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## POINT SPECTRUM OF THE OPERATOR MATRICES WITH THE FREDHOLM INTEGRAL OPERATORS

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**Abstract.** *In the present paper we consider  $2 \times 2$  and  $3 \times 3$  operator matrices, where the matrix elements are Fredholm integral operators. It is shown that these matrices has only the purely point spectrum.*

**Key words:** *block operator matrix, Fredholm integral operator, point spectrum, linear operator, eigenvalue.*

## ТОЧЕЧНЫЙ СПЕКТР ОПЕРАТОРНЫХ МАТРИЦ С ИНТЕГРАЛЬНЫМИ ОПЕРАТОРАМИ ФРЕДГОЛЬМА

**Аннотация.** *В настоящей работе рассматриваются и операторные матрицы, матричными элементами которых являются интегральные операторы Фредгольма. Показано, что эти матрицы имеют только чисто точечный спектр.*

**Ключевые слова:** *блочно-операторная матрица, интегральный оператор Фредгольма, точечный спектр, линейный оператор, собственное значение.*

## FREDHOLM INTEGRAL OPERATORLARI BILAN OPERATOR MATRITASINING NOKTA SPEKTRIMI

**Annotatsiya.** *Ushbu maqolada biz operator matritsalarini ko'rib chiqdilk. Bu matritsa elementlari Fredholm integral operatorlari hisoblanadi. Ushbu matritsalar faqat nuqta spektriga ega ekanligi ko'rsatilgan.*

**Kalit so'zlar:** *blok operator matritsasi, Fredholm integral operatori, nuqta spektri, chiziqli operator, xos qiymat.*

**Introduction.** Block operator matrices are matrices the entries which are linear operators between Banach or Hilbert spaces [1]. They arise in various areas of mathematics and its applications. One of the important class of the block operator matrices are the energy operators of a system of  $n$  non conserved number of particles. Such systems occur widely in mathematical physics, e.g. statistical physics [2 – 8], solid-state physics [6] and the theory of quantum fields [6,7].

In the theory of solid-state physics [6], quantum field theory [7] and statistical physics [2 – 8] some important problems arise where the number of quasi-particles is not fixed. The study of systems with a non conserved, but bounded, number of particles is reduced to the study of the spectral properties of self-adjoint operators acting in "the cut" subspace  $\mathcal{H}^{(n)}$ , consisting of on particle, two particle and  $n$ - particle subspaces of the Fock space [2,7,8]

In mathematics, Fredholm operators are certain operators that arise in the Fredholm theory of integral equations. They are named in honour of Erik Ivar Fredholm.

A linear operator  $\mathcal{A}$  from a Banach space  $X$  to a Banach space  $Y$  is called a Fredholm operator if

1.  $\mathcal{A}$  is closed;
2. the domain  $D(\mathcal{A})$  of  $\mathcal{A}$  is dense in  $X$ ;
3.  $\alpha(\mathcal{A})$ , the dimension of the null space  $N(\mathcal{A})$  of  $\mathcal{A}$ , is finite;
4.  $R(\mathcal{A})$ , the range of  $\mathcal{A}$ , is closed in  $Y$ ;
5.  $\beta(\mathcal{A})$ , the codimension of  $R(\mathcal{A})$  in  $Y$ , is finite.

In particular, on the spaces  $C[a; b]$  or  $L_2[a; b]$  an operator of the form

$$(\mathcal{A}\phi)(x) = \int_a^b K(x,t)\phi(t)dt, \quad (1)$$

where the kernel  $K(\cdot, \cdot)$  is continuous and hence square-integrable function on  $[a; b] \times [a; b]$ , is Fredholm. The operator of the form (1) is also called a linear Fredholm integral operator with the kernel  $K(\cdot, \cdot)$ . In the present paper we considered the case where the kernel  $K(\cdot, \cdot)$  is degenerate.

In the present paper for  $n = 2, 3$  we consider  $n \times n$  block operator matrix  $\mathcal{A}_n$  with the Fredholm integral operator.

We obtain the following results:

-determined the point spectrum of the operator matrices  $\mathcal{A}_\alpha, \alpha = 1, 2, 3$  with the Fredholm integral operators;

- $\mathcal{A}_\alpha, \alpha = 1, 2, 3$  important spectrum for operator matrices are studied;

- Determinants corresponding to matrices with operator  $\mathcal{A}_\alpha, \alpha = 1, 2, 3$  are constructed

- The numerical range of matrices with operator  $\mathcal{A}_\alpha, \alpha = 1, 2, 3$  is defined

**$n \times n (n = 2, 3)$  operator matrices with the Fredholm integral operators**

Let  $\mathbb{T}^d$  be the  $d$ -dimensional torus and  $L_2(\mathbb{T}^d)$  be the Hilbert space of square integrable symmetric (complex) functions defined on  $\mathbb{T}^d$ .

