RESEARCH ARTICLE | SEPTEMBER 13 2023

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AIP Conf. Proc. 2764, 030004 (2023)
https://doi.org/10.1063/5.0144151

Discrete Eigenvalues of a 2×2 Operator Matrix

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Abstract. We discuss a block operator matrix $\mathscr{A}_\mu$ ($\mu > 0$ is a coupling constant) of order 2 acting in a direct sum of one- and two-particle subspaces of the bosonic Fock space. Position of $\sigma_{\text{ess}}(\mathscr{A}_\mu)$ is demonstrated and its bounds are evaluated. It is shown the existence of the critical values $\mu_l^0(\gamma)$ with $\gamma > 0$ and $\mu_r^0(\gamma)$ with $\gamma < 12$ of the coupling constant $\mu > 0$ for all $\gamma > 0$ ($\gamma < 12$) so that the matrix $\mathscr{A}_\mu$ with $\mu = \mu_l^0(\gamma)$ ($\mu = \mu_r^0(\gamma)$ has infinitely many eigenvalues on the l.h.s. (r.h.s.) of the its main spectrum. We demonstrate that for all $\mu \notin \{\mu_l^0(\gamma), \mu_r^0(\gamma)\}$ the matrix $\mathscr{A}_\mu$ and has a finitely many discrete eigenvalues on the l.h.s. and r.h.s. of its essential spectrum.

INTRODUCTION.

One of the most important result for standard (continuous) and discrete three-particle Schrödinger operators is the *Efimov effect*. In [1] this effect was first stated by V.Efimov for continuous case. In [2] Yafaev give a rigorous mathematical proof of the existence Efimov's effect for continuous case and then many authors are studied this subject, see for instance [3, 4, 5, 6].

Sobolev's investigation [6] is an asymptotics of the form $\mathscr{U}_0|\log|\lambda||$ for the number $N(\lambda)$ of eigenvalues on the left of λ, $\lambda < 0$, where the coefficient $\mathscr{U}_0$ is independent on the two-particle potentials v_α for continuous case.

In the problems of modern mathematical physics [7, 8, 9] arise the spectral problems related with the discrete Schrödinger operators. The existence of the Efimov effect for lattice case was established in [10, 11, 12, 13].

In [10, 11], an asymptotics for lattice case analogous to [6] for the number of eigenvalues $N(\lambda)$ was found.

All of the above-mentioned articles are devoted to Efimov's effect, systems with a conserved bounded number of particles have been studied. However, the problems statistical physics [9, 14], solid-state physics [8] and the theory of quantum fields [15] involve the systems with bounded numbers of particles that aren't conserved. The study of these systems' spectral qualities simplifies to the study of self-adjoint operator matrices' spectral properties [16] acting in truncated subspace $\mathscr{H}^{(n)}$, consisting of one-particle, two-particle and n-particle subspaces of the Fock space [8, 14].

The presence of Efimov's effect for the operator matrices in relation to the energy operator of a system of three non conserved number of particles was proved in [17, 18, 19, 20, 21, 22, 23]. Especially, the asymptotic formula for the number of eigenvalues were studied in [17, 20, 21, 23]. Furthermore, the infiniteness of the number of eigenvalues placed inside, in the gap, and below the bottom of the essential spectrum of the associated operator matrices was demonstrated in several natural settings [20, 21].

In the article we have dealt with discrete spectrum analysis for the 2×2 matrix $\mathscr{A}_\mu$ acting in the direct sum of one- and two-particle subspaces of a bosonic Fock space. In this case, the functions with spectral parameter $\gamma \in \mathbf{R}$ of the diagonal elements of $\mathscr{A}_\mu$ has the special forms. We demonstrate the existence of important values of $\mu_l^0(\gamma)$ with $\gamma > 0$ and $\mu_r^0(\gamma)$ with $\gamma < 12$ of the parameter $\mu > 0$ such that only for $\mu = \mu_l^0(\gamma)$ (resp. $\mu = \mu_r^0(\gamma)$) the matrix $\mathscr{A}_\mu$ has an infinitely many eigenvalues lying on the l.h.s. of 0 accumulating to 0 for all $\gamma > 0$ (resp. r.h.s. of 18 accumulating to 18 for all $\gamma < 12$). In the condition of $\gamma = 6$ it is proved that there are a lot of eigenvalues located in the both sides of the essential spectrum of $\mathscr{A}_\mu$ (so-called **two-sided Efimov's effect**). For the number $N_{(a;b)}(\mathscr{A}_\mu)$ of eigenvalue of $\mathscr{A}_\mu$ located in $(a;b) \subset \mathbf{R} \setminus \sigma_{\text{ess}}(\mathscr{A}_\mu)$ we establish

$$\lim_{z \nearrow 0} \frac{N_{(-\infty;z)}(\mathscr{A}_{\mu_l^0(\gamma)})}{|\log|z||} = \mathscr{U}_0 \quad \text{with} \quad \gamma > 0;$$

$$\lim_{z \searrow 18} \frac{N_{(z;+\infty)}(\mathscr{A}_{\mu_r^0(\gamma)})}{|\log|z-18||} = \mathscr{U}_0 \quad \text{with} \quad \gamma < 12;$$

for some $\mathscr{U}_0 \in (0;+\infty)$. It is shown that for $\gamma = 6$ the equality $\mu_0 := \mu_l^0(\gamma) = \mu_r^0(\gamma)$ holds and

$$\lim_{z \nearrow 0} \frac{N_{(-\infty;z)}(\mathscr{A}_{\mu_0})}{|\log|z||} = \lim_{z \searrow 18} \frac{N_{(z;+\infty)}(\mathscr{A}_{\mu_0})}{|\log|z-18||} = \mathscr{U}_0. \tag{1}$$

Advanced Technologies in Chemical, Construction and Mechanical Sciences
AIP Conf. Proc. 2764, 030004-1–030004-10; https://doi.org/10.1063/5.0144151
Published by AIP Publishing. 978-0-7354-4650-2/$30.00

We show that for $\mu \notin \{\mu_l^0(\gamma), \mu_r^0(\gamma)\}$ the sets

$$(-\infty; \min \sigma_{\text{ess}}(\mathscr{A}_\mu)) \cap \sigma_{\text{disc}}(\mathscr{A}_\mu) \text{ and } (\max \sigma_{\text{ess}}(\mathscr{A}_\mu); +\infty) \cap \sigma_{\text{disc}}(\mathscr{A}_\mu)$$

are finite. The case $\gamma = 6$ is also analyzed in [24].

