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Original article

The first Schur complement for a lattice spin-boson model with at most two photons

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ABSTRACT In the present paper, we consider a lattice spin-boson model $\mathcal{A}_2$ with a fixed atom and at most two photons. We construct the first Schur complement $S_1(\lambda)$ with spectral parameter λ corresponding to $\mathcal{A}_2$. We prove the Birman–Schwinger principle for $\mathcal{A}_2$ with respect to $S_1(\lambda)$. We investigate an important properties of $S_1(\lambda)$ related to the number of eigenvalues of $\mathcal{A}_2$ for all dimensions d of the torus $\mathbb{T}^{\mathrm{d}}$ and for any coupling constant $\alpha > 0$.

KEYWORDS lattice spin-boson model, Schur complement, bosonic Fock space, essential spectrum, number of eigenvalues, Birman–Schwinger principle.

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1. Introduction and statement of the problem

The spin-boson model is a well-known quantum-mechanical model which describes the interaction between two-level atom and photon field. We refer to [1] and [2] for excellent reviews respectively from physical and mathematical perspectives. Despite the formal simplicity of the spin-boson model (from the physics viewpoint), its dynamics is rather complicated and rigorous spectral and scattering results are usually very difficult to obtain, especially in the case when the number of photons is unbounded. In this connection, it is natural to consider truncated models [3–6] with at most m ($m \in \mathbb{N}$) photons.

The truncated model in $\mathbb{R}^{\mathrm{d}}$ with $m = 1, 2$ was completely studied in [4] for small values of the parameter α and the case $m = 3$ is considered in [6]. The existence of wave operators and their asymptotic completeness are proven there. In [3,5], the case of arbitrary m are investigated. It should be mentioned that in these papers the smallness of the coupling constant α is important, in our analysis (in the lattice case) this constant can be arbitrary. A lattice spin-boson model $\mathcal{A}_m$ with $m = 1, 2$ is considered in [7, 8]. In particular, in [7] the location of the essential spectrum of $\mathcal{A}_2$ is described; for any coupling constant the finiteness of the number of eigenvalues below the bottom of the essential spectrum of $\mathcal{A}_2$ is established (with a sketch of the proof). The paper [8] is devoted to the study of the geometrical structure of the branches of the essential spectrum of $\mathcal{A}_2$.

Let us introduce a lattice spin-boson model with at most two photons. Let $\mathbb{T}^{\mathrm{d}}$ be the d–dimensional torus, $L_2((\mathbb{T}^{\mathrm{d}})$ be the Hilbert space of square integrable (complex) functions defined on $\mathbb{T}^{\mathrm{d}}$, $\mathbb{C}^2$ be the state of the two-level atom and $\mathcal{F}_{\mathrm{b}}(L_2(\mathbb{T}^{\mathrm{d}}))$ be the symmetric Fock space for bosons, that is,

$$\mathcal{F}_{\mathrm{b}}(L_2(\mathbb{T}^{\mathrm{d}})) := \mathbb{C} \oplus L_2(\mathbb{T}^{\mathrm{d}}) \oplus L_2^{\mathrm{sym}}((\mathbb{T}^{\mathrm{d}})^2) \oplus \ldots .$$

Here $L_2^{\mathrm{sym}}((\mathbb{T}^{\mathrm{d}})^n)$ is the Hilbert space of symmetric functions of $n \geq 2$ variables. For $m = 1, 2$ we denote $\mathcal{L}_m := \mathbb{C}^2 \otimes \mathcal{F}_{\mathrm{b}}^{(m)}(L_2(\mathbb{T}^{\mathrm{d}}))$, where

$$\mathcal{F}_{\mathrm{b}}^{(1)}(L_2(\mathbb{T}^{\mathrm{d}})) := \mathbb{C} \oplus L_2(\mathbb{T}^{\mathrm{d}}), \quad \mathcal{F}_{\mathrm{b}}^{(2)}(L_2(\mathbb{T}^{\mathrm{d}})) := \mathbb{C} \oplus L_2(\mathbb{T}^{\mathrm{d}}) \oplus L_2^{\mathrm{sym}}((\mathbb{T}^{\mathrm{d}})^2).$$

We write elements F of the space $\mathcal{L}_2$ in the form $F = \{f_0^{(\mathrm{s})}, f_1^{(\mathrm{s})}(k_1), f_2^{(\mathrm{s})}(k_1, k_2); \mathrm{s} = \pm\}$. Then the norm in $\mathcal{L}_2$ is given by

$$\|F\|^2 := \sum_{\mathrm{s}=\pm} \left(|f_0^{(\mathrm{s})}|^2 + \int\limits_{\mathbb{T}^{\mathrm{d}}} |f_1^{(\mathrm{s})}(k_1)|^2 dk_1 + \frac{1}{2} \int\limits_{(\mathbb{T}^{\mathrm{d}})^2} |f_2^{(\mathrm{s})}(k_1, k_2)|^2 dk_1 dk_2 \right). \tag{1.1}$$

We recall that the lattice spin-boson model with at most two photons $\mathcal{A}_2$ is acting in $\mathcal{L}_2$ as the 3×3 tridiagonal block operator matrix

$$\mathcal{A}_2 := \begin{pmatrix} A_{00} & A_{01} & 0 \\ A_{01}^* & A_{11} & A_{12} \\ 0 & A_{12}^* & A_{22} \end{pmatrix},$$

where matrix elements A_{ij} are defined by

$$A_{00} f_0^{(\mathrm{s})} = \mathrm{s}\varepsilon f_0^{(\mathrm{s})}, \quad A_{01} f_1^{(\mathrm{s})} = \alpha \int\limits_{\mathbb{T}^{\mathrm{d}}} v(t) f_1^{(-\mathrm{s})}(t) dt,$$

$$(A_{11} f_1^{(\mathrm{s})})(k_1) = (\mathrm{s}\varepsilon + w(k_1)) f_1^{(\mathrm{s})}(k_1), \quad (A_{12} f_2^{(\mathrm{s})})(k_1) = \alpha \int\limits_{\mathbb{T}^{\mathrm{d}}} v(t) f_2^{(-\mathrm{s})}(k_1, t) dt,$$

$$(A_{22} f_2^{(\mathrm{s})})(k_1, k_2) = (\mathrm{s}\varepsilon + w(k_1) + w(k_2)) f_2^{(\mathrm{s})}(k_1, k_2), \quad f = \{f_0^{(\mathrm{s})}, f_1^{(\mathrm{s})}, f_2^{(\mathrm{s})}; \mathrm{s} = \pm\} \in \mathcal{L}_2.$$

Here A_{ij}^* denotes the adjoint operator to A_{ij} for $i < j$ with $i, j = 0, 1, 2$; $w(k)$ is the dispersion of the free field, $\alpha v(k)$ is the coupling between the atoms and the field modes, $\alpha > 0$ is a real number, so-called the coupling constant. We assume that $v(\cdot)$ and $w(\cdot)$ are the real-valued continuous functions on $\mathbb{T}^{\mathrm{d}}$. Under these assumptions the lattice spin-boson model with at most two photons $\mathcal{A}_2$ is bounded and self-adjoint in the complex Hilbert space $\mathcal{L}_2$.

