AIP Conference Proceedings

RESEARCH ARTICLE | OCTOBER 07 2025

Spectrum and resolvent of the operator matrix associated to a system describing at most four quantum particles on a lattice FREE

Tulkin Rasulov ✉; Elyor Dilmurodov; Hakimboy Latipov

AIP Conf. Proc. 3377, 030013 (2025)
https://doi.org/10.1063/5.0299534

Spectrum and Resolvent of the Operator Matrix Associated to a System Describing at Most Four Quantum Particles on a Lattice

Tulkin Rasulov[1, a)], Elyor Dilmurodov[1, 2, b)] and Hakimboy Latipov[1, c)]

[1]*Bukhara state university, Bukhara, Uzbekistan.*
[2]*V.I.Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences, Tashkent, Uzbekistan*

[a)] *Corresponding author: t.h.rasulov@buxdu.uz*
[b)]*e.b.dilmurodov@buxdu.uz*
[c)]*h.m.latipov@buxdu.uz*

Abstract. This study investigates the tridiagonal operator matrix $\mathcal{A}$ of order 4, which is defined within the truncated four-particle subspace of the Fock space. The essential spectrum of the operator $\mathcal{A}$ is derived. The main block components are constructed, and the relationships between their spectra are analyzed. To determine the eigenvalues of the operator matrix $\mathcal{A}$, the corresponding Fredholm determinant is computed. Additionally, it is demonstrated that the resolvent operator associated with the operator matrix $\mathcal{A}$ is also represented in matrix form.

INTRODUCTION

Many problems of the modern mathematics are reduced to the determination of the spectral properties operator matrices consisting of linear bounded operators acting on the Banach or Hilbert space [1]. Operator matrices are frequently encountered in various fields, including solid-state physics, quantum field theory, statistical physics, magnetohydrodynamics, quantum mechanics, and others [2-5]. As an illustration, the Hamiltonian for a system of particles, where the particle number is not conserved and does not exceed four, is represented by a fourth-order operator matrix.

In [6], the generalized Friedrichs model was considered, and its numerical range was studied using the boundary eigenvalues and virtual levels. An approximation of essential spectrum from above and below of the 2×2 operator matrix with respect to the size of the torus and coupling constant is carried out in [7-10].

In addition, in the spectral theory op operator matrices, the main attention is paid to the issues of determining the structure of essential spectrum, studying the limited number of discrete eigenvalues and developing the resolvent operator [6-21].

Let $\mathcal{H} := \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4$, where $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ and $\mathcal{H}_4$ Hilbert spaces. It is known from the spectral theory of operators that any linear bounded operator $\mathcal{A}$ acting in $\mathcal{H}$ Hilbert space is always expressed as a fourth-order operator matrix [1]:

$$\mathcal{A} = \begin{pmatrix} A_{00} & A_{01} & A_{02} & A_{03} \\ A_{10} & A_{11} & A_{12} & A_{13} \\ A_{20} & A_{21} & A_{22} & A_{23} \\ A_{30} & A_{31} & A_{32} & A_{33} \end{pmatrix}. \tag{1}$$

Here $A_{ij}: \mathcal{H}_j \to \mathcal{H}_i,\ i, j = 0, 1, 2, 3$ are linear bounded operators. The operator matrix $\mathcal{A}$ defined in the form of equation (1) is self-adjoint only if the following equality holds: $A_{ij}^* = A_{ji}\ i, j = 0, 1, 2, 3$, i.e. $\mathcal{A}^* = \mathcal{A} \Leftrightarrow A_{ij}^* = A_{ji}$, $i, j = 0, 1, 2, 3$.

Second International Scientific and Practical Conference on Actual Problems of Mathematical Modeling and Information Technology
AIP Conf. Proc. 3377, 030013-1–030013-7; https://doi.org/10.1063/5.0299534

In this article, we will look at the following instance: $\mathcal{H}_0 \coloneqq \mathbb{C}$ is the one-dimensional complex space, and for $n = 1,2,3$, let $\mathcal{H}_n := L_2\left(\left(\mathbb{T}^{\mathrm{d}}\right)^n\right)$ is the Hilbert space of square-integrable (complex) functions defined on $\left(\mathbb{T}^{\mathrm{d}}\right)^n$, where $\mathbb{T}^{\mathrm{d}}$ considered as a d-dimensional torus. It is usually, referred to as a four-particle truncated subspace of the Fock space built on the $\mathcal{H}$ Hilbert space $L_2\left(\mathbb{T}^{\mathrm{d}}\right)$.

