ALI QUSHCHINING 620 YILLIK TAVALLUDIGA BAG‘ISHLANGAN “ALI QUSHCHI – MIRZO ULUG‘BEK ILMIY MAKTABINING BUYUK ELCHISI”

MAVZUSIDAGI XALQARO ILMIY ANJUMAN

MATERIALLARI

2023-yil 21-22 sentyabr
Samarqand, O‘zbekiston

“ALI QUSHJI – AN OUTSTANDING AMBASSADOR OF THE SCIENTIFIC SCHOOL OF ULUGH BEG” CELEBRATING THE 620TH ANNIVERSARY OF ALI QUSHJI’S BIRTH

INTERNATIONAL CONFERENCE

September 21-22, 2023
Samarkand, Uzbekistan

Ali Qushchi tavalludining 620 yilligi hamda O‘zbekiston respublikasi fanlar akademiyasining 80 yilligiga bag‘ishlangan “Ali Qushchi – Mirzo Ulug’bek ilmiy maktabining buyuk elchisi” mavzusidagi xalqaro ilmiy anjuman materiallari (Samarqand, 2023-yil 21-22 sentyabr). – Samarqand: SamDU nashriyoti, 2023.– 498 b.

Tahrir hay’ati:

Sharof Rashidov nomidagi Samarqand davlat universiteti Kengashining 2023 yil qarori bilan nashrga tavsiya etilgan.

About a regular point of the lattice spin-boson model with at most two photons

E.B. Dilmurodov

Bukhara State University, Bukhara, Uzbekistan

Bukhara branch of the Institute of Mathematics named after V.I.Romanovskiy, Bukhara, Uzbekistan

E-mail: e.b.dilmurodov@buxdu.uz

The spin-boson model is a well-known quantum-mechanical model which describes the interaction between a two-level atom and a photon field. We refer to [1] and [2] for excellent reviews respectively from physical and mathematical perspectives.

Let T^{d} be the d–dimensional torus, $L_2((T^{\mathrm{d}})$ be the Hilbert space of square integrable (complex) functions defined on T^{d}, $\mathbb{C}^2$ be the state of the two-level atom and $\mathcal{F}_{\mathrm{b}}(L_2(T^{\mathrm{d}}))$ be the symmetric Fock space for bosons, that is,

$$\mathcal{F}_{\mathrm{b}}(L_2(T^{\mathrm{d}})) := \mathbb{C} \oplus L_2(T^{\mathrm{d}}) \oplus L_2^{sym}((T^{\mathrm{d}})^2) \oplus \ldots.$$

Here $L_2^{sym}((T^{\mathrm{d}})^n)$ is the Hilbert space of symmetric functions of $n \geq 2$ variables. For $m = 1,2$ we denote $\mathcal{L}_m := \mathbb{C}^2 \otimes \mathcal{F}_{\mathrm{b}}^{(m)}(L_2(T^{\mathrm{d}}))$, where

$$\mathcal{F}_{\mathrm{b}}^{(1)}(L_2(T^{\mathrm{d}})) := \mathbb{C} \oplus L_2(T^{\mathrm{d}}), \quad \mathcal{F}_{\mathrm{b}}^{(2)}(L_2(T^{\mathrm{d}})) := \mathbb{C} \oplus L_2(T^{\mathrm{d}}) \oplus L_2^{sym}((T^{\mathrm{d}})^2).$$

We write elements F of the space $\mathcal{L}_2$ in the form $F = \{f_0^{(\mathrm{s})}, f_1^{(\mathrm{s})}(k_1), f_2^{(\mathrm{s})}(k_1,k_2); \mathrm{s} = \pm\}$. Then the norm in $\mathcal{L}_2$ is given by

$$\| F \|^2 := \sum_{\mathrm{s}=\pm} \left(|f_0^{(\mathrm{s})}|^2 + \int_{T^{\mathrm{d}}} |f_1^{(\mathrm{s})}(k_1)|^2 dk_1 + \frac{1}{2} \int_{(T^{\mathrm{d}})^2} |f_2^{(\mathrm{s})}(k_1,k_2)|^2 dk_1 dk_2 \right). \quad (1)$$