A main goal of the paper is the study of the main spectral properties of $\mathcal{A}_2$ related to the number of eigenvalues. More precisely, the following results are obtained: using a unitary dilation the lattice model $\mathcal{A}_2$ of radiative decay with a fixed atom and at most two photons is reduced to the diagonal operator and it's spectrum is described; the first Schur complement corresponding to the both diagonal entries of $\mathcal{A}_2$ is constructed; the relation between the eigenvalues of $\mathcal{A}_2$ and the first Schur complement (Birman–Schwinger principle) is established.

Throughout the paper, we use the notation $\sigma(\cdot)$, $\sigma_{\mathrm{ess}}(\cdot)$, $\sigma_{\mathrm{p}}(\cdot)$ and $\sigma_{\mathrm{disc}}(\cdot)$, respectively, for the spectrum, the essential spectrum, the point spectrum and the discrete spectrum of bounded self-adjoint operator.

2. The first Schur complement corresponding to $\mathcal{A}_2$

To study the spectral properties of $\mathcal{A}_2$, we introduce the following two bounded self-adjoint operators $\mathcal{A}_2^{(\mathrm{s})}$, $\mathrm{s} = \pm$, which act in $\mathcal{F}_{\mathrm{b}}^{(2)}(L_2(\mathbb{T}^{\mathrm{d}}))$ as

$$\mathcal{A}_2^{(\mathrm{s})} := \begin{pmatrix} \widehat{A}_{00}^{(\mathrm{s})} & \widehat{A}_{01} & 0 \\ \widehat{A}_{01}^* & \widehat{A}_{11}^{(\mathrm{s})} & \widehat{A}_{12} \\ 0 & \widehat{A}_{12}^* & \widehat{A}_{22}^{(\mathrm{s})} \end{pmatrix}$$

with the entries

$$\widehat{A}_{00}^{(\mathrm{s})} f_0 = \mathrm{s}\varepsilon f_0, \quad \widehat{A}_{01} f_1 = \alpha \int\limits_{\mathbb{T}^{\mathrm{d}}} v(t) f_1(t) dt,$$

$$(\widehat{A}_{11}^{(\mathrm{s})} f_1)(k_1) = (-\mathrm{s}\varepsilon + w(k_1)) f_1(k_1), \quad (\widehat{A}_{12} f_2)(k_1) = \alpha \int\limits_{\mathbb{T}^{\mathrm{d}}} v(t) f_2(k_1, t) dt,$$

$$(\widehat{A}_{22}^{(\mathrm{s})} f_2)(k_1, k_2) = (\mathrm{s}\varepsilon + w(k_1) + w(k_2)) f_2(k_1, k_2), \quad (f_0, f_1, f_2) \in \mathcal{F}_{\mathrm{b}}^{(2)}(L_2(\mathbb{T}^{\mathrm{d}})).$$

It is easy to check that

$$(\widehat{A}_{01}^* f_0)(k_1) = \alpha v(k_1) f_0;$$
$$(\widehat{A}_{12}^* f_1)(k_1, k_2) = \alpha \left(v(k_1) f_1(k_2) + v(k_2) f_1(k_1) \right), \quad (f_0, f_1) \in \mathcal{F}_{\mathrm{b}}^{(1)}(L_2(\mathbb{T}^{\mathrm{d}})).$$

In order to describe the essential spectrum of $\mathcal{A}_2$, we define an analytic function $\Delta^{(\mathrm{s})}(\cdot)$ in $\mathbb{C} \setminus [\mathrm{s}\varepsilon + m; \mathrm{s}\varepsilon + M]$ by

$$\Delta^{(\mathrm{s})}(\lambda) := -\mathrm{s}\varepsilon - \lambda - \alpha^2 \int\limits_{\mathbb{T}^{\mathrm{d}}} \frac{v^2(t) dt}{\mathrm{s}\varepsilon + w(t) - \lambda},$$

where the numbers m and M are defined by

$$m := \min_{p \in \mathbb{T}^{\mathrm{d}}} w(p), \quad M := \max_{p \in \mathbb{T}^{\mathrm{d}}} w(p).$$

Let $\sigma^{(\mathrm{s})}$ be the set of all complex numbers $\lambda \in \mathbb{C}$ such that the equality $\Delta^{(\mathrm{s})}(\lambda - w(k_1)) = 0$ holds for some $k_1 \in \mathbb{T}^{\mathrm{d}}$. Then (see [9]) for the essential spectrum of $\mathcal{A}_2^{(\mathrm{s})}$, we have

$$\sigma_{\mathrm{ess}}(\mathcal{A}_2^{(\mathrm{s})}) = \sigma^{(\mathrm{s})} \cup [\mathrm{s}\varepsilon + 2m; \mathrm{s}\varepsilon + 2M].$$

Consider the permutation operator $\Phi : \mathcal{L}_2 \to \mathcal{F}_{\rm b}^{(2)}(L_2(\mathbb{T}^{\rm d})) \oplus \mathcal{F}_{\rm b}^{(2)}(L_2(\mathbb{T}^{\rm d}))$ defined as

$$\Phi : (f_0^{(+)}, f_0^{(-)}, f_1^{(+)}, f_1^{(-)}, f_2^{(+)}, f_2^{(-)}) \to (f_0^{(+)}, f_1^{(-)}, f_2^{(+)}, f_0^{(-)}, f_1^{(+)}, f_2^{(-)}).$$

One can trivially verify that the operator Φ is unitary. From the construction of $\mathcal{A}_2$, $\mathcal{A}_2^{(\rm s)}$ and Φ, it follows that the following equality takes place $\Phi\mathcal{A}_2\Phi^{-1} = {\rm diag}\{\mathcal{A}_2^{(+)}, \mathcal{A}_2^{(-)}\}$. The latter facts mean that the operators $\mathcal{A}_2$ and ${\rm diag}\{\mathcal{A}_2^{(+)}, \mathcal{A}_2^{(-)}\}$ are unitarily equivalent. Therefore, $\sigma(\mathcal{A}_2) = \sigma(\mathcal{A}_2^{(+)}) \cup \sigma(\mathcal{A}_2^{(-)})$. Moreover,

$$\sigma_{\rm ess}(\mathcal{A}_2) = \sigma_{\rm ess}(\mathcal{A}_2^{(+)}) \cup \sigma_{\rm ess}(\mathcal{A}_2^{(-)}); \quad \sigma_{\rm p}(\mathcal{A}_2) = \sigma_{\rm p}(\mathcal{A}_2^{(+)}) \cup \sigma_{\rm p}(\mathcal{A}_2^{(-)}).$$