It's worth noting that claim (1) appears to be a first for lattice operator matrices in a bosonic Fock space. This assumption is typical for the lattice situation; in fact, there are no equivalents in standard (continuous) case.

STATEMENT OF THE PROBLEM.

Let us denote by $\mathbf{T}^3$ the 3D torus and X be the direct sum of complex Hilbert spaces $X := L_2(\mathbf{T}^3)$ and $X := L_2^s((\mathbf{T}^3)^2)$, that is, $X := X_1 \oplus X_2$.

In this article, we investigate the matrix

$$\mathscr{A}_\mu := \begin{pmatrix} A_{11} & \mu A_{12} \\ \mu A_{12}^* & A_{22} \end{pmatrix}$$

in X with components $A_{lk} : X_k \to X_l$, $l \leq k$, $l,k = 1,2$:

$$(A_{11}g_1)(x) = u_1(x)g_1(x), \quad (A_{12}g_2)(x) = \int_{\mathbf{T}^3} g_2(x,s)ds,$$
$$(A_{22}g_2)(x,y) = u_2(x,y)g_2(x,y), \quad x_l \in X_l, \quad l = 1,2,$$

where by A_{12}^* we denote the operator adjoint to A_{12}, i.e.

$$(A_{12}^* g_1)(x,y) = (g_1(x) + g_1(y))/2, \quad g_1 \in X_1.$$

Here $\mu > 0$; the functions $u_1(\cdot)$ and $u_2(\cdot,\cdot)$ are defined as follows

$$u_1(x) := \varepsilon(x) + \gamma, \quad u_2(x,y) := \varepsilon(x) + \varepsilon((x+y)/2) + \varepsilon(y)$$

where $\gamma \in \mathbf{R}$ and the dispersion function $\varepsilon(\cdot)$ has the form

$$\varepsilon(x) := \sum_{i=1}^{3} (1 - \cos x_l), \, x = (x_1, x_2, x_3) \in \mathbf{T}^3. \tag{2}$$

Using the corresponding definitions we may prove the boundedness and self-adjointness of the matrix $\mathscr{A}_\mu$.

ON THE ESSENTIAL AND DISCRETE SPECTRU OF THE GENERALIZED FRIEDRICHS MODEL.

The section is devoted to the spectral analysis to the so called the family of generalized Fridrichs models $\mathscr{A}_\mu(x)$, $x \in \mathbf{T}^3$, defined below.

We determine a family of matrices $\mathscr{A}_\mu(x) : X_0 \oplus X_1 \to X_0 \oplus X_1$, $x \in \mathbf{T}^3$ by

$$\mathscr{A}_\mu(x) := \begin{pmatrix} A_{00}(x) & \mu A_{01} \\ \mu A_{01}^* & A_{11}(x) \end{pmatrix},$$

with the following components:

$$A_{00}(x)g_0 = u_1(x)g_0, \, A_{01}g_1 = \frac{1}{\sqrt{2}} \int_{\mathbf{T}^3} g_1(s)ds,$$
$$(A_{11}(x)g_1)(y) = u_2(x,y)g_1(y), \quad g_l \in X_l, \quad l = 0,1.$$

Assume $\mathscr{A}_0(x) := \mathscr{A}_\mu(x)|_{\mu=0}$. It is obvious that $(\mathscr{A}_\mu(x) - \mathscr{A}_0(x))^* = \mathscr{A}_\mu(x) - \mathscr{A}_0(x)$ and $\dim\mathrm{Im}(\mathscr{A}_\mu(x) - \mathscr{A}_0(x)) = 2$. The using the Weyl theorem on the essential spectrum we obtain $\sigma_{\mathrm{ess}}(\mathscr{A}_\mu(x)) = [m(x);M(x)]$, where

$$m(x) := \min_{y\in\mathbf{T}^3} u_2(x,y), \quad M(x) := \max_{y\in\mathbf{T}^3} u_2(x,y). \tag{3}$$

From the construction of $u_2(\cdot,\cdot)$ we get that this function has a unique non-degenerate global min (respectively max) at the point $(\overline{0},\overline{0}) \in (\mathbf{T}^3)^2$ (respectively $(\overline{\pi},\overline{\pi}) \in (\mathbf{T}^3)^2$) and

$$\min_{x,y\in\mathbf{T}^3} u_2(x,y) = u_2(\overline{0},\overline{0}) = 0, \quad \max_{x,y\in\mathbf{T}^3} u_2(x,y) = u_2(\overline{\pi},\overline{\pi}) = 18,$$

where $\overline{0} := (0,0,0)$, $\overline{\pi} := (\pi,\pi,\pi) \in \mathbf{T}^3$. It is easy to see that

$$\sigma_{\mathrm{ess}}(\mathscr{A}_\mu(\overline{0})) = [0;9\frac{3}{8}]; \quad \sigma_{\mathrm{ess}}(\mathscr{A}_\mu(\overline{\pi})) = [8\frac{5}{8};18]$$

and

$$\min\{\sigma_{\mathrm{ess}}(\mathscr{A}_\mu(x)) : x \in \mathbf{T}^3\} = 0, \quad \max\{\sigma_{\mathrm{ess}}(\mathscr{A}_\mu(x)) : x \in \mathbf{T}^3\} = 18$$

In order to study the eigenvalues of $\mathscr{A}_\mu(x)$, for each fixed $x \in \mathbf{T}^3$ we determine

$$I(x;z) := \int_{\mathbf{T}^3} \frac{ds}{u_2(x,s) - z}$$

in $\mathbf{C} \setminus \sigma_{\mathrm{ess}}(\mathscr{A}_\mu(x))$.