We recall that the lattice spin-boson model with at most two photons $\mathcal{A}_2$ is acting in $\mathcal{L}_2$ as the 3×3 tridiagonal block operator matrix

$$\mathcal{A}_2 := \begin{pmatrix} A_{00} & A_{01} & 0 \\ A_{01}^* & A_{11} & A_{12} \\ 0 & A_{12}^* & A_{22} \end{pmatrix},$$

where matrix elements A_{ij} are defined by

$$A_{00}f_0^{(\mathrm{s})} = \mathrm{s}\varepsilon f_0^{(\mathrm{s})}, \quad A_{01}f_1^{(\mathrm{s})} = \alpha\int_{T^{\mathrm{d}}} v(t)f_1^{(-\mathrm{s})}(t)dt,$$

$$(A_{11}f_1^{(\mathrm{s})})(k_1) = (s\varepsilon + w(k_1))f_1^{(\mathrm{s})}(k_1), \quad (A_{12}f_2^{(\mathrm{s})})(k_1) = \alpha\int_{T^{\mathrm{d}}} v(t)f_2^{(-\mathrm{s})}(k_1,t)dt,$$

$$(A_{22}f_2^{(\mathrm{s})})(k_1,k_2) = (\mathrm{s}\varepsilon + w(k_1) + w(k_2))f_2^{(\mathrm{s})}(k_1,k_2),$$

$$f = \{f_0^{(\mathrm{s})}, f_1^{(\mathrm{s})}, f_2^{(\mathrm{s})}; \mathrm{s} = \pm\} \in \mathcal{L}_2.$$

Here A_{ij}^* denotes the adjoint operator to A_{ij} for $i < j$ with $i,j = 0,1,2$; $w(k)$ is the dispersion of the free field, $\alpha v(k)$ is the coupling between the atoms and the field modes, $\alpha > 0$ is a real number, so-called the coupling constant, real number. We assume that $v(\cdot)$ and $w(\cdot)$ are the real-valued continuous functions on T^{d}. Under these assumptions the lattice spin-boson model with at most two photons $\mathcal{A}_2$ is bounded and self-adjoint in the complex Hilbert space $\mathcal{L}_2$.

To study the spectral properties of $\mathcal{A}_2$ we introduce the following two bounded self-adjoint operators $\mathcal{A}_2^{(\mathrm{s})}$, $\mathrm{s} = \pm$, which acts in $\mathcal{F}_{\mathrm{b}}^{(2)}(L_2(T^{\mathrm{d}}))$ as

$$\mathcal{A}_2^{(\mathrm{s})} := \begin{pmatrix} \hat{A}_{00}^{(\mathrm{s})} & \hat{A}_{01} & 0 \\ \hat{A}_{01}^* & \hat{A}_{11}^{(\mathrm{s})} & \hat{A}_{12} \\ 0 & \hat{A}_{12}^* & \hat{A}_{22}^{(\mathrm{s})} \end{pmatrix}$$

with the entries

$$\hat{A}_{00}^{(\mathrm{s})}f_0 = \mathrm{s}\varepsilon f_0, \quad \hat{A}_{01}f_1 = \alpha\int_{T^{\mathrm{d}}} v(t)f_1(t)dt,$$

$$(\hat{A}_{11}^{(\mathrm{s})}f_1)(k_1) = (-\mathrm{s}\varepsilon + w(k_1))f_1(k_1), \quad (\hat{A}_{12}f_2)(k_1) = \alpha\int_{T^{\mathrm{d}}} v(t)f_2(k_1,t)dt,$$

$$(\hat{A}_{22}^{(\mathrm{s})}f_2)(k_1,k_2) = (\mathrm{s}\varepsilon + w(k_1) + w(k_2))f_2(k_1,k_2), \quad (f_0,f_1,f_2) \in \mathcal{F}_{\mathrm{b}}^{(2)}(L_2(T^{\mathrm{d}})).$$