In the Hilbert space  $L_2(\mathbb{T}^d)$  we consider the Fredholm integral operators of the form

$$(A_{ij}f_j)(x) = a_{ji}(x) \int_{\mathbb{T}^d} a_{ij}(t) f_j(t) dt, \quad f_i \in L_2(\mathbb{T}^d), \quad i \leq j, \quad i, j = 1, 2, 3,$$

where  $a_{ij}(\cdot), i, j = 1, 2, 3$  are the real-valued continuous functions on  $\mathbb{T}^d$ . Then it is easy to see that

$$A_{ij}^* = A_{ji} \text{ for all } i, j = 1, 2, 3.$$

First we investigate the spectrum of  $\mathcal{A}_1 := A_{11}$ . Direct calculations show that the operator  $\mathcal{A}_1$  has a purely point spectrum and the equality  $\sigma_{pp}(\mathcal{A}_1) = \{0, \|a_{11}\|^2\}$  holds, where the number  $\lambda = 0$  is an eigenvalue of  $\mathcal{A}_1$  with infinite multiplicity, the number  $\lambda = \|a_{11}\|^2$  is a simple eigenvalue of  $\mathcal{A}_1$ .

For the further discussions we denote

$$L_2^{(2)}(\mathbb{T}^d) := \{f = (f_1, f_2) : f_\alpha \in L_2(\mathbb{T}^d), \alpha = 1, 2\}$$

$$L_2^{(3)}(\mathbb{T}^d) := \{f = (f_1, f_2, f_3) : f_\alpha \in L_2(\mathbb{T}^d), \alpha = 1, 2, 3\}.$$

Notice that the norm and scalar product in  $L_2^{(3)}(\mathbb{T}^d)$  are defined as

$$\|f\| = \left( \int_{\mathbb{T}^d} |f_1(t)|^2 dt + \int_{\mathbb{T}^d} |f_2(t)|^2 dt + \int_{\mathbb{T}^d} |f_3(t)|^2 dt \right)^{1/2};$$

$$(f, g) = \int_{\mathbb{T}^d} f_1(t) \overline{g_1(t)} dt + \int_{\mathbb{T}^d} f_2(t) \overline{g_2(t)} dt + \int_{\mathbb{T}^d} f_3(t) \overline{g_3(t)} dt$$

for  $f = (f_1, f_2, f_3), g = (g_1, g_2, g_3) \in L_2^{(3)}(\mathbb{T}^d)$

For  $n = 2, 3$  in the Hilbert space  $L_2^{(n)}(\mathbb{T}^d)$  we consider the following  $n \times n$  operator matrix

$$\mathcal{A}_2 := \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad \mathcal{A}_3 := \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}.$$

Under these assumptions the operator matrix  $\mathcal{A}_\alpha$  is bounded and self-adjoint in  $L_2^{(\alpha)}(\mathbb{T}^d)$  for  $\alpha = 2, 3$ .

Operators of this type arise in the process of constructing the Faddeev equations for the eigenfunctions of the model operators corresponding to the Hamiltonians of a three-particle system on a lattice [9,10].

Note that all matrix elements  $A_{ij}$  of  $\mathcal{A}_3$  are one-dimensional operators, and hence depending on the functions  $a_{ij}(\cdot), i, j = 1, 2, 3$  the operator matrix  $\mathcal{A}_3$  is an at most 9-dimensional operator. Analogously, the operator matrix  $\mathcal{A}_2$  is an at most 4-dimensional operator. Since  $L_2(\mathbb{T}^d), L_2^{(2)}(\mathbb{T}^d)$  and  $L_2^{(3)}(\mathbb{T}^d)$  are the infinite-dimensional Hilbert spaces, that is,

$$\dim L_2(\mathbb{T}^d) = \dim L_2^{(2)}(\mathbb{T}^d) = \dim L_2^{(3)}(\mathbb{T}^d) = \infty,$$

the equalities hold:

$$\sigma_{ess}(\mathcal{A}_1) = \sigma_{ess}(\mathcal{A}_2) = \sigma_{ess}(\mathcal{A}_3) = \{0\}$$

To study the non zero eigenvalues of the operator matrices  $\mathcal{A}_\alpha$ ,  $\alpha = 2,3$  we introduce the following functions:

$$\Delta_2(\lambda) := \begin{vmatrix} \Delta_{11}(\lambda) & \Delta_{12} & 0 & 0 \\ 0 & \Delta_{22}(\lambda) & \Delta_{24} & \Delta_{25} \\ \Delta_{12} & \Delta_{42} & \Delta_{33}(\lambda) & 0 \\ 0 & 0 & \Delta_{25} & \Delta_{55}(\lambda) \end{vmatrix},$$

$$\Delta_3(\lambda) := \begin{vmatrix} \Delta_{11}(\lambda) & \Delta_{12} & \dots & \Delta_{19} \\ \Delta_{21} & \Delta_{22}(\lambda) & \dots & \Delta_{29} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{91} & \Delta_{92} & \dots & \Delta_{99}(\lambda) \end{vmatrix},$$