It should be noted that the function

$$\Delta_\mu(x;z) := u_1(x) - z - \frac{\mu^2}{2} I(x;z),\ z \in \mathbf{C} \setminus \sigma_{\mathrm{ess}}(\mathscr{A}_\mu(x))$$

is usually called Fredholm's determinant related with the matrix $\mathscr{A}_\mu(x)$.

It is clear that [25] for any $\mu > 0$ and $x \in \mathbf{T}^3$ the quantity $z_\mu(x) \in \mathbf{C} \setminus [m(x);M(x)]$ is a discrete eigenvalue of the matrix $\mathscr{A}_\mu(x)$ iff $\Delta_\mu(x;z_\mu(x)) = 0$. Therefore, the following equality

$$\sigma_{\mathrm{disc}}(\mathscr{A}_\mu(x)) = \{z \in \mathbf{C} \setminus [m(x);M(x)] : \Delta_\mu(x;z) = 0\} \tag{4}$$

is true.

BOUNDS OF THE ESSENTIAL SPECTRUM OF $\mathscr{A}_\mu$.

Now we learn the position of the subsets so called two-particle and three-particle branches of $\sigma_{\mathrm{ess}}(\mathscr{A}_\mu)$ and estimate its bounds.

Put

$$\Lambda_\mu := \bigcup_{x\in\mathbf{T}^3} \sigma_{\mathrm{disc}}(\mathscr{A}_\mu(x)), \quad \Sigma_\mu := [0;18] \cup \Lambda_\mu.$$

We recall that

$$\bigcup_{x\in\mathbf{T}^3} \sigma_{\mathrm{ess}}(\mathscr{A}_\mu(x)) = [0;18].$$

Theorem 1. We have $\sigma_{\mathrm{ess}}(\mathscr{A}_\mu) = \Sigma_\mu$. In addition, the set Σ_μ consists of no more than 3 segments.

For the proof of Theorem 1 we refer the reader to [10].

Next, we determine new subsets (branches) of the set $\sigma_{\mathrm{ess}}(\mathscr{A}_\mu)$: We call the set $\Lambda_\mu =: \sigma_{\mathrm{two}}(\mathscr{A}_\mu)$ as the two-particle branche of $\sigma_{\mathrm{ess}}(\mathscr{A}_\mu)$ and $[0;18] =: \sigma_{\mathrm{three}}(\mathscr{A}_\mu)$ the three-particle branch of $\sigma_{\mathrm{ess}}(\mathscr{A}_\mu)$.

The properties of $u_2(\cdot,\cdot)$, yields that the integral $I(\overline{0};0)$ is convergent, see [25]. We determine the following bounds

$$E_\mu^{(1)} := \min\left\{\Lambda_\mu \cap (-\infty;0]\right\} \quad \text{for } \mu \geq \mu_l^0(\gamma);$$
$$E_\mu^{(2)} := \max\left\{\Lambda_\mu \cap [18;\infty)\right\} \quad \text{for } \mu \geq \mu_r^0(\gamma);$$

where

$$\mu_l^0(\gamma) := \sqrt{2\gamma/I(\overline{0},0)} \quad \text{for } \gamma > 0;$$
$$\mu_r^0(\gamma) := \sqrt{(24-2\gamma)/I(\overline{0},0)} \text{ for } \quad \gamma < 12.$$

The following result about the upper bound of $\sigma_{\text{ess}}(\mathscr{A}_\mu)$.

Theorem 2. (A) Suppose $\gamma < 12$.
(A_1) The equality $\max \sigma_{\text{ess}}(\mathscr{A}_\mu) = 18$ is valid for all $\mu \in (0; \mu_r^0(\gamma)]$;
(A_2) If $\mu > \mu_r^0(\gamma)$, then we have $\max \sigma_{\text{ess}}(\mathscr{A}_\mu) = E_\mu^{(2)}$ with $E_\mu^{(2)} > 18$;
(B) In the case $\gamma \geq 12$ the equality $\max \sigma_{\text{ess}}(\mathscr{A}_\mu) = E_\mu^{(2)}$ is true for all $\mu > 0$ and $E_\mu^{(2)} > 18$.

Proof. (A) Let us consider the case $\gamma < 12$. By the construction of $\Delta_\mu(x;z)$ we $\lim\limits_{z\to+\infty} \Delta_\mu(x;z) = -\infty$ for all $x \in \mathbf{T}^3$.

(A_1) Suppose that $0 < \mu \leq \mu_r^0(\gamma)$. If $z > 18$ and $x \in \mathbf{T}^3$, then $\Delta_\mu(x;z) < 0$. From the monotonicity of $\Delta_\mu(x;\cdot)$ on $(18;+\infty)$, using (4) we obtain an absence of the discrete eigenvalues of the matrix $\mathscr{A}_\mu(x)$ bigger than 18 for any $x \in \mathbf{T}^3$. Then by Theorem 1 we receive $\max \sigma_{\text{ess}}(\mathscr{A}_\mu) = 18$.

