagi xalqaro ilmiy anjuman materiallari (Samarqand, 2023-yil 21-22 sentyabr). – Samarqand: SamDU nashriyoti, 2023.– 498 b.**

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# About a regular point of the lattice spin-boson model with at most two photons

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The spin-boson model is a well-known quantum-mechanical model which describes the interaction between a two-level atom and a photon field. We refer to [1] and [2] for excellent reviews respectively from physical and mathematical perspectives.

Let  $T^d$  be the  $d$ -dimensional torus,  $L_2((T^d))$  be the Hilbert space of square integrable (complex) functions defined on  $T^d$ ,  $\mathbb{C}^2$  be the state of the two-level atom and  $\mathcal{F}_b(L_2(T^d))$  be the symmetric Fock space for bosons, that is,

$$\mathcal{F}_b(L_2(T^d)) := \mathbb{C} \oplus L_2(T^d) \oplus L_2^{\text{sym}}((T^d)^2) \oplus \dots$$

Here  $L_2^{\text{sym}}((T^d)^n)$  is the Hilbert space of symmetric functions of  $n \geq 2$  variables. For  $m = 1, 2$  we denote  $\mathcal{L}_m := \mathbb{C}^2 \otimes \mathcal{F}_b^{(m)}(L_2(T^d))$ , where

$$\mathcal{F}_b^{(1)}(L_2(T^d)) := \mathbb{C} \oplus L_2(T^d), \quad \mathcal{F}_b^{(2)}(L_2(T^d)) := \mathbb{C} \oplus L_2(T^d) \oplus L_2^{\text{sym}}((T^d)^2).$$

We write elements  $F$  of the space  $\mathcal{L}_2$  in the form  $F = \{f_0^{(s)}, f_1^{(s)}(k_1), f_2^{(s)}(k_1, k_2); s = \pm\}$ . Then the norm in  $\mathcal{L}_2$  is given by

$$\|F\|^2 := \sum_{s=\pm} \left( |f_0^{(s)}|^2 + \int_{T^d} |f_1^{(s)}(k_1)|^2 dk_1 + \frac{1}{2} \int_{(T^d)^2} |f_2^{(s)}(k_1, k_2)|^2 dk_1 dk_2 \right). \quad (1)$$

We recall that the lattice spin-boson model with at most two photons  $\mathcal{A}_2$  is acting in  $\mathcal{L}_2$  as the  $3 \times 3$  tridiagonal block operator matrix

$$\mathcal{A}_2 := \begin{pmatrix} A_{00} & A_{01} & 0 \\ A_{01}^* & A_{11} & A_{12} \\ 0 & A_{12}^* & A_{22} \end{pmatrix},$$

where matrix elements  $A_{ij}$  are defined by

$$A_{00}f_0^{(s)} = s\varepsilon f_0^{(s)}, \quad A_{01}f_1^{(s)} = \alpha \int_{T^d} v(t)f_1^{(-s)}(t)dt,$$

$$(A_{11}f_1^{(s)})(k_1) = (s\varepsilon + w(k_1))f_1^{(s)}(k_1), \quad (A_{12}f_2^{(s)})(k_1) = \alpha \int_{T^d} v(t)f_2^{(-s)}(k_1, t)dt,$$

$$(A_{22}f_2^{(s)})(k_1, k_2) = (s\varepsilon + w(k_1) + w(k_2))f_2^{(s)}(k_1, k_2),$$

$$f = \{f_0^{(s)}, f_1^{(s)}, f_2^{(s)}; s = \pm\} \in \mathcal{L}_2.$$

Here  $A_{ij}^*$  denotes the adjoint operator to  $A_{ij}$  for  $i < j$  with  $i, j = 0, 1, 2$ ;  $w(k)$  is the dispersion of the free field,  $\alpha v(k)$  is the coupling between the atoms and the field modes,  $\alpha > 0$  is a real number, so-called the coupling constant, real number. We assume that  $v(\cdot)$  and  $w(\cdot)$  are the real-valued continuous functions on  $T^d$ . Under these assumptions the lattice spin-boson model with at most two photons  $\mathcal{A}_2$  is bounded and self-adjoint in the complex Hilbert space  $\mathcal{L}_2$ .