Typically, the Hilbert space $\mathcal{H}$ is referred to as the four-particle cut subspace of the Fock space constructed over $L_2\left(\mathbb{T}^{\mathrm{d}}\right)$. As we know, an element of a Hilbert space $\mathcal{H}$ is represented as a vector-function of the form $f = (f_0, f_1, f_2, f_3,)$, where $f_i \in \mathcal{H}_i$, $i = 0, 1, 2, 3$. We choose the matrix elements A_{ji} of the operator matrix $\mathcal{A}$ in the form (1) as follows:

$$A_{00}f_0 = \varepsilon f_0;\ A_{01}f_1 = \alpha \int_{\mathbb{T}^{\mathrm{d}}} v(s) f_1(s) ds\,;\ A_{02} = 0;\ A_{03} = 0;$$
$$A_{10} = A_{01}^*;\ (A_{11}\, f_1)(x) = \big(\varepsilon + \mathfrak{w}(x)\big) f_1(x);$$
$$(A_{12}f_2)(x) = \alpha \int_{\mathbb{T}^{\mathrm{d}}} v(\xi) f_2(x,\xi) d\xi\,; (A_{13}f_3)(x) = 0;$$
$$A_{20} = 0;\ A_{21} = A_{12}^*; (A_{22}\, f_2)(x,y) = \big(\varepsilon + \mathfrak{w}(x) + \mathfrak{w}(y)\big) f_2(x,y);$$
$$(A_{23}f_3)(x,y) = \alpha \int_{\mathbb{T}^{\mathrm{d}}} v(\zeta) f_3(x,y,\zeta) d\zeta\,;$$
$$A_{30} = 0;\ A_{31} = 0;\ A_{32} = A_{23}^*;$$
$$(A_{33}\, f_3)(x,y,t) = \big(\varepsilon + \mathfrak{w}(x) + \mathfrak{w}(y) + \mathfrak{w}(t)\big) f_3(x,y,t).$$

Here $\varepsilon > 0$; the function $v(\cdot)$ and $w(\cdot)$ are continuous and real-valued functions defined on $\mathbb{T}^{\mathrm{d}}$, $\alpha > 0$ is the coupling constant.

By definition, the operator matrix $\mathcal{A}$ given by (1) is linear, self-adjoint and bounded within the Hilbert space $\mathcal{H}$. In the prowing this fact we use the following: the norm of $f \in \mathcal{H}$ is expressed as:

$$\|f\| = \left(|f_0|^2 + \int_{\mathbb{T}^{\mathrm{d}}} |f_1(s)|^2 ds + \int_{\left(\mathbb{T}^{\mathrm{d}}\right)^2} |f_2(s,\xi)|^2 ds d\xi + \int_{\left(\mathbb{T}^{\mathrm{d}}\right)^3} |f_2(s,\xi,\zeta)|^2 ds d\xi d\zeta\right)^{1/2}$$

and the scalar multiplication of $f, g \in \mathcal{H}$ is defined as

$$(f,g) = f_0 \cdot \overline{g_0} + \int_{\mathbb{T}^{\mathrm{d}}} f_1(s)\overline{g_1(s)} ds + \int_{\left(\mathbb{T}^{\mathrm{d}}\right)^2} f_1(s,\xi)\overline{g_1(s,\xi)} ds d\xi + \int_{\left(\mathbb{T}^{\mathrm{d}}\right)^3} f_1(s,\xi,\zeta)\overline{g_1(s,\xi,\zeta)} ds d\xi d\zeta$$

We use A_{ij}^* to represent the adjoint operator to A_{ij} and according to the simple considerations we have

$$(\mathrm{A}_{01}^* f_0)(x) = \alpha v(x) f_0;$$
$$(\mathrm{A}_{12}^* f_1)(x, \mathrm{y}) = \alpha v(y) f_1(x);$$
$$(\mathrm{A}_{23}^* f_2)(x,y,t) = \alpha v(t) f_2(x,y).$$

In this study, we explore the following issues:

(a) The set $\sigma_{\mathrm{ess}}(\mathcal{A})$ and its branches are established;

(b) The main block elements of the operator matrix $\mathcal{A}$ and their spectra are characterized, and the relationships between the spectra are determined;

(c) By constructing, we examine the Fredholm determinant related to the operator matrix $\mathcal{A}$ and determine its discrete spectrum;

The expression for the resolvent operator corresponding to the $\mathcal{A}$ is given.

THE SPECTRUM OF THE OPERATOR MATRIX $\mathcal{A}$

To begin with, we introduce the Hilbert space

$$\mathcal{H}^{(n,m)} = \bigoplus_{k=n}^{m} \mathcal{H}_n\,, 0 \leq n < m \leq 3$$

and the following auxiliary operator matrices:

$$h_1\colon \mathcal{H}^{(0,1)} \to \mathcal{H}^{(0,1)},\ \ h_1 := \begin{pmatrix} \mathrm{A}_{00} & \mathrm{A}_{01} \\ \mathrm{A}_{01}^* & \mathrm{A}_{11} \end{pmatrix};$$

$$h_2\colon \mathcal{H}^{(0,2)} \to \mathcal{H}^{(0,2)},\ \ h_2 := \begin{pmatrix} \mathrm{A}_{00} & \mathrm{A}_{01} & 0 \\ \mathrm{A}_{01}^* & \mathrm{A}_{11} & \mathrm{A}_{12} \\ 0 & \mathrm{A}_{12}^* & \mathrm{A}_{22} \end{pmatrix};$$