where the matrix elements are defined by

$$\begin{aligned} \Delta_{11}(\lambda) &:= \|a_{11}\|^2 - \lambda, & \Delta_{12} &:= (a_{11}, a_{21}), & \Delta_{13} &:= (a_{11}, a_{31}); \\ \Delta_{22}(\lambda) &:= -\lambda, & \Delta_{24} &:= \|a_{12}\|^2, & \Delta_{25} &:= (a_{12}, a_{22}), & \Delta_{26} &:= (a_{12}, a_{32}); \\ \Delta_{33}(\lambda) &:= -\lambda, & \Delta_{37} &:= \|a_{13}\|^2, & \Delta_{38} &:= (a_{13}, a_{23}), & \Delta_{39} &:= (a_{13}, a_{33}); \\ \Delta_{41} &:= (a_{21}, a_{11}), & \Delta_{42} &:= \|a_{21}\|^2, & \Delta_{43} &:= (a_{21}, a_{31}), & \Delta_{44}(\lambda) &:= -\lambda; \\ & & \Delta_{54} &:= (a_{22}, a_{12}), & \Delta_{55} &:= \|a_{22}\|^2 - \lambda, & \Delta_{56} &:= (a_{22}, a_{32}); \\ \Delta_{66}(\lambda) &:= -\lambda, & \Delta_{67} &:= (a_{23}, a_{13}), & \Delta_{68} &:= \|a_{23}\|^2, & \Delta_{69} &:= (a_{23}, a_{33}); \\ \Delta_{71} &:= (a_{31}, a_{11}), & \Delta_{72} &:= (a_{31}, a_{21}), & \Delta_{73} &:= \|a_{31}\|^2, & \Delta_{77}(\lambda) &:= -\lambda; \\ \Delta_{84} &:= (a_{32}, a_{12}), & \Delta_{85} &:= (a_{32}, a_{22}), & \Delta_{86} &:= \|a_{32}\|^2, & \Delta_{88}(\lambda) &:= -\lambda; \\ \Delta_{97} &:= (a_{33}, a_{13}), & \Delta_{98} &:= (a_{33}, a_{23}), & \Delta_{86} &:= \|a_{32}\|^2, & \Delta_{99}(\lambda) &:= \|a_{33}\|^2 - \lambda \\ & & & & & \Delta_{ij} &= 0, & \text{otherwise.} \end{aligned}$$

First we discuss the case  $\alpha = 2$ . In order to study the non-zero eigenvalues of  $\mathcal{A}_2$  we consider the eigenvalue equation  $\mathcal{A}_2 f = \lambda f$ ,  $f = (f_1, f_2)$  or

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \lambda f_1 \\ \lambda f_2 \end{pmatrix}.$$

We write this equation in the form of the following system of equations:

$$\begin{cases} A_{11}f_1 + A_{12}f_2 = \lambda f_1; \\ A_{21}f_1 + A_{22}f_2 = \lambda f_2. \end{cases} \quad (2)$$

Using the definition of operators  $A_{ij}$ , we transform system (2) into an equivalent system:

$$\begin{cases} a_{11}(x) \int_{T^d} a_{11}(t) f_1(t) dt + a_{21}(x) \int_{T^d} a_{12}(t) f_2(t) dt = \lambda f_1(x); \\ a_{12}(x) \int_{T^d} a_{21}(t) f_1(t) dt + a_{22}(x) \int_{T^d} a_{22}(t) f_2(t) dt = \lambda f_2(x). \end{cases} \quad (3)$$

We introduce the following definition:

$$c_{ij} = \int_{T^d} a_{ij}(t) f_j(t) dt, \quad i, j = 1, 2 \quad (4)$$

then system (3) becomes

$$\begin{cases} a_{11}(x)c_{11} + a_{21}(x)c_{12} = \lambda f_1(x); \\ a_{12}(x)c_{21} + a_{22}(x)c_{22} = \lambda f_2(x). \end{cases} \quad (5)$$

(5) dividing both equations of the system by the number  $\lambda \neq 0$ , we find the following expressions for

$f_1(x)$  and  $f_2(x)$ :

$$\begin{cases} f_1(x) = \frac{1}{\lambda} [a_{11}(x)c_{11} + a_{21}(x)c_{12}]; \\ f_2(x) = \frac{1}{\lambda} [a_{12}(x)c_{21} + a_{22}(x)c_{22}]. \end{cases} \quad (6)$$



we put equalities (6) in (4):

$$\begin{aligned} c_{11} &= \frac{1}{\lambda} \cdot \int_{T^d} [a_{11}^2(t)c_{11} + a_{11}(t) \cdot a_{21}(t)c_{12}]dt; \\ c_{12} &= \frac{1}{\lambda} \cdot \int_{T^d} [a_{12}^2(t)c_{21} + a_{12}(t) \cdot a_{22}(t)c_{22}]dt; \\ c_{21} &= \frac{1}{\lambda} \cdot \int_{T^d} [a_{21}^2(t)c_{12} + a_{11}(t) \cdot a_{21}(t)c_{11}]dt; \\ c_{22} &= \frac{1}{\lambda} \cdot \int_{T^d} [a_{22}^2(t)c_{22} + a_{12}(t) \cdot a_{22}(t)c_{21}]dt. \end{aligned}$$

The resulting system of equations is reduced to the following system of homogeneous equations:

$$\begin{cases} (\|a_{11}\|^2 - \lambda)c_{11} + (a_{11}, a_{21})c_{12} = 0 \\ -\lambda c_{12} + \|a_{12}\|^2 c_{21} + (a_{12}, a_{22})c_{22} = 0 \\ (a_{11}, a_{21})c_{11} + \|a_{21}\|^2 c_{12} - \lambda c_{21} = 0 \\ (a_{12}, a_{22})c_{21} + (\|a_{22}\|^2 - \lambda)c_{22} = 0 \end{cases} \quad (7)$$

Similarly,  $n = 3$  cases are analyzed. In order to study the non-zero eigenvalues of the operator matrix

$\mathcal{A}_3$ , we consider the eigenvalue equation  $\mathcal{A}_3 f = \lambda f$ ,  $f = (f_1, f_2, f_3)$ , or:

