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Research Article

On the Eigenvalues of a Soluble Model of Coupled Molecular and Nuclear Hamiltonians

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Abstract: In this paper, a special type of soluble model corresponding to a coupled molecular and nuclear Hamiltonians H, the so-called generalized Friedrichs model, is considered. We aim to determine and provide the most important properties of the well-known Faddeev operator corresponding to H accommodating a number of discrete eigenvalues. Furthermore, we provide a formula for counting the multiplicity of discrete eigenvalues of H.

***Keywords*:** molecular Hamiltonian, nuclear Hamiltonian, direct sum of spaces, inner product, adjoint operator, discrete spectrum, discrete eigenvalues, Faddeev operator

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1. Introduction

We consider the complex Hilbert space $\mathcal{H} := \mathcal{H}_0 \oplus \mathcal{H}_1$ (two-channel), which is the direct sum of $\mathcal{H}_0 := \mathbb{C}$ (channel 1 – a one-dimensional "molecular" space) and $\mathcal{H}_1$ (channel 2 – a "nuclear" space). The elements of $\mathcal{H}$ are vectors with two coordinates: $f = (f_0, f_1)$ with $f_0 \in \mathcal{H}_0$ and $f_1 \in \mathcal{H}_1$. For convenience, we remind that the scalar product of $f = (f_0, f_1) \in \mathcal{H}$ and $g = (g_0, g_1) \in \mathcal{H}$ is defined as

$$\langle f, g \rangle_{\mathcal{H}} := f_0 \overline{g}_0 + \langle f_1, g_1 \rangle$$

using the scalar products in the spaces $\mathcal{H}_0$ and $\mathcal{H}_1$.

In the Hilbert space $\mathcal{H}$, we consider the operator matrix

$$H := \begin{pmatrix} \omega & \langle \cdot, v \rangle \\ v & h \end{pmatrix}, \tag{1}$$

where $\omega \in \mathbb{R}$ is a triamolecular energy, $v \in \mathcal{H}_1$ provides the coupling between the channels and h is the (self-adjoint) nuclear Hamiltonian in $\mathcal{H}_1$. The Hamiltonian (1) can be considered a soluble model of coupled molecular Hamiltonian and nuclear Hamiltonian and is recognized as the generalized Friedrichs model [1, 2].

We aim to determine the special type of nuclear Hamiltonian h, the so-called Friedrichs model. Let D_i, $i = 1, \ldots, n$

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$(n \in N)$ be the bounded domains with Eucledian measure in the d-dimensional space $\mathbb{R}^d$, such that $D_i \cap D_j = \varnothing$, $i \neq j$, and $D := \bigcup_{i=1}^{n} \overline{D}_i$. In the nuclear space $\mathcal{H}_1 := L_2(D)$, we introduce Hamiltonian h as

$$(hf_1)(p) = u(p)f_1(p) - \mu \int_D K(p,t) f_1(t)dt, \tag{2}$$

in the latter formula $\mu > 0$ is a coupling constant, $K(\cdot,\cdot)$ is a real-valued bounded symmetric function on $D^2 := D \times D$, and $u(.)$ is a real-valued piecewise continuous bounded function on D. Let $v(.)$ also be a real-valued bounded function on D. Under these conditions, one can easily show that the model Hamiltonian, H is bounded and self-adjoint.

A similar Hamiltonian (1) with (2) was introduced in [3] as the generalized Friedrichs model, where its eigenvalues and resonances were studied. This model was also studied and considered in the literature [4], where the problems of the random walk were targeted and the results obtained for the generalized Friedrichs model were applied. The bound states of such families of the generalized Friedrichs models are investigated in the work [5]. The spectrum and structure of the eigenvectors of the generalized Friedrichs model for small values of the coupling constant were studied in [6]. In [7], the existence and analyticity of eigenvalues are studied for the generalized Friedrichs model. The essential spectrum corresponding to H as a 3×3 block operator matrix is studied in [8].

The main goal of our paper is to provide and characterize the complete investigations for the number of discrete eigenvalues of H. More precisely:

(i) To determine the Faddeev equation for H and prove some of its important properties corresponding to the number of eigenvalues.

(ii) To provide the formula for the multiplicity of eigenvalues of H. The formula allows us to find the asymptotics for the distribution of the discrete spectrum lying in a spectral gap. Furthermore, we show that the number of discrete eigenvalues of H is finite.

2. The construction and main properties of Faddeev equation (operator) related with H

For further convenience, we rewrite the Hamiltonian (1) with (2) in the following form:

$$H := \begin{pmatrix} H_{00} & H_{01} \\ H_{01}^* & H_{11} \end{pmatrix}, \tag{3}$$

with the entries $H_{ij} : \mathcal{H}_j \to \mathcal{H}_i, i \leq j,\ i, j = 0,1$ defined by

$$H_{00} f_0 = \omega f_0, \qquad H_{01} f_1 = \int_D v(t) f_1(t)dt,$$
$$H_{11} := H_{11}^0 - K, \quad (H_{11}^0 f_1)(p) = u(p) f_1(p), \quad \big(K f_1\big)(p) = \mu \int_D K(p,t) f_1(t)dt.$$

Here, $f_i \in \mathcal{H}_i,\ i = 0,1.$

We noticed that threshold eigenvalue, virtual level (threshold energy resonance), and threshold energy expansion for the associated Fredholm determinant of a generalized Friedrichs model with $\mu = 0$ have been studied in [9-12]. The localization and number of discrete eigenvalues of this model are investigated in [13]. In [14-16], using the spectral information about the Hamiltonian H with the rank-1 generated kernel, the number of eigenvalues located respectively in the gap, inside, and below the bottom of the essential spectrum of the operator matrices is studied.