Since the part of $\sigma_{\rm disc}(\mathcal{A}_2^{(\rm s)})$ can be located in $\sigma_{\rm ess}(\mathcal{A}_2^{(-\rm s)})$, we have

$$\sigma_{\rm disc}(\mathcal{A}_2) \subseteq \sigma_{\rm disc}(\mathcal{A}_2^{(+)}) \cup \sigma_{\rm disc}(\mathcal{A}_2^{(-)}).$$

If we set $E_{\rm min}^{(\rm s)} := \min \sigma_{\rm ess}(\mathcal{A}_2^{(\rm s)})$ for ${\rm s} = \pm$ and $E_{\rm min} := \min \sigma_{\rm ess}(\mathcal{A}_2) = \min\{E_{\rm min}^{(+)}, E_{\rm min}^{(-)}\}$, then

$$\sigma_{\rm disc}(\mathcal{A}_2) \cap (-\infty; E_{\rm min}) = \{\sigma_{\rm disc}(\mathcal{A}_2^{(+)}) \cup \sigma_{\rm disc}(\mathcal{A}_2^{(-)})\} \cap (-\infty; E_{\rm min}). \tag{2.1}$$

Next, we represent the space $\mathcal{F}_{\rm b}^{(2)}(L_2(\mathbb{T}^{\rm d}))$ as a direct sum of two Hilbert spaces $\mathcal{F}_{\rm b}^{(1)}(L_2(\mathbb{T}^{\rm d}))$ and $L_2^{\rm sym}((\mathbb{T}^{\rm d})^2)$, that is, $\mathcal{F}_{\rm b}^{(2)}(L_2(\mathbb{T}^{\rm d})) = \mathcal{F}_{\rm b}^{(1)}(L_2(\mathbb{T}^{\rm d})) \oplus L_2^{\rm sym}((\mathbb{T}^{\rm d})^2)$. Then the first Schur complement of the operator $\mathcal{A}_2^{(\rm s)}$ with respect to this decomposition (see [10]) is defined as

$$S_1^{(\rm s)}(\lambda) : \mathcal{F}_{\rm b}^{(1)}(L_2(\mathbb{T}^{\rm d})) \to \mathcal{F}_{\rm b}^{(1)}(L_2(\mathbb{T}^{\rm d})), \quad \lambda \in \rho(\widehat{A}_{22}^{(\rm s)});$$
$$S_1^{(\rm s)}(\lambda) := \begin{pmatrix} \widehat{A}_{00}^{(\rm s)} & \widehat{A}_{01} \\ \widehat{A}_{01}^{*} & \widehat{A}_{11}^{(\rm s)} \end{pmatrix} - \lambda - \begin{pmatrix} 0 \\ \widehat{A}_{12} \end{pmatrix} (\widehat{A}_{22}^{(\rm s)} - \lambda)^{-1} (0 \;\; \widehat{A}_{12}^{*}).$$

Define

$$S_{00}^{(\rm s)}(\lambda) := \widehat{A}_{00}^{(\rm s)} - \lambda, \quad S_{01}^{(\rm s)}(\lambda) := \widehat{A}_{01};$$
$$S_{10}^{(\rm s)}(\lambda) := \widehat{A}_{01}^{*}, \quad S_{11}^{(\rm s)}(\lambda) := \widehat{A}_{11}^{(\rm s)} - \lambda - \widehat{A}_{12}(\widehat{A}_{22}^{(\rm s)} - \lambda)^{-1}\widehat{A}_{12}^{*}.$$

Then the operator $S_1^{(\rm s)}(\lambda)$ has the form

$$S_1^{(\rm s)}(\lambda) = \begin{pmatrix} S_{00}^{(\rm s)}(\lambda) & S_{01}^{(\rm s)}(\lambda) \\ S_{10}^{(\rm s)}(\lambda) & S_{11}^{(\rm s)}(\lambda) \end{pmatrix}.$$

For convenience, we represent the operator $S_{11}^{(\rm s)}(\lambda)$ as a difference of two operators

$$S_{11}^{(\rm s)}(\lambda) := D^{(\rm s)}(\lambda) - K^{(\rm s)}(\lambda),$$

where the operators $D^{(\rm s)}(\lambda)$, $K^{(\rm s)}(\lambda) : L_2(\mathbb{T}^{\rm d}) \to L_2(\mathbb{T}^{\rm d})$ are defined by

$$(D^{(\rm s)}(\lambda)f)(k_1) = \Delta^{(\rm s)}(\lambda - w(k_1))f(k_1);$$
$$(K^{(\rm s)}(\lambda)f)(k_1) = \alpha^2 v(k_1) \int_{\mathbb{T}^{\rm d}} \frac{v(t)f(t)dt}{{\rm s}\varepsilon + w(k_1) + w(t) - \lambda}.$$

For a fixed $\lambda = \lambda_0 \in \rho(\widehat{A}_{22}^{(\rm s)})$ and ${\rm s} \in \{-1, 1\}$, we define

$$a := {\rm s}\varepsilon - \lambda_0,\; u(k_1) := \Delta^{(\rm s)}(\lambda_0 - w(k_1)),\; K(k_1, k_2) := \frac{\alpha^2 v(k_1)v(k_2)}{{\rm s}\varepsilon + w(k_1) + w(k_2) - \lambda_0}.$$

Then the operator matrix $S_1^{(\rm s)}(\lambda_0)$ can be written as

$$S_1^{(\rm s)}(\lambda_0) = \begin{pmatrix} H_{00} & H_{01} \\ H_{01}^{*} & H_{11}^{0} - K \end{pmatrix} \tag{2.2}$$

with

$$H_{00}f_0 = af_0,\; H_{01} := \widehat{A}_{01},\; (H_{11}^0 f_1)(k_1) = u(k_1)f_1(k_1),\; (Kf_1)(k_1) = \int_{\mathbb{T}^{\rm d}} K(k_1, t)f_1(t)dt.$$

The operator matrix of the form (2.2) is appeared in a series of problems in analysis, mathematical physics, and probability theory and known as generalized Friedrichs model. This model operator itself was introduced in [11], where its eigenvalues and "resonances" (i.e., the singularities of the analytic continuation of the resolvent) were studied. Note that, the number and location of the eigenvalues of the generalized Friedrichs model in the case where the kernel function $K(\cdot, \cdot)$ is degenerate of rank 1, was studied in [12, 13].

The Schur complement is named after Issai Schur who used it to prove Schur's lemma, although it had been used previously [14]. Haynsworth was the first to call it the Schur complement [15]. The Schur complement is a key tool in the fields of numerical analysis, statistics, and matrix analysis. The general properties of the Shur complement have been studied in many works, for detailed information see [10]. Construction of Schur's complement for exactly solvable models of mathematical physics and proof of important properties that have not been properly studied in general for such special cases, can be considered as one of the actual problems of the operator theory.