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} \lambda f_1 \\ \lambda f_2 \\ \lambda f_3 \end{pmatrix}.$$

From this, we write down the following system of equations:

$$\begin{cases} A_{11}f_1 + A_{12}f_2 + A_{13}f_3 = \lambda f_1 \\ A_{21}f_1 + A_{22}f_2 + A_{23}f_3 = \lambda f_2 \\ A_{31}f_1 + A_{32}f_2 + A_{33}f_3 = \lambda f_3 \end{cases}$$

From the definition of operators  $A_{ij}$  system

$$\begin{cases} a_{11}(x) \int_{T^d} a_{11}(t)f_1(t)dt + a_{21}(x) \int_{T^d} a_{12}(t)f_2(t)dt + a_{31}(x) \int_{T^d} a_{13}(t)f_3(t)dt = \lambda f_1(x); \\ a_{12}(x) \int_{T^d} a_{21}(t)f_1(t)dt + a_{22}(x) \int_{T^d} a_{22}(t)f_2(t)dt + a_{32}(x) \int_{T^d} a_{23}(t)f_3(t)dt = \lambda f_2(x) \\ a_{13}(x) \int_{T^d} a_{31}(t)f_1(t)dt + a_{23}(x) \int_{T^d} a_{32}(t)f_2(t)dt + a_{33}(x) \int_{T^d} a_{33}(t)f_3(t)dt = \lambda f_3(x) \end{cases}$$

appears. By assigning  $c_{ij}$ , ( $i, j = 1, 2, 3$ ) to this system of equations, we write the following equivalent system:

$$\begin{cases} a_{11}(x)c_{11} + a_{21}(x)c_{12} + a_{31}(x)c_{13} = \lambda f_1(x); \\ a_{12}(x)c_{21} + a_{22}(x)c_{22} + a_{32}(x)c_{23} = \lambda f_2(x); \\ a_{13}(x)c_{31} + a_{23}(x)c_{32} + a_{33}(x)c_{33} = \lambda f_3(x). \end{cases}$$

We divide by the number  $\lambda \neq 0$ :

$$\begin{cases} f_1(x) = \frac{1}{\lambda} [a_{11}(x)c_{11} + a_{21}(x)c_{12} + a_{31}(x)c_{13}]; \\ f_2(x) = \frac{1}{\lambda} [a_{12}(x)c_{21} + a_{22}(x)c_{22} + a_{32}(x)c_{23}]; \\ f_3(x) = \frac{1}{\lambda} [a_{13}(x)c_{31} + a_{23}(x)c_{32} + a_{33}(x)c_{33}]. \end{cases} \quad (8)$$

Let's set  $f_1(x)$ ,  $f_2(x)$ ,  $f_3(x)$  found in expression (8) to  $c_{ij}$ :

$$c_{11} = \frac{1}{\lambda} \cdot \int_{T^d} [a_{11}^2(t)c_{11} + a_{11}(t) \cdot a_{21}(t)c_{12} + a_{11}(t) \cdot a_{31}(t)c_{13}]dt;$$

$$c_{12} = \frac{1}{\lambda} \cdot \int_{T^d} [a_{12}^2(t)c_{21} + a_{12}(t) \cdot a_{22}(t)c_{22} + a_{12}(t) \cdot a_{32}(t)c_{23}]dt;$$

$$c_{13} = \frac{1}{\lambda} \cdot \int_{T^d} [a_{13}^2(t)c_{31} + a_{13}(t) \cdot a_{23}(t)c_{32} + a_{13}(t) \cdot a_{33}(t)c_{33}]dt;$$

$$c_{21} = \frac{1}{\lambda} \cdot \int_{T^d} [a_{21}^2(t)c_{12} + a_{11}(t) \cdot a_{21}(t)c_{11} + a_{21}(t) \cdot a_{31}(t)c_{13}]dt;$$

$$c_{22} = \frac{1}{\lambda} \cdot \int_{T^d} [a_{22}^2(t)c_{22} + a_{12}(t) \cdot a_{22}(t)c_{21} + a_{22}(t) \cdot a_{32}(t)c_{23}]dt;$$

$$c_{23} = \frac{1}{\lambda} \cdot \int_{T^d} [a_{23}^2(t)c_{32} + a_{23}(t) \cdot a_{13}(t)c_{31} + a_{23}(t) \cdot a_{33}(t)c_{33}]dt;$$

$$c_{31} = \frac{1}{\lambda} \cdot \int_{T^d} [a_{31}^2(t)c_{13} + a_{31}(t) \cdot a_{11}(t)c_{11} + a_{31}(t) \cdot a_{21}(t)c_{12}]dt;$$

$$c_{32} = \frac{1}{\lambda} \cdot \int_{T^d} [a_{32}^2(t)c_{23} + a_{12}(t) \cdot a_{32}(t)c_{21} + a_{22}(t) \cdot a_{32}(t)c_{22}]dt;$$

$$c_{33} = \frac{1}{\lambda} \cdot \int_{T^d} [a_{33}^2(t)c_{33} + a_{13}(t) \cdot a_{33}(t)c_{31} + a_{23}(t) \cdot a_{33}(t)c_{32}]dt.$$