Since $K(\cdot,\cdot)$ is a bounded function on D^2, the operator K is a Hilbert-Schmidt. The Weyl theorem yields the equalities

$$\sigma_{\text{ess}}(H) = \sigma(H_{11}^0) = \overline{\text{Ran}(u)}.$$

In the rest of the work, we suppose that K is a positive operator and that it is an element of the trace class.

Lemma 1. The relation

$$(K^{1/2}f)(p) = \sqrt{\mu}\int_D \tilde{K}(p,t)f(t)dt$$

is valid. Here, the kernel of $K^{1/2}$ is formally denoted by $\tilde{K}(\cdot,\cdot)$ and it is a square-integrable on D^2.

Proof. Since K is positive, every nontrivial eigenvalue λ_m is a positive number. Applying the Hilbert-Schmidt theorem [17], we have that

$$K = \sum_m \lambda_m \langle \varphi_m, \cdot\rangle \varphi_m$$

with $\sum_m \lambda_m < \infty$; where φ_m is the eigenfunction corresponding to the eigenvalue λ_m. For positive square root $K^{1/2}$, we have

$$K^{1/2} = \sum_m \sqrt{y_m}\langle \varphi_m, \cdot\rangle\ \varphi_m.$$

Since we know that $\sum_m y_m < \infty$, $K^{1/2}$ belongs to the Hilbert-Schmidt operator, therefore, the function $\tilde{K}(\cdot,\cdot)$ is an element of $L_2(D^2)$.

Let I_i be the identity operator on $\mathcal{H}_i$, $i = 0,1$ and $R_{11}^0(z) := (H_{11}^0 - zI_1)^{-1}$. We consider

$$T(z) := \begin{pmatrix} T_{00}(z) & T_{01}(z) \\ T_{10}(z) & T_{11}(z) \end{pmatrix}, \quad z \in \mathbb{C}\setminus\sigma(H_{11}^0)$$

in the Hilbert space $\mathcal{H}$, where $T_{ij}(z): \mathcal{H}_j \to \mathcal{H}_i$, $i, j = 0,1$ has the following formulations:

$$T_{00}(z) := \left(1+z\right)I_0 - H_{00} + H_{01}R_{11}^0(z)H_{01}^*, \quad T_{01}(z) := -H_{01}R_{11}^0(z)K^{1/2},$$
$$T_{10}(z) := -K^{1/2}R_{11}^0(z)H_{01}^*, \quad T_{11}(z) := K^{1/2}R_{11}^0(z)K^{1/2}.$$

The following statement describes a well-known Birman-Schwinger principle [11, 12, 14-16] (relation between the eigenvalues of H and $T(z)$.

Lemma 2. The quantity $z \in \mathbb{C}\setminus\sigma(H_{11}^0)$ is a discrete eigenvalue of H if the quantity $\lambda = 1$ is an discrete eigenvalue of $T(z)$. Moreover, the multiplicities of eigenvalues z and 1 are the same.

Proof. Assume $f = \left(f_0, f_1\right) \in \mathcal{H}$ is the eigenvector of H corresponding to the eigenvalue $z \in \mathbb{C}\setminus\sigma(H_{11}^0)$. Then, the coordinates f_0 and f_1 satisfy

$$\begin{cases} (H_{00} - zI_0)f_0 + H_{01}f_1 = 0; \\ H_{01}^* f_0 + (H_{11}^0 - zI_1)f_1 - Kf_1 = 0. \end{cases} \tag{4}$$

Since $z \in \mathbb{C}\setminus\sigma(H_{11}^0)$, then from the second relation of (4) for the coordinate f_1, we find that

$$f_1 = R_{11}^0(z)Kf_1 - R_{11}^0(z)H_{01}^* f_0. \tag{5}$$

Next, we find the action of $K^{1/2}$ to the coordinate f_1 defined by (5):

$$K^{1/2}f_1 = K^{1/2}R_{11}^0(z)Kf_1 - K^{1/2}R_{11}^0(z)H_{01}^* f_0.$$

Setting $\tilde{f}_1 := K^{1/2}f_1$, from the last equality, we obtain that

$$\tilde{f}_1 = K^{1/2} R_{11}^0(z) K^{1/2} \tilde{f}_1 - K^{1/2} R_{11}^0(z) H_{01}^* f_0. \tag{6}$$