$$\mathcal{A}_1\colon \mathcal{H}^{(2,3)} \to \mathcal{H}^{(2,3)},\ \ \mathcal{A}_1 := \begin{pmatrix} \mathrm{A}_{22} & \mathrm{A}_{23} \\ \mathrm{A}_{23}^* & \mathrm{A}_{33} \end{pmatrix};$$

$$\mathcal{A}_2\colon \mathcal{H}^{(1,3)} \to \mathcal{H}^{(1,3)},\ \ \mathcal{A}_2 = \begin{pmatrix} \mathrm{A}_{11} & \mathrm{A}_{12} & \mathrm{A}_{13} \\ \mathrm{A}_{12}^* & \mathrm{A}_{22} & \mathrm{A}_{23} \\ \mathrm{A}_{13}^* & \mathrm{A}_{23}^* & \mathrm{A}_{33} \end{pmatrix};$$

Using the Weyl theorem about stability the essential spectrum under finite rank perturbations we obtain

$$\sigma_{\mathrm{ess}}(h_1) = [\varepsilon + m; \varepsilon + \mathrm{M}]$$

Here the number m and M are identified as

$$m = \min_{x\in\mathbb{T}^{\mathrm{d}}} w(x), \qquad M = \max_{x\in\mathbb{T}^{\mathrm{d}}} w(x).$$

In order to define the discrete spectrum, we consider the regular function on the domain in $\mathbb{C}\backslash[\varepsilon + m;\ \varepsilon + M]$

$$\Delta_1(z) := \varepsilon - z - \alpha^2 \int_{\mathbb{T}^{\mathrm{d}}} \frac{v^2(s)ds}{\varepsilon + \mathfrak{w}(s) - z}.$$

The function $\Delta_1(\cdot)$ is generally referred to as the Fredholm determinant of the operator matrix h_1.

For any fixed $\alpha > 0$, in order for the operator matrix h_1 to have an eigenvalue $z \in \mathbb{C} \setminus [\varepsilon + m; \varepsilon + M]$ if and only if $\Delta_1(z) = 0$.

The set $\sigma_{\mathrm{disc}}(h_1)$ satisfies the given equality:

$$\sigma_{\mathrm{disc}}(h_1) = \{z \in \mathbb{C} \setminus [\varepsilon + m; \varepsilon + M] \colon \Delta_1(z) = 0\}.$$

The equality

$$\sigma_{\mathrm{ess}}(h_2) = [\varepsilon + m; \varepsilon + M] \cup \bigcup_{x\in\mathbb{T}^{\mathrm{d}}} \{\mathfrak{w}(x) + \sigma_{\mathrm{disc}}(h_1)\}$$

applies to the essential spectrum of h_2. Furthermore, the set $\sigma_{\mathrm{ess}}(h_2)$ comprises no more than three closed intervals.

To investigate $\sigma_{\mathrm{disc}}(h_2)$, we introduce the regular function $\Delta_2(\cdot)$, which is defined on the domain $\mathbb{C} \setminus \sigma_{\mathrm{ess}}(h_2)$ by the relation

$$\Delta_2(z) := \varepsilon - z - \alpha^2 \int_{\mathbb{T}^{\mathrm{d}}} \frac{v^2(s)ds}{\Delta_1\big(z - \mathfrak{w}(s)\big)}.$$

For any fixed $\alpha > 0$, in order for the operator matrix h_2 to have an eigenvalue $z \in \mathbb{C} \setminus \sigma_{\mathrm{ess}}(h_2)$ if and only if $\Delta_2(z) = 0$.

Consequently

$$\sigma_{\mathrm{disc}}(h_2) = \{z \in \mathbb{C} \setminus \sigma_{\mathrm{ess}}(h_2) \colon \Delta_2(z) = 0\}.$$

According to the definition, the function $\Delta_2(\cdot)$ is a decresing function on $\mathbb{R} \setminus \sigma_{\mathrm{ess}}(h_2)$, and thus, the operator matrix h_2 has at most four simple eigenvalues.

Lemma 1. *The operator matrix $\mathcal{A}_1$ has an entirely essential spectrum and*

$$\sigma(\mathcal{A}_1) = [\varepsilon + 3m; \varepsilon + 3M] \cup \bigcup_{x,y\in\mathbb{T}^{\mathrm{d}}} \{\mathfrak{w}(x) + \mathfrak{w}(y) + \sigma_{\mathrm{disc}}(h_1)\}.$$

Moreover, the set $\sigma(\mathcal{A}_1)$ consist of up to three closed intervals in union.

Lemma 2. *The operator matrix $\mathcal{A}_2$ possesses a purely essential spectrum, satisfying the equality*

$$\sigma(\mathcal{A}_1) = [\varepsilon + 3m; \varepsilon + 3M] \cup \bigcup_{x,y\in\mathbb{T}^{\mathrm{d}}} \{\mathfrak{w}(x) + \mathfrak{w}(y) + \sigma_{\mathrm{disc}}(h_1)\}.$$

Additionally, the set $\sigma(\mathcal{A}_2)$ is composed of a union of at most seven closed intervals.