3. Main properties of the first Schur complement

In this Section, we will study some important properties of the first Schur complement

$$S_1(\lambda) := \mathrm{diag}\{S_1^{(+)}(\lambda), S_1^{(-)}(\lambda)\}$$

for the lattice spin-boson model with at most two photons $\mathcal{A}_2$.

Proposition 3.1. *The number $\lambda \in \mathbb{C} \setminus \sigma_{\mathrm{ess}}(\mathcal{A}_2)$ is an eigenvalue of the operator $\mathcal{A}_2$ if and only if the operator $S_1(\lambda)$ has an eigenvalue equal to zero. Moreover, the eigenvalues λ and 0 have the same multiplicities.*

Proof. Since $\sigma_{\mathrm{p}}(\mathcal{A}_2) = \sigma_{\mathrm{p}}(\mathcal{A}_2^{(+)}) \cup \sigma_{\mathrm{p}}(\mathcal{A}_2^{(-)})$, it is enough to prove the assertion of the Proposition for the operators $\mathcal{A}_2^{(\mathrm{s})}$ and $S_1^{(\mathrm{s})}(\lambda)$.

Let the number $\lambda \in \mathbb{C} \setminus \sigma_{\mathrm{ess}}(\mathcal{A}_2)$ be an eigenvalue of the operator $\mathcal{A}_2^{(\mathrm{s})}$ and $f = (f_0, f_1, f_2) \in \mathcal{F}_{\mathrm{b}}^{(2)}(L_2(\mathbb{T}^{\mathrm{d}}))$ be the corresponding eigenvector. Then elements f_0, f_1 and f_2 satisfy the system of equations

$$\begin{cases} (\widehat{A}_{00}^{(\mathrm{s})} - \lambda) f_0 + \widehat{A}_{01} f_1 = 0; \\ \widehat{A}_{01}^* f_0 + (\widehat{A}_{11}^{(\mathrm{s})} - \lambda) f_1 + \widehat{A}_{12} f_2 = 0; \\ \widehat{A}_{12}^* f_1 + (\widehat{A}_{22}^{(\mathrm{s})} - \lambda) f_2 = 0. \end{cases} \tag{3.1}$$

Since $\lambda \in \mathbb{C} \setminus \sigma_{\mathrm{ess}}(\mathcal{A}_2)$, from the third equation of system (3.1) for f_2 we have

$$f_2 = -(\widehat{A}_{22}^{(\mathrm{s})} - \lambda)^{-1} \widehat{A}_{12}^* f_1. \tag{3.2}$$

Substituting the expression (3.2) for f_2 into the second equation of system (3.1), we obtain the following system of equations

$$\begin{cases} (\widehat{A}_{00}^{(\mathrm{s})} - \lambda) f_0 + \widehat{A}_{01} f_1 = 0; \\ \widehat{A}_{01}^* f_0 + (\widehat{A}_{11}^{(\mathrm{s})} - \lambda - \widehat{A}_{12} (\widehat{A}_{22}^{(\mathrm{s})} - \lambda)^{-1} \widehat{A}_{12}^*) f_1 = 0. \end{cases} \tag{3.3}$$

System of equations (3.3) has nontrivial solution if and only if the equation

$$S_1^{(\mathrm{s})}(\lambda) \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

has nontrivial solution $(f_0, f_1) \in \mathcal{F}_{\mathrm{b}}^{(1)}(L_2(\mathbb{T}^{\mathrm{d}}))$.

Let now the number $\lambda \in \mathbb{C} \setminus \sigma_{\mathrm{ess}}(\mathcal{A}_2^{(\mathrm{s})})$ be an eigenvalue of $\mathcal{A}_2^{(\mathrm{s})}$ with multiplicity n and number 0 be an eigenvalue of $S_1^{(\mathrm{s})}(\lambda)$ with multiplicity m. We will prove that $n = m$.

Assume that $n < m$. Then there exist linearly independent eigenvectors $\widetilde{f}^{(i)} = (f_0^{(i)}, f_1^{(i)}) \in \mathcal{F}_{\mathrm{b}}^{(1)}(L_2(\mathbb{T}^{\mathrm{d}}))$, $i = \overline{1, m}$, corresponding to the eigenvalue 0 of the operator $S_1^{(\mathrm{s})}(\lambda)$. For each $i \in \{1, \ldots, m\}$ we put $f^{(i)} := (f_0^{(i)}, f_1^{(i)}, f_2^{(i)})$, where the function $f_2^{(i)}$ is determined by formula (3.2), with $f_1^{(i)}$ taken instead of f_1. Then the vector $f^{(i)}$ satisfies the equation $\mathcal{A}_2^{(\mathrm{s})} f^{(i)} = \lambda f^{(i)}$ for $i = 1, \ldots, m$ and hence it is an eigenvector of $\mathcal{A}_2^{(\mathrm{s})}$ corresponding to the eigenvalue λ. Since $n < m$, the eigenvectors $f^{(i)}$, $i = \overline{1, m}$ are linearly dependent. Therefore, there is a non-zero vector $(\alpha_1, ..., \alpha_m) \in \mathbb{C}^m$ such that $\sum\limits_{i=1}^m \alpha_i f^{(i)} = (0, 0, 0)^t$, but at the same time, it satisfies the inequality $\left(\sum\limits_{i=1}^m \alpha_i f_0^{(i)}, \sum\limits_{i=1}^m \alpha_i f_1^{(i)} \right) \neq (0, 0)^t$ (since $(f_0^{(i)}, f_1^{(i)})$, $i = \overline{1, m}$ are linearly independent). From the last two assertions and the construction of $f^{(i)}$, we have

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \sum_{i=1}^m \alpha_i f^{(i)} = \begin{pmatrix} \sum\limits_{i=1}^m \alpha_i f_0^{(i)} \\ \sum\limits_{i=1}^m \alpha_i f_1^{(i)} \\ -R_{22}^{(\mathrm{s})}(\lambda) \widehat{A}_{12}^* \left(\sum\limits_{i=1}^m \alpha_i f_1^{(i)} \right) \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

where $R_{22}^{(\mathrm{s})}(\lambda) := (\widehat{A}_{22}^{(\mathrm{s})} - \lambda)^{-1}$. This contradiction shows that the inequality $n < m$ is not true.