Now, we rewrite the equality (5) in the form

$$f_1 = R_{11}^0(z) K^{1/2} \tilde{f}_1 - R_{11}^0(z) H_{01}^* f_0. \tag{7}$$

Substituting the relation (7) into the first relation of (4), we receive that (4) has a nonzero solution if the system

$$\begin{cases} (H_{00} - zI_0 - H_{01} R_{11}^0(z) H_{01}^*) f_0 + H_{01} R_{11}^0(z) K^{1/2} \tilde{f}_1 = 0 \\ K^{1/2} R_{11}^0(z) H_{01}^* f_0 + (I_1 - K^{1/2} R_{11}^0(z) K^{1/2}) \tilde{f}_1 = 0 \end{cases}$$

or the matrix equation $\Phi - T(z)\Phi = 0$, $\Phi = (f_0, \tilde{f}_1) \in \mathcal{H}$ has a nontrivial solution. It is clear that the linear subspaces of solutions of (4) and the equation $\Phi - T(z)\Phi = 0$ have the same dimension. Therefore, the multiplicities of the eigenvalues z and 1 of the operators H and $T(z)$ coincide, respectively.

Remark 1. We call the equation $T(z)\Phi = \Phi$ $z \in \mathbb{C} \setminus \sigma(H_{11}^0)$ the Faddeev equation corresponding to H. The operator $T(z)$ is the Faddeev operator.

Remark 2. The function $T(.)$ is not monotone in $z \in \mathbb{R} \setminus \sigma(H_{11}^0)$, and hence, the method of [18] is not applicable to the proof of the main theorems of this work.

It is easy to obtain information about the fact that the determinant $D(\lambda, z) := \det[I - \lambda^{-1} T(z)]$ of the operator $I - \lambda^{-1} T(z)$ is well-defined and is analytic for $\lambda \neq 0$ where $I := \operatorname{diag}\{I_0, I_1\}$. The following lemma is implied by Theorem XIII.105 in [17].

Lemma 3. Let $z \in \mathbb{C} \setminus \sigma(H_{11}^0)$. The quantity $\lambda \neq 0$ is a discrete eigenvalue of $T(z)$ if $D(\lambda, z) = 0$.

By Lemmas 2 and 3, we obtain the following lemma.

Lemma 4. The quantity $z \in \mathbb{C} \setminus \sigma(H_{11}^0)$ is a discrete eigenvalue of H if $D(1, z) = 0$.

We remember that for $f_1, g_1 \in L_2(T^d)$, the inner product is defined as

$$\langle f_1, g_1 \rangle = \int_{T^d} f_1(t) \overline{g_1(t)} dt.$$

Note that the operator matrix $T(z)$ is determined for any $z \in \mathbb{C} \setminus \sigma(H_{11}^0)$.

Lemma 5. If the quantity $\lambda \neq 0$ is a discrete eigenvalue of $T(z)$ for some $z \in \mathbb{C} \setminus \sigma(H_{11}^0)$, in this case, $\operatorname{Im} z = 0$.

Proof. Assume $\Phi = (\varphi_0, \varphi_1) \in \mathcal{H}$ is the eigenvector with norm-1 corresponding to the discrete eigenvalue $\lambda \neq 0$ of $T(z)$ for some $z \in \mathbb{C} \setminus \sigma(H_{11}^0)$. Separating the imaginary and real parts of the function $(u(p) - z)^{-1}$, we rewrite $R_{11}^0(z)$ as the sum $R_{11}^0(z) = \hat{R}_{11}^0(z) + i \operatorname{Im} z \cdot \tilde{R}_{11}^0(z)$, where $\hat{R}_{11}^0(z)$, $\tilde{R}_{11}^0(z)$ are multiplication operators to

$$\frac{u(p) - \operatorname{Re} z}{(u(p) - \operatorname{Re} z)^2 + (\operatorname{Im} z)^2}, \quad \frac{1}{(u(p) - \operatorname{Re} z)^2 + (\operatorname{Im} z)^2},$$

respectively. Therefore, $T(z)$ can be written as the sum $T(z) = \hat{T}(z) + i \operatorname{Im} z \cdot \tilde{T}(z)$, where the operators $\hat{T}(z)$, and $\tilde{T}(z)$ are defined as

$$\hat{T}(z) := \begin{pmatrix} (1 + \operatorname{Re} z) I_0 - H_{00} + H_{01} \hat{R}_{11}^0(z) H_{01}^* & -H_{01} \hat{R}_{11}^0(z) K^{1/2} \\ -K^{1/2} \hat{R}_{11}^0(z) H_{01}^* & K^{1/2} \hat{R}_{11}^0(z) K^{1/2} \end{pmatrix},$$

$$\tilde{T}(z) := \begin{pmatrix} I_0 + H_{01} \tilde{R}_{11}^0(z) H_{01}^* & -H_{01} \tilde{R}_{11}^0(z) K^{1/2} \\ -K^{1/2} \tilde{R}_{11}^0(z) H_{01}^* & K^{1/2} \tilde{R}_{11}^0(z) K^{1/2} \end{pmatrix}.$$

First, taking the scalar product of the relation $\lambda\Phi = \hat{T}(z)\Phi + i \operatorname{Im} z \cdot \tilde{T}(z)\Phi$ and the vector Φ, then using the fact that $T(z)$, $\hat{T}(z)$, and $\tilde{T}(z)$ are the self-adjoint, we obtain

$$\langle \tilde{T}(z)\Phi, \Phi\rangle_{\mathcal{H}} \ \operatorname{Im} z = 0. \tag{8}$$