It should be noted that in most cases the operator matrix $\mathcal{A}_2$ is called the channel operator corresponding to the $\mathcal{A}$ and one of its main ptoperties is the equality $\sigma_{\text{ess}}(\mathcal{A}) = \sigma(\mathcal{A}_2)$.

From the above facts we obtain the following spectral inclusions:

$$\sigma(\mathrm{A}_{33}) \subset \sigma(\mathcal{A}_1) \subset \sigma(\mathcal{A}_2) = \sigma_{\text{ess}}(\mathcal{A})$$

In the domain $\mathbb{C}\backslash\sigma_{\text{ess}}(\mathcal{A})$ we consider the following analytic function

$$\Delta(z) = \varepsilon - z - \alpha^2 \int\limits_{\mathbb{T}^{\mathrm{d}}} \frac{v^2(s)ds}{\Delta_2\big(z - \mathfrak{w}(s)\big)}.$$

Lemma 3. *A number $z \in \mathbb{C}\backslash\sigma_{ess}(\mathcal{A})$ is an eigenvalue of the operator matrix $\mathcal{A}$ if and only if $\Delta(z) = 0$.*

From this lemma, it follows that the discrete spectrum of the operator matrix $\mathcal{A}$ is characterized by the equality

$$\sigma_{\text{disc}}(\mathcal{A}) = \{z \in \mathbb{C}\backslash\sigma_{\text{ess}}(\mathcal{A}) \colon \Delta(z) = 0\}.$$

RESULTS

Theorem 1. *For each fixed number $z \in \rho(\mathcal{A})$, the corresponding resolvent operator to the operator matrix $\mathcal{A}$ is the operator matrix of the form*

$$\mathfrak{R}_z(\mathcal{A}) = \begin{pmatrix} \mathfrak{R}_{00}(z,\mathcal{A}) & \mathfrak{R}_{01}(z,\mathcal{A}) & \mathfrak{R}_{02}(z,\mathcal{A}) & \mathfrak{R}_{03}(z,\mathcal{A}) \\ \mathfrak{R}_{10}(z,\mathcal{A}) & \mathfrak{R}_{11}(z,\mathcal{A}) & \mathfrak{R}_{12}(z,\mathcal{A}) & \mathfrak{R}_{13}(z,\mathcal{A}) \\ \mathfrak{R}_{20}(z,\mathcal{A}) & \mathfrak{R}_{21}(z,\mathcal{A}) & \mathfrak{R}_{22}(z,\mathcal{A}) & \mathfrak{R}_{23}(z,\mathcal{A}) \\ \mathfrak{R}_{30}(z,\mathcal{A}) & \mathfrak{R}_{31}(z,\mathcal{A}) & \mathfrak{R}_{32}(z,\mathcal{A}) & \mathfrak{R}_{33}(z,\mathcal{A}) \end{pmatrix}$$

in the Hilbert space $\mathcal{H}^{(0,3)}$, and its matrix elements are specified as follows:

$$\mathfrak{R}_{00}(z,\mathcal{A})g_0 = \frac{g_0}{\Delta(z)};\ \mathfrak{R}_{01}(z,\mathcal{A})g_1 = -\frac{\alpha}{\Delta(z)}\int_{\mathbb{T}^{\mathrm{d}}} \frac{v(s)g_1(s)ds}{\Delta_2\big(z - \mathfrak{w}(s)\big)};$$

$$\mathfrak{R}_{02}(z,\mathcal{A})g_2 = \frac{\alpha^2}{\Delta(z)}\int_{(\mathbb{T}^{\mathrm{d}})^2} \frac{v(s)v(\xi)g_2(s,\xi)dsd\xi}{\Delta_2(z - \mathfrak{w}(s))\Delta_1(z - \mathfrak{w}(s) - \mathfrak{w}(\xi))};$$

$$\mathfrak{R}_{03}(z,\mathcal{A})g_3 = -\frac{\alpha^3}{\Delta(z)}\int_{(\mathbb{T}^{\mathrm{d}})^3} \frac{v(s)v(\xi)v(\zeta)g_3(s,\xi,\zeta)dsd\xi d\backslash zeta}{(\varepsilon + \mathfrak{w}(s) + \mathfrak{w}(\xi) + \mathfrak{w}(\zeta) - z)\Delta_1(z - \mathfrak{w}(s) - \mathfrak{w}(\xi))\Delta_2(z - \mathfrak{w}(s))};$$

$$(\mathfrak{R}_{10}(z,\mathcal{A})g_0)(x) = -\frac{\alpha v(x)g_0}{\Delta(z)\Delta_2(z - \mathfrak{w}(x))};$$