(A_2) Assume $\mu > \mu_r^0(\gamma)$. Then $\Delta_\mu(\overline{\pi};18) > 0$. From the monotonicity of $\Delta_\mu(x;\cdot)$ on $(18;+\infty)$ and from the assertion $\lim\limits_{z\to+\infty} \Delta_\mu(x;z) = -\infty$ we obtain the existence of $z_\mu^0 \in (18;+\infty)$ so that $\Delta_\mu(\overline{\pi};z_\mu^0) = 0$. Therefore, z_μ^0 is an eigenvalue of the operator $\mathscr{A}_\mu(\overline{\pi})$. It means that $\Lambda_\mu \bigcap (18;+\infty) \neq \emptyset$. Now Theorem 1 yields $\max \sigma_{\text{ess}}(\mathscr{A}_\mu) = E_\mu^{(2)}$ with $E_\mu^{(2)} > 18$.

(B) If $\gamma \geq 12$, then the inequality $\Delta_\mu(\overline{\pi};18) > 0$ is valid for any $\mu > 0$. Similarly to (A_2) we have $\max \sigma_{\text{ess}}(\mathscr{A}_\mu) = E_\mu^{(2)}$ with $E_\mu^{(2)} > 18$.

We formulate next main result of this note.

Theorem 3. (A) Assume $\gamma \leq 0$. The equality $\min \sigma_{\text{ess}}(\mathscr{A}_\mu) = E_\mu^{(1)}$ is true for any $\mu > 0$ with $E_\mu^{(1)} < 0$.
(B) Let $\gamma > 0$.
(B_1) If $\mu \in (0; \mu_l^0(\gamma)]$, then we have $\min \sigma_{\text{ess}}(\mathscr{A}_\mu) = 0$;
(B_2) Assume $\mu > \mu_l^0(\gamma)$. Then the equality $\min \sigma_{\text{ess}}(\mathscr{A}_\mu) = E_\mu^{(1)}$ is true with $E_\mu^{(1)} < 0$.

We may prove Theorem 3 similarly to Theorem 2.

BIRMAN-SCHWINGER PRINCIPLE.

For $\Omega \subset \mathbf{R}$, by $F_\Omega(\mathscr{A}_\mu)$ we set the spectral subspace of $\mathscr{A}_\mu$ that correspond to Ω. Let

$$N_{(a;b)}(\mathscr{A}_\mu) := \dim F_{(a;b)}(\mathscr{A}_\mu)X;$$

$$n(\lambda, \mathscr{A}_\mu) := \sup\{\dim E : (\mathscr{A}_\mu u, u) > \lambda, u \in E \subset X, ||u|| = 1\}.$$

If $\lambda < \max \sigma_{\text{ess}}(\mathscr{A}_\mu)$, then $n(\lambda, \mathscr{A}_\mu) = \infty$; if $n(\lambda, \mathscr{A}_\mu) < \infty$, then it is equal to the number of eigenvalues $\mathscr{A}_\mu$ greater than λ, taking into account their multiplicities.

From the determination of the quantity $N(a;b)(\mathscr{O}_\mu)$ we get that

$$N_{(z;\infty)}(\mathscr{A}_\mu) = n(z;\mathscr{A}_\mu), \ z > E_\mu^{(2)};$$
$$N_{(-\infty;z)}(\mathscr{A}_\mu) = n(-z;-\mathscr{A}_\mu), \ -z > -E_\mu^{(1)}.$$

It follows from positivity of the function $\Delta_\mu(x;z)$ (respectively, $-\Delta_\mu(x;z)$) for all $x \in \mathbf{T}^3$ and $z < E_\mu^{(1)}$ (respectively $z > E_\mu^{(2)}$), there exist its positive square.

When we study the set $\sigma_{\text{disc}}(\mathscr{A}_\mu)$ an important role is play the following Fredholm operators:

$$(T_\mu(z)f)(x) = -\frac{\mu^2}{2\sqrt{-\Delta_\mu(x;z)}}\int_{\mathbf{T}^3}\frac{f(s)ds}{\sqrt{-\Delta_\mu(s;z)}(u_2(x,s)-z)} \quad \text{for} \quad z > E_\mu^{(2)};$$
$$(T_\mu(z)f)(x) = \frac{\mu^2}{2\sqrt{\Delta_\mu(x;z)}}\int_{\mathbf{T}^3}\frac{f(s)ds}{\sqrt{\Delta_\mu(s;z)}(u_2(x,s)-z)} \quad \text{for} \quad z < E_\mu^{(1)}$$

in $L_2(\mathbf{T}^3)$. For $z \in \mathbf{R}\setminus[E_\mu^{(1)};E_\mu^{(2)}]$ they are compact integral operators.

Now we formulate one of the main lemma (Birman-Schwinger's principle) of the present article.

Lemma 1. For any $z \in \mathbf{R}\setminus[E_\mu^{(1)};E_\mu^{(2)}]$ the integral operator $T_\mu(z)$ is completely continuous operator and continuous in z and

$$N_{(z;\infty)}(\mathscr{A}_\mu) = n(1,T_\mu(z)) \quad \text{for} \quad z > E_\mu^{(2)};$$
$$N_{(-\infty;z)}(\mathscr{A}_\mu) = n(1,T_\mu(z)) \quad \text{for} \quad z < E_\mu^{(1)}.$$

Proof. We write the matrix $\mathscr{A}_\mu$ in the form

$$\mathscr{A}_\mu = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} + \mu \begin{pmatrix} 0 & A_{12} \\ A_{12}^* & 0 \end{pmatrix}.$$

On complex Hilbert space $X_i, i=1,2$ by $I_i, i=1,2$ we denote the identity operator and put $I = \text{diag}\{I_1, I_2\}$.

From the positivity and invertibility of $A_{ii} - zI_i, i=1,2$ for $z < E_\mu^{(1)}$ we receive the existence of a square root $R_{ii}^{1/2}(z)$ of the resolvent $R_{ii}(z) = (A_{ii} - zI_i)^{-1}$ of A_{ii}, $i=1,2$.