Now, we show that Imz = 0. Since $\|\Phi\| = 1$, for $\Phi = (\varphi_0, \varphi_1)$, we receive $\varphi_0 = 0$ or $\varphi_0 \neq 0$. If $\varphi_0 = 0$, then $\|\Phi\| = \|\varphi_1\| = 1$. In this case, we obtain that

$$\langle \tilde{T}(z)\Phi, \Phi\rangle_{\mathcal{H}} = \langle K^{1/2}\tilde{R}_{11}^{0}(z)K^{1/2}\varphi_1, \varphi_1\rangle = \langle \tilde{R}_{11}^{0}(z)K^{1/2}\varphi_1, K^{1/2}\varphi_1\rangle.$$

From definition of $\tilde{R}_{11}^{0}(z)$, it follows that $\langle \tilde{R}_{11}^{0}(z)K^{1/2}\varphi_1, K^{1/2}\varphi_1\rangle = 0$ if and only if $K^{1/2}\varphi_1 = 0$. On the other hand, the equality $K^{1/2}\varphi_1 = 0$ contradicts the fact that the quantity $\lambda \neq 0$ is a discrete eigenvalue of $T(z)$. Therefore, $\langle \tilde{T}(z)\Phi, \Phi\rangle_{\mathcal{H}} > 0$.

Assume that $\varphi_0 \neq 0$. Furthermore, we denote

$$\phi_0 := H_{10}\varphi_0, \quad \phi_1 := K^{1/2}\varphi_1;$$
$$\hat{\phi}_j(p) := \operatorname{Re}(\phi_j(p)), \quad \tilde{\phi}_j(p) := \operatorname{Im}(\phi_j(p)), \quad j = 0,1.$$

Then, using the identity, we have that

$$\left|\phi_0(p)^2\right| + |\phi_1(p)|^2 - 2\operatorname{Re}(\phi_1(p)\overline{\phi_0(p)}) = \left(\left|\hat{\phi}_0(p)\right| - \left|\hat{\phi}_0(p)\right|\right)^2 + \left(\left|\tilde{\phi}_0(p)\right| - \left|\tilde{\phi}_0(p)\right|\right)^2,$$

after simple calculations, we have that

$$\begin{aligned}\langle \tilde{T}(z)\Phi, \Phi\rangle_{\mathcal{H}} &= |\varphi_0|^2 + \langle \tilde{R}_{11}^{0}(z)\phi_0, \phi_0\rangle - 2\operatorname{Re}\langle \tilde{R}_{11}^{0}(z)\phi_1, \phi_0\rangle + \langle \tilde{R}_{11}^{0}(z)\phi_1, \phi_1\rangle \\ &= |\varphi_0|^2 + \int_D \frac{1}{(u(p) - \operatorname{Re} z)^2 + (\operatorname{Im} z)^2}\left\{|\phi_0(p)|^2 + |\phi_1(p)|^2 - 2\operatorname{Re}(\phi_1(p)\overline{\phi_0(p)})\right\} dp \geq |\varphi_0|^2 > 0.\end{aligned}$$

So, $\langle \tilde{T}(z)\Phi, \Phi\rangle_{\mathcal{H}} > 0$. Therefore, the equality (8) imply $\operatorname{Im} z = 0$.

In the complex Hilbert space $\mathcal{H}$, we consider the bounded self-adjoint operator B. For the real number λ, we set $\mathcal{H}_B(\lambda) \subset \mathcal{H}$ a subspace such that $\langle Bf, f\rangle_{\mathcal{H}} > \lambda\|f\|^2$ for any $f \in \mathcal{H}_B(\lambda)$, and determine [19] the quantity $n(\lambda, B)$ by

$$n(\lambda, B) := \sup_{\mathcal{H}_B(\lambda)} \dim \mathcal{H}_B(\lambda).$$

The quantity $n(\lambda, B)$ is equal to infinity if λ is smaller than $\max\sigma_{\text{ess}}(B)$; if $n(\lambda, B)$ is a finite quantity, then it is equal to the number of the eigenvalues of B bigger than λ.

Lemma 6. The quantity $z \in \mathbb{R} \setminus \sigma(H_{11}^{0})$ is a regular point of H if $n(1, T(.))$ is continuous at $z = z_0$.