$$(\mathfrak{R}_{11}(z,\mathcal{A})g_1)(x) = \frac{g_1(x)}{\Delta_2(z - \mathfrak{w}(x))} + \frac{\alpha^2 v(x)}{\Delta(z)\Delta_2(z - \mathfrak{w}(x))}\int_{\mathbb{T}^{\mathrm{d}}} \frac{v(s)g_1(s)}{\Delta_2(z - \mathfrak{w}(s))}ds;$$

$$(\mathfrak{R}_{12}(z,\mathcal{A})g_2)(x) = -\frac{\alpha^3 v(x)}{\Delta(z)\Delta_2\big(z - \mathfrak{w}(x)\big)}\int_{(\mathbb{T}^{\mathrm{d}})^2} \frac{v(s)v(\xi)g_2(s,\xi)dsd\xi}{\Delta_2\big(z - \mathfrak{w}(s)\big)\Delta_1(z - \mathfrak{w}(s) - \mathfrak{w}(\xi))}$$
$$- \frac{\mu}{\Delta_2(z - \mathfrak{w}(x))}\int_{\mathbb{T}^{\mathrm{d}}} \frac{v(s)g_2(x,s)ds}{\Delta_1(z - \mathfrak{w}(x) - \mathfrak{w}(s))};$$

$$(\mathfrak{R}_{13}(z,\mathcal{A})g_3)(x) = \frac{\alpha^4 v(x)}{\Delta(z)\Delta_2\big(z - \mathfrak{w}(x)\big)}\int_{(\mathbb{T}^{\mathrm{d}})^3} \frac{v(s)v(\xi)v(\zeta)g_3(s,\xi,\zeta)dsd\xi d\zeta}{(\varepsilon + \mathfrak{w}(s) + \mathfrak{w}(\xi) + \mathfrak{w}(\zeta) - z)\Delta_1(z - \mathfrak{w}(s) - \mathfrak{w}(\xi))\Delta_2(z - \mathfrak{w}(s))}$$
$$+ \frac{\alpha^2}{\Delta_2(z - \mathfrak{w}(x))}\int_{(\mathbb{T}^{\mathrm{d}})^2} \frac{v(\xi)v(\zeta)g_3(x,\xi,\zeta)d\xi d\zeta}{(\varepsilon + \mathfrak{w}(x) + \mathfrak{w}(\xi) + \mathfrak{w}(\zeta) - z)\Delta_1(z - \mathfrak{w}(x) - \mathfrak{w}(\xi))}$$

$$(\mathfrak{R}_{20}(z,\mathcal{A})g_0)(x,y) = \frac{\alpha^2 v(x)v(y)g_0}{\Delta(z)\Delta_1(z - \mathfrak{w}(x) - \mathfrak{w}(y))\Delta_2(z - \mathfrak{w}(x))};$$

$$(\mathfrak{R}_{21}(z,\mathcal{A})g_1)(x,y) = -\frac{\alpha v(y)g_1(x)}{\Delta_1(z - \mathfrak{w}(x) - \mathfrak{w}(y))\Delta_2\big(z - \mathfrak{w}(x)\big)}$$

$$-\frac{\alpha^3 v(x)v(y)}{\Delta(z)\Delta_1(z-\mathfrak{w}(x)-\mathfrak{w}(y))\Delta_2(z-\mathfrak{w}(x))}\int_{\mathbb{T}^d}\frac{v(s)g_1(s)}{\Delta_2(z-\mathfrak{w}(s))}ds;$$

$$(\mathfrak{R}_{22}(z,\mathcal{A})g_2)(x,y)=\frac{g_2(x,y)}{\Delta_1(z-\mathfrak{w}(x)-\mathfrak{w}(y))}$$

$$+\frac{\alpha^4 v(x)v(y)}{\Delta(z)\Delta_1(z-\mathfrak{w}(x)-\mathfrak{w}(y))\Delta_2(z-\mathfrak{w}(x))}\int_{(\mathbb{T}^d)^2}\frac{v(s)v(\xi)g_2(s,\xi)dsd\xi}{\Delta_1(z-\mathfrak{w}(s)-\mathfrak{w}(\xi))\Delta_2(z-\mathfrak{w}(s))}$$

$$+\frac{\alpha^2 v(y)}{\Delta_1(z-\mathfrak{w}(x)-\mathfrak{w}(y))\Delta_2(z-\mathfrak{w}(x))}\int_{\mathbb{T}^d}\frac{v(s)g_2(x,s)ds}{\Delta_1(z-\mathfrak{w}(x)-\mathfrak{w}(s))};$$

$$(\mathfrak{R}_{23}(z,\mathcal{A})g_3)(x,y)=-\frac{\alpha}{\Delta_1(z-\mathfrak{w}(x)-\mathfrak{w}(y))}\int_{\mathbb{T}^d}\frac{v(\zeta)g_3(x,y,\zeta)d\zeta}{\varepsilon+\mathfrak{w}(x)+\mathfrak{w}(y)+\mathfrak{w}(\zeta)-z}$$