We determine the components $M_{\alpha\beta}(z)$ of $\mathscr{M}(z)$, $z < E_\mu^{(1)}$ as

$$M_{\alpha\alpha}(z) = 0, \quad \alpha = 1,2, \quad M_{12}(z) := -R_{11}^{1/2}(z)A_{12}R_{22}^{1/2}(z), \quad M_{21}(z) := M_{12}^*(z).$$

The inequality $((\mathscr{A}_\mu - zI)g, g) < 0$, $g = (g_1, g_2) \in X$ is valid iff the inequality $((\mu\mathscr{M}(z) - I)f, f) > 0$, $f = (f_1, f_2)$ is valid with $f_i = (A_{ii} - zI_i)^{1/2}g_i$, $i = 1,2$.

As conclusion from above we get

$$N_{(-\infty;z)}(\mathscr{A}_\mu) = n(1, \mu\mathscr{M}(z)). \tag{5}$$

Further for $z < E_\mu^{(1)}$ we determine the operator $V_\mu(z) := \mu^2 M_{12}(z)M_{21}(z)$ in X. Define a subspace $E \subset X_1$ from the condition $\dim E = n(1, V(z))$. In that case for all elements $g_1 \in E$ and $f = (g_1, \mu M_{21}(z)g_1)$ we get

$$((\mu\mathscr{M}(z) - I)f, f) = ((V_\mu(z) - I_1)g_1, g_1)$$

From here

$$n(1, \mu\mathscr{M}(z)) = n(1, V_\mu(z)). \tag{6}$$

The statement $((V_\mu(z) - I_1)\psi, \psi) > 0$, $\psi \in X_1$ is valid iff

$$((A_{11} - zI_1)\varphi, \varphi) < (A_{12}R_{22}(z)A_{12}^*\varphi, \varphi)$$

is valid for $\varphi = R_{11}^{1/2}(z)\psi$. It implies

$$n(1, V_\mu(z)) = n(-z, G_\mu(z)), \tag{7}$$

where $G_\mu(z) := \mu^2 A_{12}R_{22}(z)A_{12}^* - A_{11}$.

If we determine the operators B_1 and B_2 acting from $L_2(\mathbf{T}^3)$ to $L_2((\mathbf{T}^3)^2)$ as

$$(B_1 g_1)(x,y) = \frac{1}{2}g_1(y), \quad (B_2 g_1)(x,y) = \frac{1}{2}g_1(x), \quad f_1 \in L_2(\mathbf{T}^3),$$

then $A_{12}^* = B_1 + B_2$. The operator $D_\mu(z) := A_{11} - z - \mu^2 A_{12} R_{22}(z) B_2$ is invertible, if $z < E_\mu^{(1)}$. Because this is the multiplication operator by the positive function $\Delta_\mu(\cdot;z)$ on $\mathbf{T}^3$. By construction we have $(D_\mu^{-1/2}(z)f)(x) = \Delta_\mu^{-1/2}(x;z)f(x)$.

Summarizing we conclude that $(G_\mu(z)\psi,\psi) > -z(\psi,\psi)$ is true iff $(T_\mu(z)\zeta,\zeta) > (\zeta,\zeta)$ is valid for $\zeta = D_\mu^{1/2}(z)\psi$. So

$$n(-z, G_\mu(z)) = n(1, T_\mu(z)). \tag{8}$$

Taking into account (5), (6), (7) and (8) we receive $N_{(-\infty;z)}(\mathscr{A}_\mu) = n(1, T_\mu(z))$.

From above mentioned assertions we obtain completely continuity and continuity in z of $T_\mu(z)$ for $z < E_\mu^{(1)}$.

The same conclusion can be drawn for the case $z > E_\mu^{(2)}$.

FINITENESS OF $\sigma_{\text{disc}}(\mathscr{A}_\mu)$.

Here, we discuss the finiteness of the subset of $\sigma_{\text{disc}}(\mathscr{A}_\mu)$ located in $(-\infty; E_\mu^{(1)})$ and $(E_\mu^{(2)}; +\infty)$, respectively.

Next we assume that the values of $C_1, C_2, C_3 > 0$ are different and for $\delta >$ put

$$U_\delta(x_0) := \{x \in \mathbf{T}^3 : |x - x_0| < \delta\}$$

with $x_0 \in \mathbf{T}^3$.

Lemma 2. For some positive quantities C_1, C_2, C_3, δ we have
(A) $C_1(|x|^2 + |y|^2) \le u_2(x,y) \le C_2(|x|^2 + |y|^2)$, $x, y \in U_\delta(\overline{0})$;
(B) $C_1(|x - \overline{\pi}|^2 + |y - \overline{\pi}|^2) \le 18 - u_2(x,y) \le C_2(|x - \overline{\pi}|^2 + |y - \overline{\pi}|^2)$, $x, y \in U_\delta(\overline{\pi})$;
(C) $u_2(x,y) \ge C_3$, $(x,y) \notin U_\delta(\overline{0}) \times U_\delta(\overline{0})$;
(D) $18 - u_2(x,y) \ge C_3$, $(x,y) \notin U_\delta(\overline{\pi}) \times U_\delta(\overline{\pi})$.

Proof. We extend the function $u_2(\cdot,\cdot)$ to the Taylor series at the point $(\overline{0},\overline{0}) \in (\mathbf{T}^3)^2$ $((\overline{\pi},\overline{\pi}) \in (\mathbf{T}^3)^2)$ of a unique global non-degenerate min (max). For this reason we may fine positive quantities C_1, C_2, C_3, δ, so (A) - (D) are valid.

From the determination of $\Delta_\mu(\cdot;\cdot)$ for $\mu > \mu_r^0(\gamma)$ with $\gamma < 12$ the function $\Delta_\mu(\cdot, E_\mu^{(2)})$ is an analytic function on the compact set $\mathbf{T}^3$. Therefore, the set $\{x \in \mathbf{T}^3 : \Delta_\mu(x, E_\mu^{(2)}) = 0\}$ is finite.