Proof. *Necessity*. For the regular point $z = z_0 \in \mathbb{R} \setminus \sigma(H_{11}^{0})$ of H using Lemma 2, one can know that $I - T(z_0)$ is invertible. It follows from the continuity of the operator-function $\lambda^{-1} T(z)$ with respect to (λ, z) in $(0, \infty) \times \mathbb{R} \setminus \sigma(H_{11}^{0})$ and from the compactness of $T(z)$ that $I - \lambda^{-1} T(z)$ is an invertible for any pair (λ, z) in some neighborhood of $(1, z_0)$. Therefore, it means that for some $\rho > 0$, the identity $\sigma(T(z)) \cap (1-\rho, 1+\rho) = \varnothing$ is valid for any $z \in [z_0 - \rho, z_0 + \rho]$. By the determination of $n(a, T(.))$, yields

$$n(1 \pm \delta, T(z_0 \pm \xi)) = n(1, T(z_0 \pm \xi))$$

for any $\xi, \delta \in [0, \rho)$. Furthermore, applying Weyl inequality [2], we have that

$$n(a_1 + a_2, A_1 + A_2) \leq n(a_1, A_1) + n(a_2, A_2) \tag{9}$$

for the sum of compact self-adjoint and positive operators A_1 and A_2, this allows to have that

$$n(1+\eta_0, T(z_0 \pm \xi)) \leq n(1, T(z_0 \pm \xi)) + n(\eta_0, T(z_0 \pm \xi) - T(z_0)) = n(1, T(z_0))$$

for a fixed $\eta_0 \in (0, \delta)$ and for small $\xi > 0$. Similarly, for small values of $\xi > 0$, we obtain that $n(1, T(z_0 \pm \xi)) = n(1, T(z_0))$. Therefore, the relation $n(1, T(z_0 \pm \xi)) = n(1, T(z_0))$ holds for all small $\xi > 0$, i.e., it means that n (1, T (.)) is continuous at $z = z_0$.

Sufficiency. On the contrary, suppose that n (1, T (.)) is continuous at $z = z_0$, point z_0 is a discrete eigenvalue of H.

Arguing as above and using the Weyl inequality (9), we verify that there exist quantities $\delta_0 > 0$ and $c_0 = c_0(\delta_0) > 0$, so that the relations

$$n(1, T(z_0)) = n(1+\delta_0, T(z_0)) = n(1+\delta_0 / 4, T(z_0 + \varepsilon)) \tag{10}$$

hold for all $\varepsilon \in [-c_0, c_0]$. By Lemma 4, we have $D(1, z_0) = 0$. Let Γ_δ be the boundary of the complex $\delta-$neighborhood U_δ (z_0) of the point z_0. In this case, by virtue of the smallness of δ, we have that $D(1, z_0) \neq 0$ for all $z \in \Gamma_\delta$. We set

$$d := \min_{z\in\Gamma_\delta} |D(1, z)|, \quad \psi_\varepsilon(z) := D(1+\varepsilon, z) - D(1, z).$$

By the continuity of $D(\cdot,\cdot)$, we find $\rho = \rho(\delta) > 0$, so that the condition $|\psi_\varepsilon(z)| < d$ is vanish for any quantity $\varepsilon \in [-\rho, \rho]$ and for any point $z \in \Gamma_\delta$. Therefore, for fixed $\varepsilon \in [-\rho, \rho]$, the maps $D(1, .)$ and $\psi_\varepsilon(.)$ determined on the closed set $\overline{U_\delta(z_0)}$ satisfies all main assertions of theorem of Rouché. Therefore, the number of zeros $D(1, .)$ and $D(1+\varepsilon, .)$ belonging to open set $U_\delta(z_0)$ coincides. Assume that $D(1+\varepsilon, z_\varepsilon) = 0$, $\varepsilon > 0$ for some fixed point $z_\varepsilon \in U_\delta(z_0)$. Using Lemma 4, one can prove that the quantity $1+\varepsilon$ is a discrete eigenvalue of the operator $T(z_\varepsilon)$ for any quantity $\varepsilon \in (0, \rho)$. Applying Lemma 5, we obtain that the quantity z_ε is a real quantity. This, together with the relation (10), implies the inequalities

$$n(1, T(z_\varepsilon)) - n(1, T(z_0)) \geq n(1+\varepsilon/2, T(z_\varepsilon)) - n(1, T(z_0)) \geq 1 + n(1+\delta_0/4, T(z_\varepsilon)) - n(1, T(z_0)) = 1$$

for all $\varepsilon \in (0, \rho)$. Consequently, $n(1, T(z_0)) \neq \lim_{\varepsilon\to 0} n(1, T(z_\varepsilon))$, i.e., the function n (1, T(.)) is not continuous at the point $z = z_0$, which contradicts our assumption.

Lemma 7. Let $z_0 \in \sigma_{\text{disc}}(H)$. Then, for all small $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\begin{aligned}\text{card}\{z \in U_\delta(z_0) : D(1+\varepsilon, z) = 0\} &= \text{card}\{z \in U_\delta(z_0) : D(1-\varepsilon, z) = 0\} \\ &= \text{card}\{z \in U_\delta(z_0) : D(1, z) = 0\},\end{aligned}$$

where card M denotes the cardinality of M.

Proof. If $z_0 \in \sigma_{\text{disc}}(H)$, then by Lemma 4, we have that $D(1, z_0) = 0$. Then, there exists the quantity $\delta_0 > 0$, so that $D(1, z_0) \neq 0$ or any point $z \in \Gamma_\delta$. For that case, we show during the proof of the assertion of Lemma 6 for small quantity $\rho > 0$, the number of zeros of $D(1, .)$ belonging to the open set $U_\delta(z_0)$, and the number of zeros of $D(1+\varepsilon, .) = \psi_\varepsilon(.) + D(1, .)$ belonging to the open set $U_\delta(z_0)$ coincides for all fixed quantity $\varepsilon \in [-\rho, \rho]$.