The following system of homogeneous equations is written from the resulting system of equations:

$$\begin{cases} (\|a_{11}\|^2 - \lambda)c_{11} + (a_{11}, a_{21})c_{12} + (a_{11}, a_{31})c_{13} = 0; \\ -\lambda c_{12} + (a_{12}, a_{22})c_{22} + (a_{12}, a_{32})c_{23} = 0; \\ -\lambda c_{13} + \|a_{13}\|^2 c_{31} + (a_{13}, a_{23})c_{32} + (a_{13}, a_{33})c_{33} = 0; \\ (a_{11}, a_{21})c_{11} + \|a_{21}\|^2 c_{12} + (a_{21}, a_{31})c_{13} - \lambda c_{21} = 0; \\ (a_{12}, a_{22})c_{21} + (\|a_{22}\|^2 - \lambda)c_{22} + (a_{22}, a_{32})c_{23} = 0; \\ -\lambda c_{23} + (a_{23}, a_{13})c_{31} + \|a_{23}\|^2 c_{32} + (a_{23}, a_{33})c_{33} = 0; \\ (a_{11}, a_{31})c_{11} + (a_{31}, a_{21})c_{12} + \|a_{31}\|^2 c_{13} - \lambda c_{31} = 0; \\ (a_{12}, a_{32})c_{21} + (a_{32}, a_{22})c_{22} + \|a_{32}\|^2 c_{23} - \lambda c_{32} = 0; \\ (a_{13}, a_{33})c_{31} + (a_{33}, a_{23})c_{32} + (\|a_{33}\|^2 - \lambda)c_{33} = 0. \end{cases}$$

In the following theorem we describe the point spectrum of  $\mathcal{A}_\alpha$ ,  $\alpha = 2,3$ .

**Theorem 1.** For  $\alpha = 2,3$  the operator matrix  $\mathcal{A}_\alpha$  has a purely point spectrum and

$$\sigma_{pp}(\mathcal{A}_\alpha) = \{0\} \cup \{\lambda \in \mathbb{R}: \Delta_\alpha(\lambda) = 0\}.$$

Moreover, the number  $\lambda = 0$  is an eigenvalue of  $\mathcal{A}_\alpha$  with infinite multiplicity.

It can be seen that the function  $\Delta_2(\cdot)$  is a polynomial of order 4 with respect to  $\lambda$ . Therefore, it has at most 4 real zeros (taking into account the multiplicity). Therefore, by virtue of Theorem 1, an operator matrix  $\mathcal{A}_2$  can have at most 4 (taking into account the multiplicity) eigenvalues with finite multiplicity.

Analogously, an operator matrix  $\mathcal{A}_3$  can have at most 9 (taking into account the multiplicity) eigenvalues with finite multiplicity.

Using Theorem 1 and the fact about  $\sigma_{pp}(\mathcal{A}_1)$  it is possible to find an exact representation of the numerical range of the operator  $\mathcal{A}_\alpha$ ,  $\alpha = 1,2,3$ . It should be noted that since the operator  $\mathcal{A}_\alpha$ ,  $\alpha = 1,2,3$  has a purely point spectrum, its numerical range  $W(\mathcal{A}_\alpha)$  always a bounded (closed) segment and for  $\alpha = 1,2,3$  the equality

$$W(\mathcal{A}_\alpha) = [\min \sigma_{pp}(\mathcal{A}_\alpha); \max \sigma_{pp}(\mathcal{A}_\alpha)]$$

is valid. In particular, we have  $W(\mathcal{A}_1) = [0; \|a_{11}\|^2]$ . The study of quadratic numerical range of  $\mathcal{A}_2$  and cubic numerical range of  $\mathcal{A}_3$  needs an additional investigations.

**$\mathcal{A}_1$  operator resolvent.** In order to determine the resolvent operator of the operator  $\mathcal{A}_1$ , we consider the equation

$$(\mathcal{A}_1 - \lambda)f = g$$

with respect to the functions  $f, g \in L_2(\mathbb{T}^d)$  for the number  $\lambda$ .

$\mathcal{A}_1$  is the last equality according to the definition of the operator

$$a_{11}(x) \cdot c_{11} - \lambda f(x) = g(x)$$

appears. Here, the value of  $c_{11}$  is determined using equation (4). From the above equation, we find the following expression for the function  $f(x)$ :

$$f(x) = \frac{1}{\lambda} [a_{11}(x) \cdot c_{11} - g(x)] \tag{9}$$

We put the expression (9) found for  $f(x)$  in (4), and

$$c_{11} = \frac{1}{\lambda} \int_{\mathbb{T}^d} a_{11}(t) [a_{11}(t) \cdot c_{11} - g(t)] dt$$

we will have equality. From this we find  $c_{11}$  and put it in the expression found for  $f(x)$  and

$$f(x) = \frac{1}{\lambda} \left( \frac{a_{11}(x)}{\|a_{11}\|^2 - \lambda} (a_{11}, g) - g(x) \right)$$

we form the equation. The expression on the right side of this equality indicates the action formula of the resolvent operator  $R_{\mathcal{A}_1}(\lambda)$  corresponding to the operator  $\mathcal{A}_1$ . Which

$$R_{\mathcal{A}_1}(\lambda)g = \frac{1}{\lambda} \left( \frac{a_{11}(x)}{\|a_{11}\|^2 - \lambda} \int_{\mathbb{T}^d} a_{11}(t) \cdot g_1(t) dt - g_1(x) \right)$$