Let now $n > m$. In this case, there exist linearly independent eigenvectors $f^{(i)} = (f_0^{(i)}, f_1^{(i)}, f_2^{(i)})$, $i = \overline{1,n}$ corresponding to the eigenvalue λ of the operator $\mathcal{A}_2^{(\mathrm{s})}$. One can easily show that $\widetilde{f}^{(i)} = (f_0^{(i)}, f_1^{(i)})$, $i = \overline{1,n}$ is an eigenvector corresponding to the eigenvalue 0 of the operator $S_1^{(\mathrm{s})}(\lambda)$. Arguing similarly, from the inequality $n > m$, we obtain that there exists nonzero vector $(\beta_1, \cdots, \beta_n) \in \mathbb{C}^n$ so that $\sum\limits_{i=1}^{n} \beta_i \widetilde{f}^{(i)} = (0,0)^t$. At the same time $\sum\limits_{i=1}^{n} \beta_i f^{(i)} \neq (0,0,0)^t$. From the last two assertions and the construction of $f^{(i)}$, and also linearity of the operators $\widehat{A}_{12}^{*}$ and $R_{22}^{(\mathrm{s})}(\lambda)$, we have

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \neq \sum_{i=1}^{n} \beta_i f^{(i)} = \begin{pmatrix} \sum\limits_{i=1}^{n} \beta_i f_0^{(i)} \\ \sum\limits_{i=1}^{n} \beta_i f_1^{(i)} \\ -R_{22}^{(\mathrm{s})}(\lambda) \widehat{A}_{12}^{*} \left(\sum\limits_{i=1}^{n} \beta_i f_1^{(i)} \right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This contradiction shows that the inequality $n > m$ is not valid. Therefore, $n = m$. Proposition 3.1 is proved. □

Proposition 3.2. $\lambda \in \sigma_{\mathrm{ess}}(\mathcal{A}_2) \setminus \sigma(A_{22})$ *if and only if* $0 \in \sigma_{\mathrm{ess}}(S_1(\lambda))$.

Proof. We prove that $\lambda \in \sigma_{\mathrm{ess}}(\mathcal{A}_2^{(\mathrm{s})}) \setminus \sigma(\widehat{A}_{22}^{(\mathrm{s})})$ if and only if $0 \in \sigma_{\mathrm{ess}}(S_1^{(\mathrm{s})}(\lambda))$. Let $\mathrm{Ran}(u)$ be the range of the function $u(\cdot)$. Since for any fixed $\lambda \in \mathbb{R} \setminus \sigma(\widehat{A}_{22}^{(\mathrm{s})})$, the kernel of the integral operator $K^{(\mathrm{s})}(\lambda)$ is continuous in $(\mathbb{T}^{\mathrm{d}})^2$, it is the Hilbert-Schmidt operator. By the Weyl theorem on the invariance of the essential spectrum under compact perturbations and by the continuity of the function $\Delta^{(\mathrm{s})}(\lambda - w(\cdot))$ as $\lambda \in \mathbb{R} \setminus \sigma(\widehat{A}_{22}^{(\mathrm{s})})$ on the compact set $\mathbb{T}^{\mathrm{d}}$, we obtain

$$\sigma_{\mathrm{ess}}(S_1^{(\mathrm{s})}(\lambda)) = \mathrm{Ran}(\Delta^{(\mathrm{s})}(\lambda - w(\cdot))). \tag{3.4}$$

Let $\lambda_0 \in \sigma_{\mathrm{ess}}(\mathcal{A}_2^{(\mathrm{s})}) \setminus \sigma(\widehat{A}_{22}^{(\mathrm{s})})$. Then $\lambda_0 \in \sigma^{(\mathrm{s})} \setminus \sigma(\widehat{A}_{22}^{(\mathrm{s})})$. From the definition of $\sigma^{(\mathrm{s})}$, it follows that there exists point $p_0 \in \mathbb{T}^{\mathrm{d}}$ such that $\Delta^{(\mathrm{s})}(\lambda_0 - w(k_0)) = 0$. Taking into account equality (3.4), we have $0 \in \sigma_{\mathrm{ess}}(S_1^{(\mathrm{s})}(\lambda_0))$.

Let now $0 \in \sigma_{\mathrm{ess}}(S_1^{(\mathrm{s})}(\lambda_1))$ for some $\lambda_1 \in \mathbb{R} \setminus \sigma(\mathcal{A}_{22}^{(\mathrm{s})})$. Then by virtue of equality (3.4), there exists a point $p_1 \in \mathbb{T}^{\mathrm{d}}$ such that $\Delta^{(\mathrm{s})}(\lambda_1 - w(p_1)) = 0$. Using the construction of $\sigma^{(\mathrm{s})}$, we obtain $\lambda_1 \in \sigma^{(\mathrm{s})} \subset \sigma_{\mathrm{ess}}(\mathcal{A}_2^{(\mathrm{s})})$. □

From Propositions 3.1 and 3.2, we obtain the following two corollaries.

Corollary 3.3. *Let* $\lambda \in \mathbb{C} \setminus \sigma_{\mathrm{ess}}(\mathcal{A}_2)$. *Then* $\lambda \in \rho(\mathcal{A}_2) \Longleftrightarrow 0 \in \rho(S_1(\lambda))$.

Corollary 3.4. *Let* $\lambda_0 \in \mathbb{R} \setminus \sigma_{\mathrm{ess}}(\mathcal{A}_2)$. *If* $(\lambda_0; \lambda_0 + \gamma) \in \rho(\mathcal{A}_2)$ (*resp.* $(\lambda_0 - \gamma; \lambda_0) \in \rho(\mathcal{A}_2)$) *for some* $\gamma > 0$, *then there exists a number* $\delta = \delta(\gamma) > 0$ *such that* $(0; \delta) \in \rho(S_1(\lambda_0))$ (*resp.* $(-\delta; 0) \in \rho(S_1(\lambda_0))$).

The definition of the set $\sigma^{(\mathrm{s})}$ implies that the inequality $\Delta^{(\mathrm{s})}(\lambda - w(k_1)) > 0$ holds for all $k_1 \in \mathbb{T}^{\mathrm{d}}$ and $\lambda < E_{\min}^{(\mathrm{s})}$. Therefore, for such λ, the inclusion $\sigma_{\mathrm{ess}}(S_1^{(\mathrm{s})}(\lambda)) \subset (0; +\infty)$ holds.

For a bounded self-adjoint operator A acting in a Hilbert space $\mathcal{H}$ we denote by $N_{(-\infty;\lambda)}(A)$ the number of eigenvalues of A to the left of λ, $\lambda \leq \min \sigma_{\mathrm{ess}}(A)$.

Theorem 3.5. *For any* $\lambda < E_{\min}^{(\mathrm{s})}$, *the equality*

$$N_{(-\infty;\lambda)}(\mathcal{A}_2^{(\mathrm{s})}) = N_{(-\infty;0)}(S_1^{(\mathrm{s})}(\lambda))$$

holds.

Proof. For any $\lambda < E_{\min}^{(\mathrm{s})}$, the operator $\widehat{A}_{22}^{(\mathrm{s})} - \lambda$ is positive and invertible and hence the square root $(R_{22}^{(\mathrm{s})}(\lambda))^{1/2}$ of the resolvent $R_{22}^{(\mathrm{s})}(\lambda)$ of $\widehat{A}_{22}^{(\mathrm{s})}$ exists.