3. Formula for the number of eigenvalues of *H*

The number of discrete eigenvalues of H belonging to the interval $(a, b) \subset \mathbb{R} \setminus \sigma(H_{11}^0)$ are denote by $N_{(a,b)}(H)$.

The formulation of the main theorem of the present article is given as.

Theorem 1. The quantity $z \in \mathbb{R} \setminus \sigma(H_{11}^0)$ is a discrete eigenvalue of H if n (1, T(.)) and is discontinuous at $z = z_0$. Moreover, the multiplicity k of the discrete eigenvalue z_0 satisfies the identity

$$k = \lim_{\xi\to+0} [n(1, T(z_0+\xi)) - n(1, T(z_0))] + \lim_{\xi\to+0} [n(1, T(z_0-\xi)) - n(1, T(z_0))].$$

Proof. Using the assertion of Lemma 6, we receive that the quantity $z \in \mathbb{R} \setminus \sigma(H_{11}^0)$ is a discrete eigenvalue of H if

$n(1, T(.))$ is discontinuous at $z = z_0$.

By k, we denote the multiple of the eigenvalue z_0 of H. We prove the formula

$$k = \lim_{\xi \to +0}[n(1, T(z_0+\xi)) + n(1, T(z_0-\xi)) - 2n(1, T(z_0))]. \tag{11}$$

Since $T(z_0)$ is a compact operator, there exists $\eta_0 > 0$ such that $n(1, T(z_0)) = n(1+\eta_0, T(z_0))$. Then, from the Weyl inequality (9), it follows that

$$n(1+\eta_0, T(z_0)) \le n(1, T(z_0+\xi)) + n(\eta_0, T(z_0) - T(z_0+\xi)) = n(1, T(z_0+\xi)),$$

for small $|\xi|$. Therefore, $n(1, T(z_0)) \le n(1, T(z_0+\xi))$ for small $|\xi|$ and the right-hand side of (11) is nonnegative.

Using Lemma 2, we obtain that the quantity $\lambda = 1$ is a k–multiple eigenvalue of $T(z_0)$. The function $T(z)\Phi$ is a vector-valued analytic function of z in the neighborhood of $z = z_0$ for every $\Phi \in \mathcal{H}$. Therefore, by Theorem XII.13 in [17], the operator $T(z)$ has exactly k eigenvalues $\lambda_1(z), ..., \lambda_k(z)$ (counting multiplicities) in the neighborhood of $z = z_0$. By Lemma 5, if the numbers $\lambda_1(z), ..., \lambda_k(z)$ are real, then z is also real. Let $\delta > 0$ be a number such that the inclusion $\lambda_i(z_0+\xi) \in (1-\delta, 1+\delta)$ holds with $\xi \in (-c_\delta, c_\delta)$ for all $i \in \{1, \ldots, k\}$ and for some constant $c_\delta > 0$. Since the multiplicity of the zero of the determinant $D(1, z)$ at the point $z = z_0$ is at least equal to the geometric multiplicity of discrete eigenvalue 1 of $T(z)$, Lemma 7 implies the relation

$$\mathrm{card}\{z \in (-c_\delta, c_\delta) : D(1 \pm \varepsilon, z) = 0\} = \mathrm{card}\{z \in (-c_\delta, c_\delta) : D(1, z) = 0\} = k + s,$$

where s is a nonnegative integer. It follows that

$$\mathrm{card}\{i : \lambda_i(z) = 1-\varepsilon\} = \mathrm{card}\{i : \lambda_i(z) = 1+\varepsilon\} = k.$$

Taking into account $\lambda_i(z_0) = 1$ and $\lambda_i(z) \ne 1$ as $z \ne z_0$, we can split the set $\{1, ..., k\}$ into the two nonintersecting subsets

$$\{M \nearrow\} := \{i : \lambda_i(z_0 - c_\delta) < \lambda_i(z_0) < \lambda_i(z_0 + c_\delta)\},$$
$$\{M \searrow\} := \{i : \lambda_i(z_0 - c_\delta) > \lambda_i(z_0) > \lambda_i(z_0 + c_\delta)\}.$$

Then,

$$\mathrm{card}\{i : \lambda_i(z_0+\xi) > 1, \quad \xi \in (0, c_\delta)\} = \mathrm{card}\{M \nearrow\},$$
$$\mathrm{card}\{i : \lambda_i(z_0-\xi) > 1, \quad \xi \in (0, c_\delta)\} = \mathrm{card}\{M \searrow\}.$$

Consequently,

$$\mathrm{card}\{i : \lambda_i(z_0+\xi) > 1, \quad \xi \in (0, c_\delta)\} + \mathrm{card}\{i : \lambda_i(z_0-\xi) > 1, \quad \xi \in (0, c_\delta)\} = k.$$

On the other hand, we have the relations

$$\mathrm{card}\{i : \lambda_i(z_0+\xi) > 1, \quad \xi \in (0, c_\delta)\} = n(1, T(z_0+\xi)) - n(1+\delta/2, T(z_0+\xi)),$$
$$\mathrm{card}\{i : \lambda_i(z_0-\xi) > 1, \quad \xi \in (0, c_\delta)\} = n(1, T(z_0-\xi)) - n(1+\delta/2, T(z_0-\xi)).$$

From here, while making use of

$$n(1+\delta/2, T(z_0-\xi)) = n(1+\delta/2, T(z_0+\xi)) = n(1, T(z_0))$$

with $|\xi| < c_\delta$, we derive (11).