**$\mathcal{A}_2$  operator resolvent.** We define the resolvent operator of the matrix with operator  $\mathcal{A}_2$ . For this, we consider the following equation with respect to  $f = (f_1, f_2), g = (g_1, g_2) \in L_2(\mathbb{T}^d)$  vector functions:

$$(\mathcal{A}_2 - \lambda)f = g$$

According to the definition of the matrix with operator  $\mathcal{A}_2$ , we write the following equation:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} - \lambda \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}.$$

We form the following system of equations from this equation:

$$\begin{cases} A_{11}f_1 + A_{12}f_2 - \lambda f_1 = g_1; \\ A_{21}f_1 + A_{22}f_2 - \lambda f_2 = g_2. \end{cases}$$

$$\begin{cases} A_{11}f_1 + A_{12}f_2 - \lambda f_1 = g_1; \\ A_{21}f_1 + A_{22}f_2 - \lambda f_2 = g_2. \end{cases}$$

According to the definition of operators  $A_{ij}$ , this system of equations takes the following form:

$$\begin{cases} a_{11}(x)c_{11} + a_{21}(x)c_{12} - \lambda f_1(x) = g_1(x); \\ a_{12}(x)c_{21} + a_{22}(x)c_{22} - \lambda f_2(x) = g_2(x). \end{cases}$$

$$\begin{cases} a_{11}(x)c_{11} + a_{21}(x)c_{12} - \lambda f_1(x) = g_1(x); \\ a_{12}(x)c_{21} + a_{22}(x)c_{22} - \lambda f_2(x) = g_2(x). \end{cases}$$

We can find  $f_1(x)$  from the first equation of the last system of equations, and  $f_2(x)$  from the second equation. Which

$$\begin{cases} f_1(x) = \frac{1}{\lambda} [a_{11}(x)c_{11} + a_{21}(x)c_{12} - g_1(x)]; \\ f_2(x) = \frac{1}{\lambda} [a_{12}(x)c_{21} + a_{22}(x)c_{22} - g_2(x)]. \end{cases}$$

Let's define the last expressions found in (4):

$$c_{11} = \frac{1}{\lambda} \cdot \int_{\mathbb{T}^d} [a_{11}^2(t)c_{11} + a_{11}(t) \cdot a_{21}(t)c_{12} - a_{11}(t) \cdot g_1(t)] dt;$$

$$c_{12} = \frac{1}{\lambda} \cdot \int_{\mathbb{T}^d} [a_{12}^2(t)c_{21} + a_{12}(t) \cdot a_{22}(t)c_{22} - a_{12}(t) \cdot g_2(t)] dt;$$

$$c_{21} = \frac{1}{\lambda} \cdot \int_{\mathbb{T}^d} [a_{21}^2(t)c_{12} + a_{11}(t) \cdot a_{21}(t)c_{11} - a_{21}(t) \cdot g_1(t)] dt;$$

$$c_{22} = \frac{1}{\lambda} \cdot \int_{\mathbb{T}^d} [a_{22}^2(t)c_{22} + a_{12}(t) \cdot a_{22}(t)c_{21} - a_{22}(t) \cdot g_2(t)] dt$$

or,

$$\begin{cases} (\|a_{11}\|^2 - \lambda)c_{11} + (a_{11}, a_{21})c_{12} - (a_{11}, g_1) = 0; \\ -\lambda c_{12} + \|a_{12}\|^2 c_{21} + (a_{12}, a_{22})c_{22} - (a_{12}, g_2) = 0; \\ (a_{11}, a_{21})c_{11} + \|a_{21}\|^2 c_{12} - \lambda c_{21} - (a_{21}, g_1) = 0; \\ (a_{12}, a_{22})c_{21} + (\|a_{22}\|^2 - \lambda)c_{22} - (a_{22}, g_2) = 0. \end{cases} \quad (10)$$

we form a system of equations. We are looking for the solution of the system of equations (10). For this, we introduce the following definitions:

$$\Delta_2(\lambda) = \begin{vmatrix} \|a_{11}\|^2 - \lambda & (a_{11}, a_{21}) & 0 & 0 \\ 0 & -\lambda & \|a_{12}\|^2 & (a_{12}, a_{22}) \\ (a_{11}, a_{21}) & \|a_{21}\|^2 & -\lambda & 0 \\ 0 & 0 & (a_{12}, a_{22}) & \|a_{22}\|^2 - \lambda \end{vmatrix}; \quad G = \begin{pmatrix} (a_{11}, g_1) \\ (a_{12}, g_2) \\ (a_{21}, g_1) \\ (a_{22}, g_2) \end{pmatrix};$$

$$\Delta_2^{(1)}(\lambda) = \begin{vmatrix} (a_{11}, g_1) & (a_{11}, a_{21}) & 0 & 0 \\ (a_{12}, g_2) & -\lambda & \|a_{12}\|^2 & (a_{12}, a_{22}) \\ (a_{21}, g_1) & \|a_{21}\|^2 & -\lambda & 0 \\ (a_{22}, g_2) & 0 & (a_{12}, a_{22}) & \|a_{22}\|^2 - \lambda \end{vmatrix} =$$

$$= (a_{11}, g_1) \cdot B_1(\lambda) - (a_{11}, a_{21}) \cdot G_1(\lambda);$$

$$\Delta_2^{(2)}(\lambda) = \begin{vmatrix} \|a_{11}\|^2 - \lambda & (a_{11}, g_1) & 0 & 0 \\ 0 & (a_{12}, g_2) & \|a_{12}\|^2 & (a_{12}, a_{22}) \\ (a_{11}, a_{21}) & (a_{21}, g_1) & -\lambda & 0 \\ 0 & (a_{22}, g_2) & (a_{12}, a_{22}) & \|a_{22}\|^2 - \lambda \end{vmatrix} =$$