Denote

$$\{x \in \mathbf{T}^3 : \Delta_\mu(x, E_\mu^{(2)}) = 0\} = \{x_1, x_2, \ldots, x_N\},\ N < \infty.$$

Lemma 3. Let $i \in \{1, 2, \ldots, N\}$ and $\mu > \mu_r^0(\gamma)$ with $\gamma < 12$. There are some $C_1, C_2, C_3, \delta > 0$ with

$$C_1|x - x_i|^2 \le |\Delta_\mu(x; E_\mu^{(2)})| \le C_2|x - x_i|^2, \quad x \in U_\delta(x_i).$$

We may prove Lemma 3 using the proof of Lemma 10 of [13].

Now we study some properties of the limit operator $T_\mu(E_\mu^{(2)})$.

Lemma 4. Let $\gamma \ge 12$ and $\mu > 0$ be an arbitrary or $\mu \ne \mu_r^0(\gamma)$ for any $\gamma < 12$. Then integral operator $T_\mu(z)$ is completely continuous and continuous at $z = E_\mu^{(2)}$.

Proof. Let $0 < \mu < \mu_r^0(\gamma)$ for any $\gamma < 12$. Under these assumptions by Theorem 2 we receive $E_\mu^{(2)} = 18$.

From the positivity and continuity of $\Delta_\mu(\cdot; E_\mu^{(2)})$ on $\mathbf{T}^3$ we get $C_1 \le \Delta_\mu(x; E_\mu^{(2)}) \le C_2, \quad x \in \mathbf{T}^3$ for some $C_1, C_2 > 0$. Then from the statements (C) and (D) of Lemma 2 for kernel of the limit integral operator $T_\mu(18)$ the following estimate is valid

$$\int_{(\mathbf{T}^3)^2} \frac{dxdy}{\Delta_\mu(x;18)\Delta_\mu(y;18)(u_2(x,y) - 18)^2} \le C_1 \int_{(U_\delta(\overline{0}))^2} \frac{dxdy}{(x^2 + y^2)^2} + C_2.$$

Passing on to a spherical coordinate system, then to a polar coordinate system in the last integral we get

$$C_1 \int_{(U_\delta(\overline{0}))^2} \frac{dxdy}{(x^2 + y^2)^2} \le C_1 \int_0^\delta \int_0^\delta \frac{\rho_1^2 \rho_2^2 d\rho_1 d\rho_2}{(\rho_1^2 + \rho_2^2)^2} \le C_1 \int_0^{\pi/2} sin^2 2\varphi d\varphi \int_0^{2\delta} \frac{\rho^5}{\rho^4} dr < C_1.$$

Let now $\gamma \geq 12$ and $\mu > 0$ be an arbitrary or $\mu > \mu_r^0(\gamma)$ for all $\gamma < 12$. Then from Theorem 2 we conclude that $E_\mu^{(2)} > 18$. The continuity of $u_2(\cdot;\cdot)$ on $\mathbf{T}^3$ and the statement $u_2(x,y) - E_\mu^{(2)} < 0,\ x,y \in \mathbf{T}^3$, imply

$$C_1 \leq \frac{1}{|u_2(x,y) - E_\mu^{(2)}|} \leq C_2 \tag{9}$$

for some $C_1, C_2 > 0$.

By virtue of estimate (9) and Lemma 3 for kernel of the operator $T_\mu(E_\mu^{(2)})$ we have

$$\begin{aligned}&\int_{(\mathbf{T}^3)^2} \frac{dxdy}{\Delta_\mu(x;E_\mu^{(2)})\Delta_\mu(y;E_\mu^{(2)})(u_2(x,y)-E_\mu^{(2)})^2}\\ &\quad\leq C_1 \sum_{i,j=1}^{N} \int_{U_\delta(x_i)} \frac{dx}{\Delta_\mu(x;E_\mu^{(2)})} \int_{U_\delta(x_j)} \frac{dx}{\Delta_\mu(x;E_\mu^{(2)})} + C_2\\ &\quad\leq C_1 \sum_{i,j=1}^{N} \int_{U_\delta(x_i)} \frac{dx}{|x-x_i|^2} \int_{U_\delta(x_j)} \frac{dx}{|x-x_j|^2} + C_2 = C_1 N^2 \left(\int_{U_\delta(\bar{0})} \frac{dx}{x^2}\right)^2 + C_2.\end{aligned}$$

Using the system of spherical coordinate we get

$$\int_{U_\delta(\bar{0})} \frac{dx}{x^2} \leq C_1 \int_0^\delta \frac{\rho_1^2 d\rho_1}{\rho_1^2} \leq C_1.$$

Hence for all $\gamma \geq 12$ and $\mu > 0$ or $\mu \neq \mu_r^0(\gamma)$ for all $\gamma < 12$ the $T_\mu(z),\ z \geq E_\mu^{(2)}$ is completely continuous.

The continuity of the kernel of $T_\mu(z)$ in $x,y \in \mathbf{T}^3$ for $z > E_\mu^{(2)}$ is obvious. Now the continuity of $T_\mu(z)$ from right upwards $z = E_\mu^{(2)}$ follows from the Lebesgue theorem.

We now formulate the result on the finiteness of $\sigma_{\mathrm{disc}}(\mathscr{A}_\mu)$.

Theorem 4. (A) Let $\gamma \geq 12$ and $\mu > 0$ or $\mu \neq \mu_r^0(\gamma)$ for $\gamma < 12$. Then the subset of $\sigma_{\mathrm{disc}}(\mathscr{A}_\mu)$ located in $(E_\mu^{(2)};+\infty)$ is finite.
(B) Let $\gamma \leq 0$ and $\mu > 0$ or $\mu \neq \mu_l^0(\gamma)$ for $\gamma > 0$. Then the subset of $\sigma_{\mathrm{disc}}(\mathscr{A}_\mu)$ located in $(-\infty;E_\mu^{(1)})$ is finite.