It is easy to check that

$$(\hat{A}_{01}^*f_0)(k_1) = \alpha v(k_1)f_0;$$

$$(\hat{A}_{12}^*f_1)(k_1,k_2) = \alpha(v(k_1)f_1(k_2) + v(k_2)f_1(k_1)), \quad (f_0,f_1) \in \mathcal{F}_{\mathrm{b}}^{(1)}(L_2(T^{\mathrm{d}})).$$

In order to describe the essential spectrum of $\mathcal{A}_2$ we define an analytic function $\Delta^{(\mathrm{s})}(\cdot)$ in $\mathbb{C}\backslash[s\varepsilon + m; s\varepsilon + M]$ by

$$\Delta^{(\mathrm{s})}(\lambda):=-\mathrm{s}\varepsilon-\lambda-\alpha^2\int_{T^{\mathrm{d}}}\frac{v^2(t)dt}{\mathrm{s}\varepsilon+w(t)-\lambda},$$

where the numbers m and M are defined by

$$m:=\min_{p\in T^{\mathrm{d}}} w(p),\quad M:=\max_{p\in T^{\mathrm{d}}} w(p).$$

Let $\sigma^{(\mathrm{s})}$ be the set of all complex numbers $\lambda\in\mathbb{C}$ such that the equality $\Delta^{(\mathrm{s})}(\lambda-w(k_1))=0$ holds for some $k_1\in T^{\mathrm{d}}$. Then for the essential spectrum of $\mathcal{A}_2^{(\mathrm{s})}$ we have

$$\sigma_{ess}(\mathcal{A}_2^{(\mathrm{s})})=\sigma^{(\mathrm{s})}\cup[\mathrm{s}\varepsilon+2m;\mathrm{s}\varepsilon+2M].$$

We set $E_{\min}^{(s)}\coloneqq\min\sigma_{ess}(\mathcal{A}_2^{(s)})$, for $s=\pm$ and $E_{\min}\coloneqq\min\{E_{\min}^{(+)},E_{\min}^{(-)}\}$.

Next, we represent the space $\mathcal{F}_{\mathrm{b}}^{(2)}(L_2(T^{\mathrm{d}}))$ as a direct sum of two Hilbert spaces $\mathcal{F}_{\mathrm{b}}^{(1)}(L_2(T^{\mathrm{d}}))$ and $L_2^{sym}((T^{\mathrm{d}})^2)$, that is, $\mathcal{F}_{\mathrm{b}}^{(2)}(L_2(T^{\mathrm{d}}))=\mathcal{F}_{\mathrm{b}}^{(1)}(L_2(T^{\mathrm{d}}))\oplus L_2^{sym}((T^{\mathrm{d}})^2)$. Then the first Schur complement of the operator $\mathcal{A}_2^{(\mathrm{s})}$ with respect to this decomposition is defined as

$$S_1^{(\mathrm{s})}(\lambda):\mathcal{F}_{\mathrm{b}}^{(1)}(L_2(T^{\mathrm{d}}))\to\mathcal{F}_{\mathrm{b}}^{(1)}(L_2(T^{\mathrm{d}})),\quad \lambda\in\rho(\hat{A}_{22}^{(\mathrm{s})});$$

$$S_1^{(\mathrm{s})}(\lambda):=\begin{pmatrix}\hat{A}_{00}^{(\mathrm{s})} & \hat{A}_{01}\\ \hat{A}_{01}^{*} & \hat{A}_{11}^{(\mathrm{s})}\end{pmatrix}-\lambda-\begin{pmatrix}0\\ \hat{A}_{12}\end{pmatrix}(\hat{A}_{22}^{(\mathrm{s})}-\lambda)^{-1}(0\quad \hat{A}_{12}^{*}).$$

Define

$$S_{00}^{(\mathrm{s})}(\lambda):=\hat{A}_{00}^{(\mathrm{s})}-\lambda,\quad S_{01}^{(\mathrm{s})}(\lambda):=\hat{A}_{01};$$

$$S_{10}^{(\mathrm{s})}(\lambda):=\hat{A}_{01}^{*},\quad S_{11}^{(\mathrm{s})}(\lambda):=\hat{A}_{11}^{(\mathrm{s})}-\lambda-\hat{A}_{12}(\hat{A}_{22}^{(\mathrm{s})}-\lambda)^{-1}\hat{A}_{12}^{*}.$$