$$-\frac{\alpha^5 v(x)v(y)}{\Delta(z)\Delta_1(z-\mathfrak{w}(x)-\mathfrak{w}(y))\Delta_2(z-\mathfrak{w}(x))}\int_{(\mathbb{T}^d)^3}\frac{v(s)v(\xi)v(\zeta)g_3(s,\xi,\zeta)dsd\xi d\zeta}{(\varepsilon+\mathfrak{w}(s)+\mathfrak{w}(\xi)+\mathfrak{w}(\zeta)-z)\Delta_1(z-\mathfrak{w}(s)-\mathfrak{w}(\xi))\Delta_2(z-\mathfrak{w}(s))}$$

$$-\frac{\alpha^3}{\Delta_1(z-\mathfrak{w}(x)-\mathfrak{w}(y))\Delta_2(z-\mathfrak{w}(x))}\int_{(\mathbb{T}^d)^2}\frac{v(\xi)v(\zeta)g_3(x,\xi,\zeta)d\xi d\zeta}{(\varepsilon+\mathfrak{w}(x)+\mathfrak{w}(\xi)+\mathfrak{w}(\zeta)-z)\Delta_1(z-\mathfrak{w}(x)-\mathfrak{w}(\xi))}$$

$$(\mathfrak{R}_{30}(z,\mathcal{A})g_0)(x,y,t)=-\frac{\alpha v(t)}{\varepsilon+\mathfrak{w}(x)+\mathfrak{w}(y)+\mathfrak{w}(t)-z}(\mathfrak{R}_{20}(z,\mathcal{A})g_0)(x,y);$$

$$(\mathfrak{R}_{31}(z,\mathcal{A})g_1)(x,y,t)=-\frac{\alpha v(t)}{\varepsilon+\mathfrak{w}(x)+\mathfrak{w}(y)+\mathfrak{w}(t)-z}(\mathfrak{R}_{21}(z,\mathcal{A})g_1)(x,y);$$

$$(\mathfrak{R}_{32}(z,\mathcal{A})g_2)(x,y,t)=-\frac{\alpha v(t)}{\varepsilon+\mathfrak{w}(x)+\mathfrak{w}(y)+\mathfrak{w}(t)-z}(\mathfrak{R}_{22}(z,\mathcal{A})g_2)(x,y);$$

$$(\mathfrak{R}_{33}(z,\mathcal{A})g_3)(x,y,t)=\frac{g_3(x,y,t)}{\varepsilon+\mathfrak{w}(x)+\mathfrak{w}(y)+\mathfrak{w}(t)-z}-\frac{\alpha v(t)}{\varepsilon+\mathfrak{w}(x)+\mathfrak{w}(y)+\mathfrak{w}(t)-z}(\mathfrak{R}_{23}(z,\mathcal{A})g_3)(x,y).$$

Proof. To find the form of the resolvent operator of $\mathcal{A}$, we look for the solution to the equation

$$(\mathcal{A}_\mu - zI)f = g$$

for $g=(g_0,g_1(x),g_2(x,y),g_3(x,y,t))\in\mathcal{H}^{(0,3)}$. We can reformulate this equation into the following system of equations:

$$\begin{cases}(\varepsilon-z)f_0+\alpha\int_{\mathbb{T}^d}v(s)f_1(s)ds=g_0\\ \alpha v(x)f_0+(\varepsilon+\mathfrak{w}(x)-z)f_1(x)+\alpha\int_{\mathbb{T}^d}v(\xi)f_2(x,\xi)d\xi=g_1(x)\\ \alpha v(y)f_1(x)+(\varepsilon+\mathfrak{w}(x)+\mathfrak{w}(y)-z)f_2(x,y)+\alpha\int_{\mathbb{T}^d}v(\zeta)f_3(x,y,\zeta)d\zeta=g_2(x,y)\\ \alpha v(t)f_2(x,y)+(\varepsilon+\mathfrak{w}(x)+\mathfrak{w}(y)+\mathfrak{w}(t)-z)f_3(x,y,t)=g_3(x,y,t)\end{cases} \quad (2)$$

Since $z\notin\sigma(\mathcal{A})$ for any $x,y,t\in\mathbb{T}^d$ we have

$$\varepsilon+u(x)+u(y)+u(t)-z\neq 0,\ \Delta_1(z-\mathfrak{w}(x)-\mathfrak{w}(y))\neq 0,\ \Delta_2(z-\mathfrak{w}(x))\neq 0,\ \Delta(z)\neq 0.$$

From the fourth equation of the system (2), we derive the equation for $f_3(x,y,t)$:

$$f_3(x,y,t)=\frac{g_3(x,y,t)}{\varepsilon+\mathfrak{w}(x)+\mathfrak{w}(y)+\mathfrak{w}(t)-z}-\frac{\alpha v(t)f_2(x,y)}{\varepsilon+\mathfrak{w}(x)+\mathfrak{w}(y)+\mathfrak{w}(t)-z}. \quad (3)$$