Let $V^{(\mathrm{s})}(\lambda)$, $\lambda < E_{\min}^{(\mathrm{s})}$ be the 3×3 block operator matrix in $\mathcal{F}_{\mathrm{b}}^{(2)}(L_2(\mathbb{T}^{\mathrm{d}}))$ with entries

$$\begin{aligned} &V_{00}^{(\mathrm{s})}(\lambda) := \widehat{A}_{00}^{(\mathrm{s})} - \lambda I_0, \quad V_{01}^{(\mathrm{s})}(\lambda) := \widehat{A}_{01}^{(\mathrm{s})}, \quad V_{02}^{(\mathrm{s})}(\lambda) := 0; \\ &V_{10}^{(\mathrm{s})}(\lambda) := \widehat{A}_{10}^{(\mathrm{s})}, \quad V_{11}^{(\mathrm{s})}(\lambda) := \widehat{A}_{11}^{(\mathrm{s})} - \lambda I_1, \quad V_{12}(\lambda) := \widehat{A}_{12}^{(\mathrm{s})} (R_{22}^{(\mathrm{s})}(\lambda))^{1/2}; \\ &V_{20}^{(\mathrm{s})}(\lambda) := 0, \quad V_{21}(\lambda) := (R_{22}^{(\mathrm{s})}(\lambda))^{1/2} \widehat{A}_{12}^{*}, \quad V_{22}^{(\mathrm{s})}(\lambda) := I_2, \end{aligned}$$

where $\mathrm{diag}\,\{I_0, I_1, I_2\}$ is an identity operator on $\mathcal{F}_{\mathrm{b}}^{(2)}(L_2(\mathbb{T}^{\mathrm{d}}))$. A simple calculation shows that $(\mathcal{A}_2^{(\mathrm{s})} f, f) < \lambda(f, f)$, $f = (f_0, f_1, f_2) \in \mathcal{F}_{\mathrm{b}}^{(2)}(L_2(\mathbb{T}^{\mathrm{d}}))$ if and only if $(V^{(\mathrm{s})}(\lambda) g, g) < 0$, $g = (f_0, f_1, (\widehat{A}_{22}^{(\mathrm{s})} - \lambda I_2)^{1/2} f_2)$. It follows that

$$N_{(-\infty;\lambda)}(\mathcal{A}_2^{(\mathrm{s})}) = N_{(-\infty;0)}(V^{(\mathrm{s})}(\lambda)). \tag{3.5}$$

For a bounded self-adjoint operator B acting in a Hilbert space $\mathcal{H}$, let us denote by $\mathcal{H}_B(\lambda) \subset \mathcal{H}$, $\lambda \in \mathbb{R}$, a subspace such that $(Bf, f) > \lambda \|f\|^2$ for any $f \in \mathcal{H}_B(\lambda)$. Assume $\widetilde{f} := (f_0, f_1) \in \mathcal{H}_{-S_1^{(\mathrm{s})}(\lambda)}(0)$, i.e., $(S_1^{(\mathrm{s})}(\lambda)\widetilde{f}, \widetilde{f}) < 0$. Then for any

$$g := (f_0, f_1, -V_{21}^{(\mathrm{s})}(\lambda) f_1) \in \mathcal{F}_{\mathrm{b}}^{(2)}(L_2(\mathbb{T}^{\mathrm{d}})),$$

we have

$$(V^{(\mathrm{s})}(\lambda) g, g) = (S_1^{(\mathrm{s})}(\lambda)\widetilde{f}, \widetilde{f}) < 0.$$

Therefore, $g \in \mathcal{H}_{-V^{(\mathrm{s})}(\lambda)}(0)$, and one has

$$N_{(-\infty;0)}(S_1^{(\mathrm{s})}(\lambda)) \le N_{(-\infty;0)}(V^{(\mathrm{s})}(\lambda)). \tag{3.6}$$

For any $\widetilde{f} := (f_0, f_1) \in \mathcal{F}_{\mathrm{b}}^{(1)}(L_2(\mathbb{T}^{\mathrm{d}}))$ and $f = (f_0, f_1, f_2) \in \mathcal{F}_{\mathrm{b}}^{(2)}(L_2(\mathbb{T}^{\mathrm{d}}))$ the equality

$$(S_1^{(\mathrm{s})}(\lambda)\widetilde{f}, \widetilde{f}) = (V^{(\mathrm{s})}(\lambda) f, f) - (V_{12}^{(\mathrm{s})}(\lambda) V_{21}^{(\mathrm{s})}(\lambda) f_1, f_1) - (V_{21}^{(\mathrm{s})}(\lambda) f_1, f_2) - (V_{12}^{(\mathrm{s})}(\lambda) f_2, f_1) - (f_2, f_2)$$

holds. Then we obtain

$$(S_1^{(\mathrm{s})}(\lambda)\widetilde{f}, \widetilde{f}) = (V^{(\mathrm{s})}(\lambda) f, f) - \|f_2 + V_{21}^{(\mathrm{s})} f_1\|^2 < 0$$

for all $f = (f_0, f_1, f_2) \in \mathcal{H}_{-V^{(\mathrm{s})}(\lambda)}(0)$, i.e. $\widetilde{f} \in \mathcal{H}_{-S_1^{(\mathrm{s})}(\lambda)}(0)$. Consequently,

$$N_{(-\infty;0)}(V^{(\mathrm{s})}(\lambda)) \le N_{(-\infty;0)}(S_1^{(\mathrm{s})}(\lambda)). \tag{3.7}$$

Now inequalities (3.6), (3.7) and equality (3.5) complete the proof. $\square$

By Theorem 3.5 and equality (2.1), we obtain

$$N_{(-\infty, E_{\min})}(\mathcal{A}_2) = \sum_{\mathrm{s}=\pm} N_{(-\infty,0)}(S_1^{(\mathrm{s})}(E_{\min})). \tag{3.8}$$

Note that the compact part $K^{(\mathrm{s})}(E_{\min})$ of $S_{11}^{(\mathrm{s})}(E_{\min})$ is positive. Indeed, taking into account the identity

$$\frac{\pi}{2x^2y^2(x^2+y^2)} = \int\limits_0^\infty \frac{d\xi}{(x^4+\xi^2)(y^4+\xi^2)}$$

and the inequality

$$w(k_1) + \frac{\mathrm{s}\varepsilon - E_{\min}}{2} > 0, \quad k_1 \in \mathbb{T}^{\mathrm{d}},$$

we represent the kernel $K^{(\mathrm{s})}(\cdot,\cdot)$ of the operator $K^{(\mathrm{s})}(E_{\min})$ in the form

$$\begin{aligned} K^{(\mathrm{s})}(k_1, t) \quad &= \frac{\alpha^2 v(k_1) v(t)}{\pi} \Big[w(k_1) + \frac{\mathrm{s}\varepsilon - E_{\min}}{2}\Big]\Big[w(t) + \frac{\mathrm{s}\varepsilon - E_{\min}}{2}\Big] \\ &\times \int\limits_0^\infty \Big(\Big[w(k_1) + \frac{\mathrm{s}\varepsilon - E_{\min}}{2}\Big]^2 + \xi^2\Big)^{-1}\Big(\Big[w(t) + \frac{\mathrm{s}\varepsilon - E_{\min}}{2}\Big]^2 + \xi^2\Big)^{-1} d\xi. \end{aligned}$$