Proof. (A) Suppose that $\gamma \geq 12$ and $\mu > 0$ or $\mu \neq \mu_r^0(\gamma)$ for any $\gamma < 12$. Applying Lemma 1 we get

$$N_{(z;+\infty)}(\mathscr{A}_\mu) = n(1;T_\mu(z)) \quad \text{for} \quad z > E_\mu^{(2)},$$

and using Lemma 4 we receive the finiteness of $n(1-a;T_\mu(E_\mu^{(2)}))$ for $a \in [0;1)$.

If we have the compact operators H_1 and H_2 as well the positive λ_1 and λ_2, then $n(\lambda_1+\lambda_2;H_1+H_2) \leq n(\lambda_1;H_1) + n(\lambda_2;H_2)$ (Weyl's inequality). So for $z > E_\mu^{(2)}$ and $a \in (0;1)$ the statement

$$N_{(z;+\infty)}(\mathscr{A}_\mu) = n(1;T_\mu(z)) \leq n(1-a;T_\mu(E_\mu^{(2)})) + n(a;T_\mu(z) - T_\mu(E_\mu^{(2)}))$$

is valid.

Using the continuity of $T_\mu(z)$ at $z = E_\mu^{(2)}$ (see Lemma 4) for $a \in (0;1)$ we receive the two-sided estimate

$$\lim_{z \searrow E_\mu^{(2)}} N_{(z;+\infty)}(\mathscr{A}_\mu) \leq N_{(E_\mu^{(2)};+\infty)}(\mathscr{A}_\mu) \leq n(1-a;T_\mu(E_\mu^{(2)})).$$

So

$$N_{(E_\mu^{(2)};+\infty)}(\mathscr{A}_\mu) \leq n(1-a;T_\mu(E_\mu^{(2)})) < \infty.$$

Assertion (A) of Theorem 4 is proved.

Assertion (B) of Theorem 4 can be proven similarly.

"TWO-SIDED EFIMOV'S EFFECT" FOR $\mathscr{A}_\mu$ AND DISCRETE SPECTRUM ASYMPTOTICS.

The section is devoted to the proof of the existence of the "two-sided Efimov's effect" for $\mathscr{A}_\mu$.

By $\mathbf{B}^2$ we denote the unit sphere in 3D space $\mathbf{R}^3$. We determine an integral operator

$$T_\rho : L_2((0,\rho),\sigma_0) \to L_2((0,\rho),\sigma_0), \quad \rho > 0, \quad \sigma_0 = L_2(\mathbf{B}^2)$$

with kernel

$$T(\xi;\zeta) := \frac{25}{8\pi^2\sqrt{6}} \frac{1}{5\cosh(\zeta)+\xi}, \quad \xi = t-t', \quad t,t' \in (0,\rho), \quad \xi = (x,y), \quad x,y \in \mathbf{B}^2.$$

For positive λ, determine

$$U(\lambda) = \frac{1}{2}\lim_{\rho\to\infty}\frac{n(\lambda,T_\rho)}{\rho}.$$

In the paper [6] Sobolev prove the existence of $U(\lambda)$ with $U(1) > 0$.

We present Lemma from [6].

Lemma 5. Suppose $H(z) := H_0(z)+H_1(z)$. Here $H_0(z)$ $(H_1(z))$ is completely continuous as well continuous when $z > 18$ (when $z \geq 18$). From the existence of

$$\lim_{z\searrow 18} g(z)\, n(\xi, H_0(z)) = p(\xi)$$

and from continuity of $p(\cdot)$ in $(0;+\infty)$ for some $g(\cdot)$ with $g(z) \to 0$ when $z \searrow 18$ we obtain the same limit for $H(z)$ and

$$\lim_{z\searrow 18} g(z)\, n(\zeta, H(z)) = p(\xi).$$

Now we formulate important lemma about decompositions for $\Delta_{\mu_l^0(\gamma)}(x;z)$ and $\Delta_{\mu_r^0(\gamma)}(x;z)$.

Lemma 6. Following decompositions are hold:

$$\Delta_{\mu_l^0(\gamma)}(x;z) = \frac{32}{25}\pi^2(\mu_l^0(\gamma))^2\sqrt{6|x|^2-10z}+O(|x|^2)+O(|z|),$$

with $\gamma > 0$ as $|x| \to 0$, $z \nearrow 0$;

$$\begin{aligned}\Delta_{\mu_r^0(\gamma)}(x;z) =& -\frac{32}{25}\pi^2(\mu_r^0(\gamma))^2\sqrt{6|x-\overline{\pi}|^2+10(18-z)}\\ &+O(|x-\overline{\pi}|^2)+O(|z-18|),\end{aligned}$$

with $\gamma < 12$ as $|p-\overline{\pi}| \to 0, \quad z \searrow 18$.

Lemma 6 can be proved like Lemma 4 in [25] if we replace μ_0 by $\mu_l^0(\gamma)$ with $\gamma < 0$ and $\mu_r^0(\gamma)$ with $\gamma > 12$, respectively. Therefore, to avoid repetition, it is not given here.

Theorem 5. (A) If $\mu = \mu_r^0(\gamma)$ with $\gamma < 12$, then the matrix $\mathscr{A}_\mu$ has infinitely many eigenvalues $\{E_\mu^{(2,n)}\} \subset (18;+\infty)$ with $E_\mu^{(2,n)} \to E_\mu^{(2)} = 18$ as $n \to +\infty$. In addition

$$\lim_{z\searrow 18}\frac{N_{(z;+\infty)}(\mathscr{A}_{\mu_r^0(\gamma)})}{|\log|z-18||} = U(1). \tag{10}$$

(B) If $\mu = \mu_l^0(\gamma)$ with $\gamma > 0$, then the matrix $\mathscr{A}_\mu$ has infinitely many eigenvalues $\{E_\mu^{(1,n)}\} \subset (-\infty;0)$ with $E_\mu^{(1,n)} \to E_\mu^{(1)} = 0$ as $n \to +\infty$. In addition

$$\lim_{z\nearrow 0}\frac{N_{(-\infty;z)}(\mathscr{A}_{\mu_l^0(\gamma)})}{|\log|z||} = U(1).$$

Proof. (A) From the positiveness of the quantity $U(1)$ and using (10) we receive the infinity of $\sigma_{\rm disc}(\mathscr{A}_{\mu_r^0(\gamma)}\cap(18;+\infty)$. Thus, it is enough to derive (10). First, let us find an asymptotic expression for $n(1,T_{\mu_r^0(\gamma)}(z))$ at $z\searrow 18$. Here we use some facts about the integral operator $T^{(1)}(\rho)$, which was considered in [17].