To study the spectral properties of  $\mathcal{A}_2$  we introduce the following two bounded self-adjoint operators  $\mathcal{A}_2^{(s)}$ ,  $s = \pm$ , which acts in  $\mathcal{F}_b^{(2)}(L_2(T^d))$  as

$$\mathcal{A}_2^{(s)} := \begin{pmatrix} \hat{A}_{00}^{(s)} & \hat{A}_{01} & 0 \\ \hat{A}_{01}^* & \hat{A}_{11}^{(s)} & \hat{A}_{12} \\ 0 & \hat{A}_{12}^* & \hat{A}_{22}^{(s)} \end{pmatrix}$$

with the entries

$$\hat{A}_{00}^{(s)}f_0 = s\varepsilon f_0, \quad \hat{A}_{01}f_1 = \alpha \int_{T^d} v(t)f_1(t)dt,$$

$$(\hat{A}_{11}^{(s)}f_1)(k_1) = (-s\varepsilon + w(k_1))f_1(k_1), \quad (\hat{A}_{12}f_2)(k_1) = \alpha \int_{T^d} v(t)f_2(k_1, t)dt,$$

$$(\hat{A}_{22}^{(s)}f_2)(k_1, k_2) = (s\varepsilon + w(k_1) + w(k_2))f_2(k_1, k_2), \quad (f_0, f_1, f_2) \in \mathcal{F}_b^{(2)}(L_2(T^d)).$$

It is easy to check that

$$(\hat{A}_{01}^*f_0)(k_1) = \alpha v(k_1)f_0;$$

$$(\hat{A}_{12}^*f_1)(k_1, k_2) = \alpha(v(k_1)f_1(k_2) + v(k_2)f_1(k_1)), \quad (f_0, f_1) \in \mathcal{F}_b^{(1)}(L_2(T^d)).$$

In order to describe the essential spectrum of  $\mathcal{A}_2$  we define an analytic function  $\Delta^{(s)}(\cdot)$  in  $\mathbb{C} \setminus [s\varepsilon + m; s\varepsilon + M]$  by

$$\Delta^{(s)}(\lambda) := -s\varepsilon - \lambda - \alpha^2 \int_{T^d} \frac{v^2(t)dt}{s\varepsilon + w(t) - \lambda},$$

where the numbers  $m$  and  $M$  are defined by

$$m := \min_{p \in T^d} w(p), \quad M := \max_{p \in T^d} w(p).$$

Let  $\sigma^{(s)}$  be the set of all complex numbers  $\lambda \in \mathbb{C}$  such that the equality  $\Delta^{(s)}(\lambda - w(k_1)) = 0$  holds for some  $k_1 \in T^d$ . Then for the essential spectrum of  $\mathcal{A}_2^{(s)}$  we have

$$\sigma_{\text{ess}}(\mathcal{A}_2^{(s)}) = \sigma^{(s)} \cup [s\varepsilon + 2m; s\varepsilon + 2M].$$

We set  $E_{\min}^{(s)} := \min \sigma_{\text{ess}}(\mathcal{A}_2^{(s)})$ , for  $s = \pm$  and  $E_{\min} := \min\{E_{\min}^{(+)}, E_{\min}^{(-)}\}$ .

Next, we represent the space  $\mathcal{F}_b^{(2)}(L_2(T^d))$  as a direct sum of two Hilbert spaces  $\mathcal{F}_b^{(1)}(L_2(T^d))$  and  $L_2^{\text{sym}}((T^d)^2)$ , that is,  $\mathcal{F}_b^{(2)}(L_2(T^d)) = \mathcal{F}_b^{(1)}(L_2(T^d)) \oplus L_2^{\text{sym}}((T^d)^2)$ . Then the first Schur complement of the operator  $\mathcal{A}_2^{(s)}$  with respect to this decomposition is defined as

$$S_1^{(s)}(\lambda): \mathcal{F}_b^{(1)}(L_2(T^d)) \rightarrow \mathcal{F}_b^{(1)}(L_2(T^d)), \quad \lambda \in \rho(\hat{A}_{22}^{(s)});$$

$$S_1^{(s)}(\lambda) := \begin{pmatrix} \hat{A}_{00}^{(s)} & \hat{A}_{01} \\ \hat{A}_{01}^* & \hat{A}_{11}^{(s)} \end{pmatrix} - \lambda - \begin{pmatrix} 0 \\ \hat{A}_{12} \end{pmatrix} (\hat{A}_{22}^{(s)} - \lambda)^{-1} \begin{pmatrix} 0 & \hat{A}_{12}^* \end{pmatrix}.$$