By substituting the derived expression (3) for $f_3(\cdot,\cdot,\cdot)$ into the third equation of the system (2), we obtain the expression for $f_2(x,y)$:

$$f_2(x,y)=\frac{g_2(x,y)}{\Delta_1(z-\mathfrak{w}(x)-w(y))}-\frac{\alpha}{\Delta_1(z-\mathfrak{w}(x)-\mathfrak{w}(y))}\int_{\mathbb{T}^d}\frac{v(\zeta)g_3(x,y,\zeta)d\zeta}{\varepsilon+\mathfrak{w}(x)+\mathfrak{w}(y)+\mathfrak{w}(\zeta)-z}$$
$$-\frac{\alpha v(y)f_1(x)}{\Delta_1(z-\mathfrak{w}(x)-\mathfrak{w}(y))} \quad (4)$$

Now, by inserting the expression (4) obtained for $f_2(\cdot,\cdot)$ into the second equation of the system of equations (2), we obtain the relation for $f_1(x)$:

$$f_1(x)=\frac{g_1(x)}{\Delta_2\big(z-\mathfrak{w}(x)\big)}-\frac{\alpha v(x)f_0}{\Delta_2\big(z-\mathfrak{w}(x)\big)}-\frac{\alpha}{\Delta_2\big(z-\mathfrak{w}(x)\big)}\int_{\mathbb{T}^{\mathrm{d}}}\frac{v(\xi)g_2(x,\xi)d\xi}{\Delta_1\big(z-\mathfrak{w}(x)-\mathfrak{w}(\xi)\big)} \tag{5}$$

$$+\frac{\alpha^2}{\Delta_2\big(z-\mathfrak{w}(x)\big)}\int_{(\mathbb{T}^{\mathrm{d}})^2}\frac{v(\xi)v(\zeta)g_3(x,\xi,\zeta)d\xi d\zeta}{(\varepsilon+\mathfrak{w}(x)+\mathfrak{w}(\xi)+\mathfrak{w}(\zeta)-z)\Delta_1(z-\mathfrak{w}(x)-\mathfrak{w}(\xi))}$$

We substitute the expression $f_1(\cdot)$ found above into the first equation in the system of equations (2) to obtain the following relation.

$$f_0=\frac{g_0}{\Delta(z)}-\frac{\alpha}{\Delta(z)}\int_{\mathbb{T}^{\mathrm{d}}}\frac{v(s)g_1(s)ds}{\Delta_2\big(z-\mathfrak{w}(s)\big)}+\frac{\alpha^2}{\Delta(z)}\int_{(\mathbb{T}^{\mathrm{d}})^2}\frac{v(s)v(\xi)g_2(s,\xi)dsd\xi}{\Delta_2(z-\mathfrak{w}(s))\Delta_1(z-\mathfrak{w}(s)-\mathfrak{w}(\xi))}$$

$$-\frac{\alpha^3}{\Delta(z)}\int_{(\mathbb{T}^{\mathrm{d}})^3}\frac{v(s)v(\xi)v(\zeta)g_3(s,\xi,\zeta)dsd\xi d\zeta}{(\varepsilon+\mathfrak{w}(s)+\mathfrak{w}(\xi)+\mathfrak{w}(\zeta)-z)\Delta_1(z-\mathfrak{w}(s)-\mathfrak{w}(\xi))\Delta_2\big(z-\mathfrak{w}(s)\big)} \tag{6}$$

That is, $f_0=\mathfrak{R}_{00}(z,\mathcal{A})g_0+\mathfrak{R}_{01}(z,\mathcal{A})g_1+\mathfrak{R}_{02}(z,\mathcal{A})g_2+\mathfrak{R}_{03}(z,\mathcal{A})g_3$.

That is, by substituting the expression for f_0 obtained from (6) into the expression (5), we derive the relation:

$$f_1(x)=(\mathfrak{R}_{10}(z,\mathcal{A})g_0)(x)+(\mathfrak{R}_{11}(z,\mathcal{A})g_1)(x)+(\mathfrak{R}_{12}(z,\mathcal{A})g_2)(x)+(\mathfrak{R}_{13}(z,\mathcal{A})g_3)(x)$$

By placing the derived expression for $f_1(\cdot)$into equation (4), we obtain the following expression:

$$f_2(x,y)=(\mathfrak{R}_{20}(z,\mathcal{A})g_0)(x,y)+(\mathfrak{R}_{21}(z,\mathcal{A})g_1)(x,y)+(\mathfrak{R}_{22}(z,\mathcal{A})g_2)(x,y)+(\mathfrak{R}_{23}(z,\mathcal{A})g_3)(x,y)$$

Using this expression and equation (3), we obtain

$$f_3(x,y,t)=(\mathfrak{R}_{30}(z,\mathcal{A})g_0)(x,y,t)+(\mathfrak{R}_{31}(z,\mathcal{A})g_1)(x,y,t)+(\mathfrak{R}_{32}(z,\mathcal{A})g_2)(x,y,t)+(\mathfrak{R}_{33}(z,\mathcal{A})g_3)(x,y,t)$$

This completes the proof of the theorem.