Remark 3. In [18], it was established that if the quantity z_0 is the discrete eigenvalue of H_{11} and k is its multiplicity, then

$$k = \lim_{\xi \to +0} n(1, T_{11}(z_0 + \xi)) - n(1, T_{11}(z_0)).$$

We formulate and prove two corollaries of Theorem 1.

Corollary 1. For any $(a,b) \subset \mathbb{R} \setminus \sigma(H_{11}^0)$, the equality

$$N_{(a,b)}(H) = \bigvee_a^b (n(1, T(.)))$$

holds true, where $\bigvee_a^b (f)$ is the total variation of f in (a, b). Using the monotonicity property of $n(1, T(.))$ in $(-\infty, \min \sigma(H_{11}^0))$, we have that

$$N_{(-\infty,z)}(H) = n(1, T(z)), \; z < \min \sigma(H_{11}^0). \tag{12}$$

Proof. Let the operator H have no eigenvalues on the interval (a, b), where $(a,b) \cap \sigma(H_{11}^0) = \varnothing$. In other words, let $N_{(a,b)}(H) = 0$. Then, Lemma 6 implies that $n(1, T(\zeta)) = \text{const}$ for all $\zeta \in (a,b)$. It follows that $\bigvee_a^b (n(1, T(.))) = 0$. Let

$$\sigma_{\text{disc}}(H) \cap (a_n, b_n) = \{\xi_n\}, \quad \sigma_{\text{disc}}(H) \cap (a,b) = \bigcup_n \{\xi_n\}.$$

Then, for the multiplicity k_n of the eigenvalue ξ_n, we have that

$$N_{(a,b)}(H) = \sum_n k_n$$

Theorem 1 implies that

$$\begin{aligned}\sum_n k_n &= \sum_n \left\{ \lim_{\varepsilon \to +0} [n(1, T(\xi_n + \varepsilon)) - n(1, T(\xi_n))] + \lim_{\varepsilon \to +0} [n(1, T(\xi_n - \varepsilon)) - n(1, T(\xi_n))] \right\} \\ &= \bigvee_a^b (n(1, T(.))).\end{aligned}$$

This finishes the proof.

Remark 4. In [18], it was shown that for any closed interval $(a,b) \subset \mathbb{R} \setminus \sigma(H_{11}^0)$ the following equality

$$N_{(a,b)}(H_{11}) = n(1, T_{11}(b)) - n(1, T_{11}(a))$$

holds true.

Corollary 2. Let $(a,b) \subset \mathbb{R} \setminus \sigma(H_{11}^0)$ and let the operator-function $T(.)$ be uniformly convergent to some operators $T(a)$ and $T(b)$ as $z \to a + 0$ and $z \to b - 0$, respectively. Then, H has a finite number of discrete eigenvalues located in (a, b).

Proof. The operators $T(a)$ and $T(b)$ are compact, and hence, the equality $n(1, T(\zeta)) = n(1 + \delta, T(\zeta))$ holds with $\zeta = a$ and $\zeta = b$ for some $\delta > 0$. Since $\|T(.)\|$ is continuous, there is $\varepsilon > 0$ so that $\|T(a) - T(z)\| < \delta$ for $z \in [a, a + \varepsilon]$ and $\|T(b) - T(z)\| < \delta$ for $z \in [b - \varepsilon, b]$. Therefore, the Weyl inequality (9) implies that

$$\begin{aligned} n(1, T(a)) = n(1 + \delta, T(a)) \le n(1, T(z)), \quad z \in [a, a + \varepsilon], \\ n(1, T(b)) = n(1 + \delta, T(b)) \le n(1, T(z)), \quad z \in [b - \varepsilon, b], \end{aligned}$$

i.e., the function $n(1, T(.))$ is a monotone on the intervals $(a, a+\varepsilon)$ and $(b-\varepsilon, b)$. It follows that

$$\bigvee_{a}^{a+\varepsilon}(n(1,T(.))) = n(1,T(a+\varepsilon)) - n(1,T(a)) < \infty,$$

$$\bigvee_{b-\varepsilon}^{b}(n(1,T(.))) = n(1,T(b)) - n(1,T(b-\varepsilon)) < \infty.$$

Applying the Corollary 1, we obtain that the number of discrete eigenvalues of H located in $\left(a, a+\varepsilon\right) \cup \left(b-\varepsilon, b\right)$ are finite. The analyticity of $D(1, .)$ in some complex neighborhood of the closed interval $(a+\varepsilon, b-\varepsilon)$ implies that the set

$$\{z \in [a+\varepsilon, b-\varepsilon] : D(1,z) = 0\}$$

is finite. Consequently, applying Lemma 4, we obtain that the operator H have finite number of discrete eigenvalues located in $[a+\varepsilon, b-\varepsilon]$.