Then for any $f_1 \in L_2(\mathbb{T}^{\mathrm{d}})$, we obtain

$$\langle K^{(\mathrm{s})}(E_{\min}) f_1, f_1\rangle = \frac{\alpha^2}{\pi} \int\limits_0^\infty \Big| \int\limits_{\mathbb{T}^{\mathrm{d}}} \frac{v(t)\Big[w(t) + \frac{\mathrm{s}\varepsilon - E_{\min}}{2}\Big]^{3/2} f_1(t)dt}{\Big[w(t) + \frac{\mathrm{s}\varepsilon - E_{\min}}{2}\Big]^2 + \xi^2}\Big|^2 d\xi \ge 0.$$

Therefore, $K^{(\mathrm{s})}(E_{\min}) \ge 0$.

Proposition 3.6. *The function $N_{(-\infty;0)}(S_1(\cdot))$ is monotonically increasing in $(-\infty; E_{\min})$.*

Proof. Since $N_{(-\infty;0)}(S_1(\lambda)) = N_{(-\infty;0)}(S_1^{(+)}(\lambda)) + N_{(-\infty;0)}(S_1^{(-)}(\lambda))$ for all $\lambda \in (-\infty; E_{\min})$, we show that the function $N_{(-\infty;0)}(S_1^{(\mathrm{s})}(\cdot))$, $\mathrm{s} = \pm$ is monotonically increasing in $(-\infty; E_{\min})$.

Let $\lambda_1, \lambda_2 \in (-\infty; E_{\min})$ be such that $\lambda_1 < \lambda_2$. Since for each $f_2 \in L_2^{\mathrm{sym}}((\mathbb{T}^{\mathrm{d}})^2)$ the function $(R_{22}^{(\mathrm{s})}(\cdot) f_2, f_2)$ is increasing in $(-\infty; E_{\min})$, we have

$$\begin{aligned} (S_1^{(\mathrm{s})}(\lambda_1)\widetilde{f}, \widetilde{f}) \quad &= ((\widehat{A}_{00}^{(\mathrm{s})} - \lambda_1) f_0 + \widehat{A}_{01} f_1, f_0) \\ &+ (\widehat{A}_{01}^* f_0 + (\widehat{A}_{11}^{(\mathrm{s})} - \lambda_1 - \widehat{A}_{12} R_{22}^{(\mathrm{s})}(\lambda_1) \widehat{A}_{12}^*) f_1, f_1) = ((\widehat{A}_{00}^{(\mathrm{s})} - \lambda_1) f_0, f_0) \\ &+ (\widehat{A}_{01} f_1, f_0) + (\widehat{A}_{01}^* f_0, f_1) + ((\widehat{A}_{11}^{(\mathrm{s})} - \lambda_1) f_1, f_1) - (R_{22}^{(\mathrm{s})}(\lambda_1) \widehat{A}_{12}^* f_1, \widehat{A}_{12}^* f_1) \\ &> ((\widehat{A}_{00}^{(\mathrm{s})} - \lambda_2) f_0, f_0) + (\widehat{A}_{01} f_1, f_0) + (\widehat{A}_{01}^* f_0, f_1) \\ &+ ((\widehat{A}_{11}^{(\mathrm{s})} - \lambda_2) f_1, f_1) - (R_{22}^{(\mathrm{s})}(\lambda_2) \widehat{A}_{12}^* f_1, \widehat{A}_{12}^* f_1) = (S_1^{(\mathrm{s})}(\lambda_2)\widetilde{f}, \widetilde{f}), \end{aligned}$$

where $\widetilde{f} := (f_0, f_1)$. From here, we obtain that if $\widetilde{f} \in \mathcal{H}_{S_1^{(s)}(\lambda_1)}(0)$, then $\widetilde{f} \in \mathcal{H}_{S_1^{(s)}(\lambda_2)}(0)$, that is, $\mathcal{H}_{S_1^{(s)}(\lambda_1)}(0) \subset \mathcal{H}_{S_1^{(s)}(\lambda_2)}(0)$ and hence, $N_{(-\infty;0)}(S_1^{(s)}(\lambda_1)) \leq N_{(-\infty;0)}(S_1^{(s)}(\lambda_2))$. Proposition 3.6 is completely proved. □

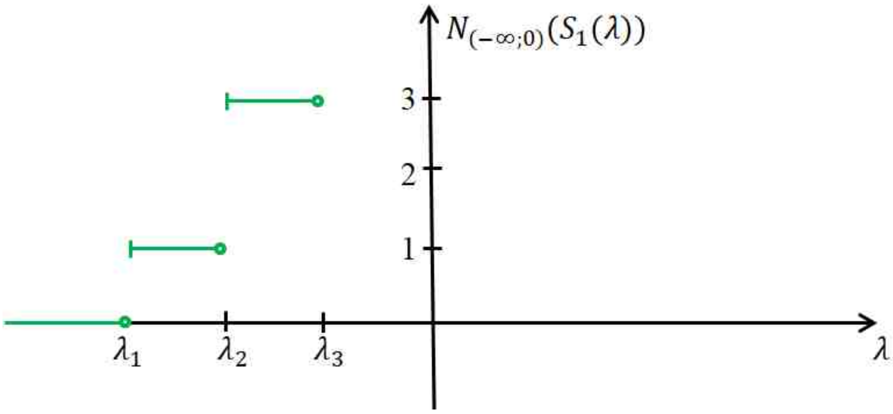

FIG. 1. The graph of $N_{(-\infty;0)}(S_1(\cdot))$ for the case λ_1 is a simple eigenvalue of $\mathcal{A}_2$ and λ_2 is an eigenvalue of $\mathcal{A}_2$ with multiplicity two.

Definition 3.7. *We denote by $E_m(\cdot)$, $m \in \mathbb{N}$ the positive definite function on the segment $[\alpha;\beta] \subset \mathbb{R} \setminus \sigma_{\rm ess}(\mathcal{A}_2)$, satisfying the condition: $E_m(\lambda)$ is the m-th eigenvalue (eigenvalues numbered in ascending order, counting their multiplicity) of the operator $S_1(\lambda)$, $\lambda \in [\alpha;\beta]$.*

Recall that the operator function $S_1(\cdot)$ is continuous on $[\alpha;\beta]$ in the sense of the uniform operator topology. Therefore for each $m \in \mathbb{N}$, the function $E_m(\cdot)$ is continuous in any segment $[\alpha;\beta] \subset \mathbb{R} \setminus \sigma_{\rm ess}(\mathcal{A}_2)$.