CONCLUSION

This paper investigates the operator matrix $\mathcal{A}$ of order four, which operates in the direct sum of the zero-particle, one-particle, two-particle, and three-particle subspaces of the Fock space. The main block elements of $\mathcal{A}$ are separated. They are denoted as $h_1, h_2, \mathcal{A}_1, \mathcal{A}_2$. The essential and discrate spectra of these elements are described. The corresponding Fredholm determinants are constructed. Using the monotonicity property of the Fredholm determinant, the number of eigenvalues, location and multiplicity are determined. A relationship between $\sigma(\mathcal{A}_1)$, $\sigma(\mathcal{A}_2)$ and $\sigma_{\text{ess}}(\mathcal{A})$ is established. The corresponding resolvent operator for $\mathcal{A}$ is constructed., and is demonstrated that the resolvent operator is also in the form of a 4th-order operator matrix.

REFERENCES

1. C.Tretter. *Spectral theory of block operator matrices and applications* (Imperial College Press, 2008).
2. R.A. Minlos, H. Spohn, The three-body problem in radioactive decay: the case of one atom and at most two photons, Topics in Statistical and Theoretical Physics, Amer. Math. Soc. Transl., 177(2), 159-193 (1996).
3. A.I. Mogilner, Hamiltonians in solid state physics as multiparticle discrete Schrodinger operators: problems and results, Advances in Sov. Math. 5, 139-194 (1991).
4. K.O.Friedrichs. *Perturbation of spectra in Hilbert space*. (Amer. Math. Soc., Providence, Rhole Island, 1965)
5. V.A. Malyshev, R.A. Minlos, *Linear infinite-particle operators*. (Translations of Mathematical Monographs, vol. 143. Providence, RI, Am. Math. Soc., 1995).
6. T.H.Rasulov, E.B.Dilmurodov. Eigenvalues and virtual levels of a family of 2×2 operator matrices. Methods of Functional Analysis and Topology. 25(3), 273-281 (2019)
7. E.B.Dilmurodov, Discrete Eigenvalues of a 2×2 Operator Matrix. AIP Conference Proceedings 2764(1) 030004 (2023)
8. T.H.Rasulov, E.B.Dilmurodov. Infinite number of eigenvalues of 2×2 operator matrices: Asymptotic discrete spectrum. Theoret. and Math. Phys. 205(3) 1564-1584 (2020).
9. T.H.Rasulov, E.B.Dilmurodov. Analysis of the spectrum of a 2×2 operator matrix. Discrete spectrum asymptotics. Nanosystems: Phys. Chem. Math., 11(2), 138-144 (2020)

10. M.I.Muminov, T.H.Rasulov, Infiniteness of the number of eigenvalues embedded in the essential spectrum of a 2×2 operator matrix. Eurasian Mathematical Journal, 5(2) 60–77 (2014).
11. Z. Muminov, F. Ismail, Z. Eshkuvatov. On the number of eigenvalues of a model operator in fermionic Fock space. Journal of Physics: Conference Series. 435, 012036 (2013)
12. T.Kh.Rasulov, Study of the essential spectrum of a matrix operator. Theoret. and Math. Phys., 164(1) 883–895 (2010).
13. M.I.Muminov, T.H.Rasulov, Embedded eigenvalues of a Hamiltonian in bosonic Fock space. Communications in Mathematical Analysis, 17(1), 1-22 (2014).
14. E.B.Dilmurodov. New branches of the essential spectrum of a 2×2 operator matrix. Uzbek Mathematical Journal. 2, 44-51 (2020)
15. E.B.Dilmurodov, B.I.Bahronov, Kh.G.Khayitova, G.G.Kurbonov, and R.T. Norchayev, On the eigenvalues of the lattice spin-boson model with at most one photon. E3S Web of Conferences 587, 01002 (2024).
16. G.H.Umirkulova, B.I.Bahronov, N.A.Tosheva, O.A.Begmurodov and N.U.Akboeva. Faddeev equation and its symmetric version for a three-particle lattice hamiltonian. E3S Web of Conferences 587, 03003 (2024)
17. N.A.Tosheva, Essential Spectrum of a Family of 3×3 Operator Matrices: Location of the Branches. AIP Conference Proceedings 2764(1) 030003 (2023)
18. T.Kh.Rasulov, On the Structure of the Essential Spectrum of a Model Many-Body Hamiltonian. Math. Notes, 83(1) 80–87 (2008).
19. Muminov M. I., Shodiev U. R. On the essential spectrum of a four-particle Schrödinger operator on a lattice //Siberian Advances in Mathematics. – 2011. – T. 21. – C. 292-303.
20. Muminov M. E., Shodiev U. R. Spectral properties of a Hamiltonian of a four-particle system on a lattice //Russian Mathematics. – 2010. – T. 54. – C. 27-37.
21. Muminov, M.I., Shadiev, U.R. On the Existence of an Eigenvalue of the Generalized Friedrichs Model. Russ Math. 68, 28–34 (2024). https://doi.org/10.3103/S1066369X24700257