In the next example, we show that in Corollary 2, the compactness of the operator $T(b)$ is a sufficient condition for the finiteness of the number of discrete eigenvalues of H on $(a,b) \subset \mathbb{R} \setminus \sigma(H_{11}^{0})$.

4. Conclusion

In this paper, we have analyzed and provided the spectral properties of a special type of soluble model of coupled molecular and nuclear Hamiltonians H, the so-called generalized Friedrichs model. We have studied several important properties of the well-known Faddeev equation corresponding to H. These properties are related to the number of discrete (isolated and with finite multiplicity) eigenvalues. We have provided a new formula for the multiplicity of the discrete eigenvalues of H. Finally, we have provided the results on the number of discrete eigenvalues of H.

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Conflict of interest

There is no conflict of interest in this study.

References

[1] Friedrichs K. Über die spektralzerlegung eines integraloperators. *Mathematische Annalen*. 1938; 115(1): 249-272. Available from: https://doi.org/10.1007/BF01448941.

[2] Friedrichs KO. On the perturbation of continuous spectra. *Communications on Pure and Applied Mathematics*. 1948; 1(4): 361-406. Available from: https://doi.org/10.1002/cpa.3160010404.

[3] Lakaev SN. Some spectral properties of the generalized Friedrichs model. *Journal of Soviet Mathematics*. 1989; 45: 1540-1563. Available from: https://doi.org/10.1007/BF01097277.

[4] Boldrighini C, Minlos RA, Pellegrinotti A. Random walks in a random (fluctuating) environment. *Russian Mathematical Surveys*. 2007; 62(4): 663. Available from: https://doi.org/10.1070/RM2007v062n04ABEH004428.

[5] Lakshtanov EL, Minlos RA. The spectrum of two-particle bound states for the transfer matrices of Gibbs fields (an isolated bound state). *Functional Analysis and Its Applications*. 2004; 38: 202-216. Available from: https://doi.org/10.1023/B:FAIA.0000042805.04113.42.

[6] Akchurin ER. Spectral properties of the generalized Friedrichs model. *Theoretical and Mathematical Physics*. 2010; 163: 414-428. Available from: https://doi.org/10.1007/s11232-010-0032-4.

[7] Lakaev SN, Latipov SM. Existence and analyticity of eigenvalues of a two-channel molecular resonance model.

Theoretical and Mathematical Physics. 2011; 169(3): 1658-1667. Available from: https://doi.org/10.1007/s11232-011-0143-6.

[8] Muminov MI, Rasulov TH. The Faddeev equation and essential spectrum of a Hamiltonian in Fock space. *Methods of Functional Analysis and Topology*. 2011; 17(1): 47-57.

[9] Rasulov TH, Dilmurodov EB. Eigenvalues and virtual levels of a family of 2×2 operator matrices. *Methods of Functional Analysis and Topology*. 2019; 25(03): 273-281.

[10] Rasulov TH, Dilmurodov EB. Analysis of the spectrum of a 2×2 operator matrix. Discrete spectrum asymptotics. *Nanosystems: Physics, Chemistry, Mathematics*. 2020; 11(2): 138-144. Available from: https://doi.org/10.17586/2220-8054-2020-11-2-138-144.

[11] Rasulov TH, Dilmurodov EB. Infinite number of eigenvalues of 2×2 operator matrices: Asymptotic discrete spectrum. *Theoretical and Mathematical Physics*. 2020; 205(3): 1564-1584. Available from: https://doi.org/10.1134/S0040577920120028.

[12] Muminov MI, Rasulov TH, Tosheva NA. Analysis of the discrete spectrum of the family of 3×3 operator matrices. *Communications in Mathematical Analysis*. 2020; 23(1): 17-37.

[13] Rasulov TH, Dilmurodov EB. Estimates for the bounds of the essential spectrum of a 2×2 operator matrix. *Contemporary Mathematics*. 2020; 1(4): 170-186. Available from: https://doi.org/10.37256/cm.142020409.

[14] Muminov MI, Rasulov TH. On the eigenvalues of a 2×2 block operator matrix. *Opuscula Mathematica*. 2015; 35(3): 371-395. Available from: http://dx.doi.org/10.7494/OpMath.2015.35.3.371.

[15] Muminov MI, Rasulov TH. Infiniteness of the number of eigenvalues embedded in the essential spectrum of a 2×2 operator matrix. *Eurasian Mathematical Journal*. 2014; 5(2): 60-77.

[16] Muminov MI, Rasulov TH. Embedded eigenvalues of an Hamiltonian in bosonic Fock space. *Communications in Mathematical Analysis*. 2014; 17(1): 1-22.

[17] Reed M, Simon B. *Methods of modern mathematical physics*. *IV: Analysis of operators*. New York: Academic Press; 1978.

[18] Muminov MÉ. Expression for the number of eigenvalues of a Friedrichs model. *Mathematical Notes*. 2007; 82: 67-74. Available from: https://doi.org/10.1134/S0001434607070097.

[19] Glazman IM. *Direct methods of the qualitative spectral analysis of singular differential operators*. Jerusalem: Israel Program for Scientific Translations; 1965.