Define

$$S_{00}^{(s)}(\lambda) := \hat{A}_{00}^{(s)} - \lambda, \quad S_{01}^{(s)}(\lambda) := \hat{A}_{01};$$

$$S_{10}^{(s)}(\lambda) := \hat{A}_{01}^*, \quad S_{11}^{(s)}(\lambda) := \hat{A}_{11}^{(s)} - \lambda - \hat{A}_{12}(\hat{A}_{22}^{(s)} - \lambda)^{-1}\hat{A}_{12}^*.$$

Then the operator  $S_1^{(s)}(\lambda)$  has form

$$S_1^{(s)}(\lambda) = \begin{pmatrix} S_{00}^{(s)}(\lambda) & S_{01}^{(s)}(\lambda) \\ S_{10}^{(s)}(\lambda) & S_{11}^{(s)}(\lambda) \end{pmatrix}.$$

We study some important properties of the first Schur complement

$$S_1(\lambda) := \text{diag}\{S_1^{(+)}(\lambda), S_1^{(-)}(\lambda)\}$$

for the lattice spin-boson model with at most two photons  $\mathcal{A}_2$ .

**Proposition 1.** *The number  $\lambda \in \mathbb{C} \setminus \sigma_{ess}(\mathcal{A}_2)$  is an eigenvalue of the operator  $\mathcal{A}_2$  if and only if the operator  $S_1(\lambda)$  has an eigenvalue equal to zero. Moreover, the eigenvalues  $\lambda$  and 0 have the same multiplicities.*

**Proposition 2.**  *$\lambda \in \sigma_{ess}(\mathcal{A}_2) \setminus \sigma(A_{22})$  if and only if  $0 \in \sigma_{ess}(S_1(\lambda))$ .*

From Propositions 1 and 2 we obtain the following two corollaries.

**Corollary 1.** *Let  $\lambda \in \mathbb{C} \setminus \sigma_{ess}(\mathcal{A}_2)$ . Then  $\lambda \in \rho(\mathcal{A}_2) \Leftrightarrow 0 \in \rho(S_1(\lambda))$ .*

**Corollary 2.** *Let  $\lambda_0 \in \mathbb{R} \setminus \sigma_{ess}(\mathcal{A}_2)$ . If  $(\lambda_0; \lambda_0 + \gamma) \subset \rho(\mathcal{A}_2)$  (resp.  $(\lambda_0 - \gamma; \lambda_0) \subset \rho(\mathcal{A}_2)$ ) for some  $\gamma > 0$ , then there exists a number  $\delta = \delta(\gamma) > 0$  such that  $(0; \delta) \subset \rho(S_1(\lambda_0))$  (resp.  $(-\delta; 0) \subset \rho(S_1(\lambda_0))$ ).*

For a bounded self-adjoint operator  $\mathcal{A}$  acting in a Hilbert space  $H$  we denote by  $N_{(-\infty; \lambda]}(\mathcal{A})$  the number of eigenvalues of  $\mathcal{A}$  to the left of  $\lambda$ ,  $\lambda \leq \min \sigma_{ess}(\mathcal{A})$ .

**Proposition 3.** *The function  $N_{(-\infty; 0)}(S_1(\cdot))$  is monotonically increasing in  $(-\infty; 0)$ .*

**Definition 1.** *We denote by  $E_m(\cdot)$ ,  $m \in \mathbb{N}$  the positive definite function on the segment  $[\alpha; \beta] \subset \mathbb{R} \setminus \sigma(\mathcal{A}_2)$ , satisfying the condition:  $E_m(\lambda)$  is the  $m$ -th eigenvalue (eigenvalues numbered in ascending order, counting their multiplicity) of the operator  $S_1(\lambda)$ ,  $\lambda \in [\alpha; \beta]$ .*

**Theorem.** *The number  $\lambda_0 < E_{\min}$  is the regular point of the operator  $\mathcal{A}_2$  if and only if the function  $N_{(-\infty; 0)}(S_1(\cdot))$  is continuous at point  $\lambda = \lambda_0$ .*

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