$$= (\|a_{11}\|^2 - \lambda) \cdot G_1(\lambda) - (a_{11}, g_1) \cdot B_2(\lambda);$$

$$\Delta_2^{(3)}(\lambda) = \begin{vmatrix} \|a_{11}\|^2 - \lambda & (a_{11}, a_{21}) & (a_{11}, g_1) & 0 \\ 0 & -\lambda & (a_{12}, g_2) & (a_{12}, a_{22}) \\ (a_{11}, a_{21}) & \|a_{21}\|^2 & (a_{21}, g_1) & 0 \\ 0 & 0 & (a_{22}, g_2) & \|a_{22}\|^2 - \lambda \end{vmatrix} =$$

$$= (\|a_{22}\|^2 - \lambda) \cdot G_2(\lambda) - (a_{22}, g_2) \cdot B_3(\lambda);$$

$$\Delta_2^{(4)}(\lambda) = \begin{vmatrix} \|a_{11}\|^2 - \lambda & (a_{11}, a_{21}) & 0 & (a_{11}, g_1) \\ 0 & -\lambda & \|a_{12}\|^2 & (a_{12}, g_2) \\ (a_{21}, a_{11}) & \|a_{21}\|^2 & -\lambda & (a_{21}, g_1) \\ 0 & 0 & (a_{12}, a_{22}) & (a_{22}, g_2) \end{vmatrix} =$$

$$= (a_{22}, g_2) \cdot B_4(\lambda) - (a_{12}, a_{22}) \cdot G_2(\lambda).$$

Here

$$B_1(\lambda) = \begin{vmatrix} -\lambda & \|a_{12}\|^2 & (a_{12}, a_{22}) \\ \|a_{21}\|^2 & -\lambda & 0 \\ 0 & (a_{12}, a_{22}) & \|a_{22}\|^2 - \lambda \end{vmatrix}; \quad B_2(\lambda) = \begin{vmatrix} 0 & \|a_{12}\|^2 & (a_{12}, a_{22}) \\ (a_{11}, a_{21}) & -\lambda & 0 \\ 0 & (a_{12}, a_{22}) & \|a_{22}\|^2 - \lambda \end{vmatrix};$$

$$B_3(\lambda) = \begin{vmatrix} \|a_{11}\|^2 - \lambda & (a_{11}, a_{21}) & 0 \\ 0 & -\lambda & (a_{12}, a_{22}) \\ (a_{11}, a_{21}) & \|a_{21}\|^2 & 0 \end{vmatrix}; \quad B_4(\lambda) = \begin{vmatrix} \|a_{11}\|^2 - \lambda & (a_{11}, a_{21}) & 0 \\ 0 & -\lambda & \|a_{12}\|^2 \\ (a_{11}, a_{21}) & \|a_{21}\|^2 & -\lambda \end{vmatrix};$$

$$G_1(\lambda) = \begin{vmatrix} (a_{12}, g_2) & \|a_{12}\|^2 & (a_{12}, a_{22}) \\ (a_{21}, g_1) & -\lambda & 0 \\ (a_{22}, g_2) & (a_{12}, a_{22}) & \|a_{22}\|^2 - \lambda \end{vmatrix}; \quad G_2(\lambda) = \begin{vmatrix} \|a_{11}\|^2 - \lambda & (a_{11}, a_{21}) & (a_{11}, g_1) \\ 0 & -\lambda & (a_{12}, g_2) \\ (a_{11}, a_{21}) & \|a_{21}\|^2 & (a_{21}, g_1) \end{vmatrix}.$$

The solutions of the system of equations (10) are as follows:

$$c_{11} = \frac{1}{\Delta_2(\lambda)} \cdot [(a_{11}, g_1) \cdot B_1(\lambda) - (a_{11}, a_{21}) \cdot G_1(\lambda)];$$

$$c_{12} = \frac{1}{\Delta_2(\lambda)} \cdot [(\| a_{11} \|^2 - \lambda) \cdot G_1(\lambda) - (a_{11}, g_1) \cdot B_2(\lambda)];$$

$$c_{21} = \frac{1}{\Delta_2(\lambda)} \cdot [(\| a_{22} \|^2 - \lambda) \cdot G_2(\lambda) - (a_{22}, g_2) \cdot B_3(\lambda)];$$

$$c_{22} = \frac{1}{\Delta_2(\lambda)} \cdot [(a_{22}, g_2) \cdot B_4(\lambda) - (a_{12}, a_{22}) \cdot G_2(\lambda)].$$