Then the operator $S_1^{(\mathrm{s})}(\lambda)$ has form

$$S_1^{(\mathrm{s})}(\lambda)=\begin{pmatrix}S_{00}^{(\mathrm{s})}(\lambda) & S_{01}^{(\mathrm{s})}(\lambda)\\ S_{10}^{(\mathrm{s})}(\lambda) & S_{11}^{(\mathrm{s})}(\lambda)\end{pmatrix}.$$

We study some important properties of the first Schur complement

$$S_1(\lambda):=\operatorname{diag}\{S_1^{(+)}(\lambda),S_1^{(-)}(\lambda)\}$$

for the lattice spin-boson model with at most two photons $\mathcal{A}_2$.

Proposition 1. *The number* $\lambda \in \mathbb{C}\backslash\sigma_{ess}(\mathcal{A}_2)$ *is an eigenvalue of the operator* $\mathcal{A}_2$ *if and only if the operator* $S_1(\lambda)$ *has an eigenvalue equal to zero. Moreover, the eigenvalues* λ *and* 0 *have the same multiplicities.*

Proposition 2. $\lambda \in \sigma_{ess}(\mathcal{A}_2)\backslash\sigma(A_{22})$ *if and only if* $0 \in \sigma_{ess}(S_1(\lambda))$.

From Propositions 1 and 2 we obtain the following two corollaries.

Corollary 1. *Let* $\lambda \in \mathbb{C}\backslash\sigma_{ess}(\mathcal{A}_2)$. *Then* $\lambda \in \rho(\mathcal{A}_2) \Leftrightarrow 0 \in \rho(S_1(\lambda))$.

Corollary 2. *Let* $\lambda_0 \in \mathbb{R}\backslash\sigma_{ess}(\mathcal{A}_2)$. *If* $(\lambda_0; \lambda_0 + \gamma) \subset \rho(\mathcal{A}_2)$ (*resp.* $(\lambda_0 - \gamma; \lambda_0) \subset \rho(A_2)$) *for some* $\gamma > 0$, *then there exists a number* $\delta = \delta(\gamma) > 0$ *such that* $(0; \delta) \subset \rho(S_1(\lambda_0))$ (*resp.* $(-\delta; 0) \subset \rho(S_1(\lambda_0))$).

For a bounded self-adjoint operator $\mathcal{A}$ acting in a Hilbert space H we denote by $N_{(-\infty;\lambda)}(\mathcal{A})$ the number of eigenvalues of $\mathcal{A}$ to the left of $\lambda, \lambda \leq \min \sigma_{\text{ess}}(\mathcal{A})$.

Proposition 3. *The function* $N_{(-\infty;0)}(S_1(\cdot))$ *is monotonically increasing in* $(-\infty; 0)$.

Definition 1. *We denote by* $E_m(\cdot), m \in \mathbb{N}$ *the positive definite function on the segment* $[\alpha; \beta] \subset \mathbb{R}\backslash\sigma(\mathcal{A}_2)$, *satisfying the condition:* $E_m(\lambda)$ *is the* m*-th eigenvalue* (*eigenvalues numbered in asceding order, counting their multiplicity*) *of the operator* $S_1(\lambda), \lambda \in [\alpha; \beta]$.

Theorem. *The number* $\lambda_0 < E_{\min}$ *is the regular point of the operator* $\mathcal{A}_2$ *if and only if the function* $N_{(-\infty;0)}(S_1(\cdot))$ *is continuous at point* $\lambda = \lambda_0$.

References

1. A.J.Leggett, S.Chakravarty, A.T.Dorsey, M.P.A.Fisher, A.Garg, W.Zwerger. Dynamics of the dissipative two-state system. Rev. Mod. Phys. 59 (Jan 1987), 1–85.

2. M.Hübner, H.Spohn. Radiative decay: nonperturbative approaches. Rev. Math. Phys. **7**:3 (1995), 363–387.

MUNDARIJA	