As kernel of the integral operator $T(\delta;|z-18|):L_2(\mathbf{T}^3)\to L_2(\mathbf{T}^3)$ we put

$$\frac{25}{8\pi^2}\,\frac{\chi_\delta(x-\overline{\pi})\chi_\delta(y-\overline{\pi})\sqrt[4]{(6|x-\overline{\pi}|^2+10|z-18|)^{-1}(6|y-\overline{\pi}|^2+10|z-18|)^{-1}}}{5|x-\overline{\pi}|^2+2(x-\overline{\pi},y-\overline{\pi})+5|y-\overline{\pi}|^2+8|z-18|}.$$

Using Lemma 6 and Lemma 2 we establish that for any $z\geq 18$ and small $\delta>0$ the difference $T_{\mu_r^0(\gamma)}(z)-T(\delta;|z-18|)$ is a Hilbert-Schmidt operator and it is continuous at $z=18$.

We investigate the subspace of all functions g whose supports lie in $U_\delta(\overline{\pi})$. From the determination we conclude that it is an invariant subspace for $T(\delta;|z-18|)$. The restriction of $T(\delta;|z-18|)$ to $L_2(U_\delta(\overline{\pi}))$ we denote by $T^{(0)}(\delta;|z-18|)$. Then it is an integral operator with kernel

$$\frac{25}{8\pi^2}\,\frac{\sqrt[4]{(6|x-\overline{\pi}|^2+10|z-18|)^{-1}(6|y-\overline{\pi}|^2+10|z-18|)^{-1}}}{5|x-\overline{\pi}|^2+2(x-\overline{\pi},y-\overline{\pi})+5|y-\overline{\pi}|^2+8|z-18|}.$$

Using unitary map

$$S_\rho:L_2(U_\delta(\overline{\pi}))\to L_2(U_\rho(\overline{0})),\quad (S_\rho g)(x)=\rho^{-\frac{3}{2}}g(\frac{\delta}{\rho}(x-\overline{\pi}))$$

we can easily see that $T^{(0)}(\delta;|z-18|)$ is unitarily equivalent to

$$T^{(1)}(\rho):L_2(U_\rho(\overline{0}))\to L_2(U_\rho(\overline{0})),$$

$$(T^{(1)}(\rho)f)(x)=\frac{25}{8\pi^2}\int_{U_\rho(\overline{0})}\frac{\sqrt[4]{(6x^2+10)^{-1}(6y^2+10)^{-1}}}{5x^2+2(x,y)+5y^2+8}f(y)dy.$$

In [17] it was shown that

$$\lim_{\rho\to\infty}\frac{n(1,T^{(1)}(r))}{|\log\rho|}=U(1).$$

Now taking into account the assertions of Lemmas 1 and 5 we receive the proof of the statement (A) of Theorem 5.

Proof of the assertion (B) of Theorem 5 is similar.

Let us mention one important consequence of the Theorem 5.

Corollary. If $\gamma=6$, then

(A) $\mu_0:=\mu_l^0(6)=\mu_r^0(6)$;

(B) $E_{\mu_0}^{(1)}=0$, $E_{\mu_0}^{(2)}=18$;

(C) $N_{(-\infty;z)}(\mathscr{A}_{\mu_0})=\infty;\quad N_{(z;\infty)}(\mathscr{A}_{\mu_0})=\infty$;

(D) $\lim\limits_{z\nearrow 0}|\log|z||^{-1}N_{(-\infty;z)}(\mathscr{A}_{\mu_0})=\lim\limits_{z\searrow 18}|\log|z-18||^{-1}N_{(z;\infty)}(\mathscr{A}_{\mu_0})=U(1)$.

CONCLUSION

In the present paper the block operator matrix $\mathscr{A}_\mu$ ($\mu>0$ is a coupling constant) of order 2 acting in a direct sum of one-particle and two-particle subspaces of the bosonic Fock space is considered. Firstly the corresponding generalized Friedrichs model is introduced. Its essential and discrete spectrum are described. The location of $\sigma_{\rm ess}(\mathscr{A}_\mu)$ is studied. The bounds of the set $\sigma_{\rm ess}(\mathscr{A}_\mu)$ are evaluated. The Birman-Schwinger principle for $\mathscr{A}_\mu$ is proved. We prove the existence of the critical values $\mu_l^0(\gamma)$ with $\gamma>0$ and $\mu_r^0(\gamma)$ with $\gamma<12$ of the coupling constant $\mu>0$ for all $\gamma>0$ ($\gamma<12$) so that the matrix $\mathscr{A}_\mu$ with $\mu=\mu_l^0(\gamma)$ ($\mu=\mu_r^0(\gamma)$ has infinitely many eigenvalues on the l.h.s. (r.h.s.) of the its essential spectrum. It is shown that for all $\mu\notin\{\mu_l^0(\gamma),\mu_r^0(\gamma)\}$ the matrix $\mathscr{A}_\mu$ and has only finitely many discrete eigenvalues on the l.h.s. and r.h.s. of its essential spectrum. In the proof of this results we use the Weyl inequality and Sobolev's method.

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