We put the expressions found for  $c_{ij}$  into the expressions found for  $f_1(x)$  and  $f_2(x)$  and form the following equations:

$$f_1(x) = \frac{1}{\lambda \cdot \Delta_2(\lambda)} \cdot [(a_{11}(x) \cdot B_1(\lambda) - a_{21}(x) \cdot B_2(\lambda)) \cdot (a_{11}, g_1) + [a_{21}(x) \times \\ \times (\| a_{11} \|^2 - \lambda) - a_{11}(x) \cdot (a_{11}, a_{21})] \cdot [(a_{12}, a_{22})^2 - \| a_{12} \|^2 \cdot (\| a_{22} \|^2 - \lambda)] \cdot (a_{21}, g_1)] + \\ + \frac{1}{\Delta_2(\lambda)} \cdot [a_{21}(x) \cdot (\| a_{11} \|^2 - \lambda) - a_{11}(x) \cdot (a_{11}, a_{21})] \times \\ \times [(a_{12}, a_{22}) \cdot (a_{22}, g_2) - (\| a_{22} \|^2 - \lambda) \cdot (a_{12}, g_2)] - \frac{1}{\lambda} g_1(x);$$

$$f_2(x) = \frac{1}{\lambda \cdot \Delta_2(\lambda)} \cdot [(a_{22}(x) \cdot B_4(\lambda) - a_{12}(x) \cdot B_3(\lambda)) \cdot (a_{22}, g_2) + [a_{12}(x) \times \\ \times (\| a_{22} \|^2 - \lambda) - a_{22}(x) \cdot (a_{12}, a_{22})] \cdot [(a_{11}, a_{21})^2 - \| a_{21} \|^2 \cdot (\| a_{11} \|^2 - \lambda)] \cdot (a_{12}, g_2)] + \\ + \frac{1}{\Delta_2(\lambda)} \cdot [a_{12}(x) \cdot (\| a_{22} \|^2 - \lambda) - a_{22}(x) \cdot (a_{12}, a_{22})] \times \\ \times [(a_{11}, a_{21}) \cdot (a_{11}, g_1) - (\| a_{11} \|^2 - \lambda) \cdot (a_{21}, g_1)] - \frac{1}{\lambda} g_2(x)$$

The expressions on the right side of the resulting equations indicate the action formula of the resolvent operator  $R_{\mathcal{A}_2}(\lambda)$  corresponding to the operator matrix  $\mathcal{A}_2$ . Which,

$$R_{\mathcal{A}_2}(\lambda) = \begin{pmatrix} R_{11}(\lambda) & R_{12}(\lambda) \\ R_{21}(\lambda) & R_{22}(\lambda) \end{pmatrix}$$

$$R_{11}(\lambda)g_1 = \frac{1}{\lambda \cdot \Delta_2(\lambda)} \cdot [(a_{11}(x) \cdot B_1(\lambda) - a_{21}(x) \cdot B_2(\lambda)) \cdot \int_{T^d} a_{11}(t) \cdot g_1(t) dt + \\ + (a_{21}(x) \cdot (\| a_{11} \|^2 - \lambda) - a_{11}(x) \cdot (a_{11}, a_{21})) \times \\ \times \left( (a_{12}, a_{22})^2 - \| a_{12} \|^2 \cdot (\| a_{22} \|^2 - \lambda) \cdot \int_{T^d} a_{21}(t) \cdot g_1(t) dt \right)] - \frac{1}{\lambda} g_1(x);$$

$$R_{12}(\lambda)g_2 = \frac{1}{\Delta_2(\lambda)} \cdot [a_{21}(x) \cdot (\| a_{11} \|^2 - \lambda) - a_{11}(x) \cdot (a_{11}, a_{21})] \times \\ \times \left[ (a_{12}, a_{22}) \cdot \int_{T^d} a_{22}(t) \cdot g_2(t) dt - (\| a_{22} \|^2 - \lambda) \cdot \int_{T^d} a_{12}(t) \cdot g_2(t) dt \right];$$

$$R_{21}(\lambda)g_1 = \frac{1}{\Delta_2(\lambda)} \cdot [a_{12}(x) \cdot (\| a_{22} \|^2 - \lambda) - a_{22}(x) \cdot (a_{12}, a_{22})] \times \\ \times \left[ (a_{11}, a_{21}) \cdot \int_{T^d} a_{11}(t) \cdot g_1(t) dt - (\| a_{11} \|^2 - \lambda) \right] \cdot \int_{T^d} a_{21}(t) \cdot g_1(t) dt;$$

$$R_{22}(\lambda)g_2 = \frac{1}{\lambda \cdot \Delta_2(\lambda)} \cdot [(a_{22}(x) \cdot B_4(\lambda) - a_{12}(x) \cdot B_3(\lambda)) \cdot \int_{T^d} a_{22}(t) \cdot g_2(t) dt + \\ + (a_{12}(x) \cdot (\| a_{22} \|^2 - \lambda) - a_{22}(x) \cdot (a_{12}, a_{22})) \times \\ \times \left( (a_{11}, a_{21})^2 - \| a_{21} \|^2 \cdot (\| a_{11} \|^2 - \lambda) \cdot \int_{T^d} a_{12}(t) \cdot g_2(t) dt \right)] - \frac{1}{\lambda} g_2(x).$$

**Conclusion.** In this article, determined the point spectrum of the operator matrices  $\mathcal{A}_\alpha, \alpha = 1, 2, 3$  with the Fredholm integral operators;  $\mathcal{A}_\alpha, \alpha = 1, 2, 3$  important spectrum for operator matrices are studied;

Determinants corresponding to matrices with operator  $\mathcal{A}_\alpha$ ,  $\alpha = 1,2,3$  are constructed; The numerical range of matrices with operator  $\mathcal{A}_\alpha$ ,  $\alpha = 1,2,3$  is determined and the resolvent operators of  $\alpha = 1,2$  cases is found.

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