Theorem 3.8. *The number $\lambda_0 < E_{\min}$ is the regular point of the operator $\mathcal{A}_2$ if and only if the function $N_{(-\infty;0)}(S_1(\cdot))$ is continuous at point $\lambda = \lambda_0$.*

Proof. **Necessity.** Let $\lambda_0 < E_{\min}$ be the regular point of the operator $\mathcal{A}_2$. Then from Proposition 3.1, it follows that $E_n(\lambda_0) \neq 0$, $n \in \mathbb{N}$.

Due to the continuity of the function $E_n(\cdot)$, there exists the number $\rho > 0$ such that for all $n \in \mathbb{N}$ and $\lambda \in [\lambda_0 - \rho; \lambda_0 + \rho] \subset (-\infty; E_{\min})$, the inequality $E_n(\lambda) \neq 0$ holds. From here, we obtain

$$\mathrm{card}\{n : E_n(\lambda_0) < 0\} = \mathrm{card}\{n : E_n(\lambda) < 0,\ \lambda \in [\lambda_0 - \rho; \lambda_0 + \rho]\},$$

where $\mathrm{card}M$ is the cardinality of the set M. Therefore, $N_{(-\infty;0)}(S_1(\lambda_0)) = N_{(-\infty;0)}(S_1(\lambda_0 + \varepsilon))$, $\varepsilon \in [-\rho;\rho]$, that is, the function $N_{(-\infty;0)}(S_1(\cdot))$ is continuous at the point $\lambda = \lambda_0$.

Sufficiency. Let the function $N_{(-\infty;0)}(S_1(\cdot))$ be the continuous one at the point $\lambda = \lambda_0$. Then for some $\varepsilon > 0$, the equalities

$$N_{(-\infty;0)}(S_1(\lambda_0 - \varepsilon)) = N_{(-\infty;0)}(S_1(\lambda_0)) = N_{(-\infty;0)}(S_1(\lambda_0 + \varepsilon)) \tag{3.9}$$

hold.

Taking into account the monotonicity of the quadratic form $(S_1(\lambda)\cdot,\cdot)$ as $\lambda \in (\lambda_0 - \varepsilon; \lambda_0 + \varepsilon)$ and arguing as in the proving of Theorem 2 of the paper [16], one can show that the function $E_1(\cdot), E_2(\cdot), \ldots$ monotonically decreases on $(\lambda_0 - \varepsilon; \lambda_0 + \varepsilon)$. From here, we have $E_n(\lambda_0) > E_n(\lambda_0 + \varepsilon)$, $n \in \mathbb{N}$. Thus,

$$E_n(\lambda_0) > E_n(\lambda_0 + \varepsilon) \geq 0, \quad \forall n \in \{n : E_n(\lambda_0 - \varepsilon) \geq 0\}.$$

By virtue of (3.9), the following equality

$$\{n : E_n(\lambda_0 + \varepsilon) < 0\} = \{n : E_n(\lambda_0 - \varepsilon) < 0\}$$

holds. Therefore,

$$E_n(\lambda_0) < E_n(\lambda_0 - \varepsilon) < 0, \quad \forall n \in \{n : E_n(\lambda_0 + \varepsilon) < 0\}.$$

In such a way for every natural $n \in \mathbb{N}$, we obtain the inequality $E_n(\lambda_0) \neq 0$, that is, the number 0 is not an eigenvalue of the operator $S_1(\lambda_0)$. According to Proposition 3.1, the number λ_0 is regular point of the operator $\mathcal{A}_2$. Theorem 3.8 is completely proved. □

References

[1] Leggett A.J., Chakravarty S., Dorsey A.T., Fisher M.P.A., Garg A., Zwerger W. Dynamics of the dissipative two-state system. *Rev. Mod. Phys.*, **59**, (Jan 1987), P. 1–85.

[2] Hübner M., Spohn H. Radiative decay: nonperturbative approaches. *Rev. Math. Phys.*, 1995, **7**(3), P. 363–387.

[3] Huebner M., Spohn H. Spectral properties of spin-boson Hamiltonian. *Ann. Inst. Henri Poincaré*, 1995, **62**(3), P. 289–323.

[4] Minlos R.A., Spohn H. The three-body problem in radioactive decay: the case of one atom and at most two photons. Topics in Statistical and Theoretical Physics, American Mathematical Society Translations–Series 2, 1996, **177**, P. 159–193.

[5] Spohn H. Ground states of the spin-boson Hamiltonian. *Comm. Math. Phys.*, 1989, **123**, P. 277–304.

[6] Zhukov Yu., Minlos R. Spectrum and scattering in a "spin-boson" model with not more than three photons. *Theor. Math. Phys.*, 1995, **103**(1), P. 398–411.

[7] Muminov M., Neidhardt H., Rasulov T. On the spectrum of the lattice spin-boson Hamiltonian for any coupling: 1D case. *Journal of Mathematical Physics*, 2015, **56**, P. 053507.

[8] Rasulov T.Kh. Branches of the essential spectrum of the lattice spin-boson model with at most two photons. *Theor. Math. Phys.*, 2016, **186**(2), P. 251–267.

[9] Lakaev S.N., Rasulov T.Kh. A model in the theory of perturbations of the essential spectrum of multi-particle operators. *Math. Notes.*, 2003, **73**(4), P. 521–528.

[10] Tretter C. *Spectral Theory of Block Operator Matrices and Applications*. Imperial College Press, London, 2008.

[11] Lakaev S.N. Some spectral properties of the generalized Friedrichs model. *J. Soviet Math.*, 1989, **45**(6), P. 1540–1565.

[12] Muminov M.I., Rasulov T.H. The Faddeev equation and essential spectrum of a Hamiltonian in Fock space. *Methods Func. Anal. Topology*, 2011, **17**(1), P. 47–57.

[13] Rasulov T.Kh. Study of the essential spectrum of a matrix operator. *Theor. Math. Phys.*, 2010, **164**(1), P. 883–895.

[14] Zhang F. T*he Schur Complement and Its Applications. Numerical Methods and Algorithms*. Vol. 4. Springer, Berlin, 2005.

[15] Haynsworth E.V. *On the Schur complement*. Basel Mathematical Notes, BNB 20, 17 pages, 1968.

[16] Muminov M.I., Rasulov T.Kh. An eigenvalue multiplicity formula for the Schur complement of a 3×3 block operator matrix. *Siberian Math. J.*, 2015, **56**(4), P. 878–895.

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