PROCEEDINGS OF SCIENTIFIC CONFERENCE "ACTUAL PROBLEMS OF STOCHASTIC ANALYSIS" dedicated to the 80th anniversary of the birth of academician Sh.K.FORMANOV

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“ACTUAL PROBLEMS OF STOCHASTIC ANALYSIS”

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Sh.K.FORMANOV

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80 ЙИЛЛИГИГА БАҒИШЛАНГАН

«СТОХАСТИК ТАҲЛИЛНИНГ ДОЛЗАРБ МУАММОЛАРИ»

МАВЗУСИДАГИ ИЛМИЙ КОНФЕРЕНЦИЯ

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Now we calculate the limits of r_n.

Case of even n. From (2) it is easy to see that

$$\lim_{n\to\infty}|f^n(x)|_p=\lim_{n\to\infty}r_n=\begin{cases}0, & \text{if } r<\alpha\\ \alpha, & \text{if } r=\alpha\\ +\infty, & \text{if } r>\alpha\end{cases}$$

Case of odd n. In this case we have

$$\lim_{n\to\infty}|f^n(x)|_p=\lim_{n\to\infty}r_n=\begin{cases}+\infty, & \text{if } r<\alpha\\ \alpha, & \text{if } r=\alpha\\ 0, & \text{if } r>\alpha\end{cases}$$

Summarizing above-mentioned results we obtain the following theorem:

Theorem 1. If $p\geq 3$ and α is defined by (1). Then

1. if $x\in U_\alpha(0)$ then

$$\lim_{k\to\infty}f^{2k}(x)=0,\quad \lim_{k\to\infty}|f^{2k-1}(x)|_p=+\infty.$$

2. if $x\in S_\alpha(0)$ then $f^n(x)\in S_\alpha(0)$, $n\geq 1$.
3. if $x\in \mathcal{C}_p\setminus V_\alpha(0)$ then

$$\lim_{k\to\infty}|f^{2k}(x)|_p=+\infty,\quad \lim_{k\to\infty}f^{2k-1}(x)=0.$$

References

1. *Albeverio S., Rozikov U.A., Sattarov I.A.,* p-adic $(2,1)$-rational dynamical systems. *Jour. Math. Anal. Appl.* **398**(2) (2013), 553–566.

2. *Anashin V., Khrennikov A.,* Applied algebraic dynamics, de Gruyter Expositions in Mathematics vol 49, Walter de Gruyter (Berlin - New York), 2009.

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p-adic dynamical systems of a non-linear function

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We study p-adic dynamical systems generated by a rational function. For motivation of such investigations see [1], [2] and references therein.

It is known that the completion of the set of rational numbers $\mathcal{Q}$ with respect to p-adic norm $|\cdot|_p$ defines the p-adic field which is denoted by $\mathcal{Q}_p$ (see [3]).

The algebraic completion of $\mathcal{Q}_p$ is denoted by $\mathcal{C}_p$ and it is called *complex p-adic numbers.*

For any $a \in \mathcal{C}_p$ and $r > 0$ denote

$$U_r(a) = \{x \in \mathcal{C}_p : |x-a|_p < r\}, \quad V_r(a) = \{x \in \mathcal{C}_p : |x-a|_p \leq r\},$$

$$S_r(a) = \{x \in \mathcal{C}_p : |x-a|_p = r\}.$$

Let x_0 be a fixed point of a function $f(x)$, i.e. $f(x_0) = x_0$.

Put $\lambda = f'(x_0)$. The point x_0 is attractive if $0 < |\lambda|_p < 1$, *indifferent* if $|\lambda|_p = 1$, and repelling if $|\lambda|_p > 1$.

The ball $U_r(x_0)$ is said to be a *Siegel disk* if each sphere $S_\rho(x_0)$, $\rho < r$ is an invariant sphere of $f(x)$, i.e. if $x \in S_\rho(x_0)$ then all iterated points $f^n(x) \in S_\rho(x_0)$ for all $n = 1, 2 \dots$. The union of all Siegel desks with the center at x_0 is said to *a maximum Siegel disk* and is denoted by $SI(x_0)$.

Consider the dynamical system associated with function $f : \mathcal{C}_p \to \mathcal{C}_p$ defined by

$$f(x) = \frac{a}{x-2b}, \quad a \neq 0, \quad a, b \in \mathcal{C}_p, \tag{1}$$

where $x \neq 2b$.

Our goal here is to present the behavior of trajectories $\{f^n(x),\ x \in \mathcal{C}_p\}$ of (1) in the complex p-adic filed $\mathcal{C}_p$.

Remark. The case $b = 0$ is simple: in this case any point $x \in \mathcal{C}_p \setminus \{-b\}$ is two periodic. That is $f(f(x)) = x$. Indeed,

$$f(f(x)) = \frac{a}{\frac{a}{x}} = a \cdot \frac{x}{a} = x.$$

Therefore, below we consider the case $b \neq 0$.

Since $\mathcal{C}_p$ is algebraic closed, this function (for $ab \neq 0$) has two fixed points:

$$f(x) = x \quad \Rightarrow \quad x^2 - 2bx - a = 0 \quad \Rightarrow \quad x_1 = b - \sqrt{b^2 + a}, \quad x_2 = b + \sqrt{b^2 + a}.$$

Denote:

$$\mathcal{P} = \{x \in \mathcal{C}_p : \exists n \in \mathcal{N} \cup \{0\}, f^n(x) = 2b\}.$$

For example, $x = \hat{x} = 2b + \frac{a}{2b} \in \mathcal{P}$, because $f(\hat{x}) = 2b$.

The following proposition describes the set $\mathcal{P}$

Proposition. The set $\mathcal{P}$ is the following

$$\mathcal{P} = \{2b\} \cup \left\{ \frac{b_n - 2bd_n}{2bc_n - a_n} : n \geq 1 \right\},$$

where a_n, b_n, c_n, d_n are coefficients of f^n.

For (1) we have

$$f'(x) = -\frac{a}{(x-2b)^2} = -\frac{1}{a}\left(\frac{a}{(x-2b)}\right)^2 = -\frac{1}{a}(f(x))^2.$$

Using this formula and $x_1 x_2 = -a$ we get

$$|f'(x_1)|_p = \frac{|x_1|_p}{|x_2|_p}, \quad |f'(x_2)|_p = \frac{|x_2|_p}{|x_1|_p},$$

i.e., if the point x_1 (resp. x_2) is repeller then x_2 (resp. x_1) is attractive. Moreover, x_1 is indifferent iff x_2 is indifferent. Thus we need to compare $|x_1|_p = |b - \sqrt{b^2+a}|_p$ and $|x_2|_p = |b + \sqrt{b^2+a}|_p$.

Case: $b^2 + a = 0$. In this case $x_1 = x_2$, i.e. the function has unique fixed point $x_1 = b$. Moreover, $|f'(x_1)|_p = 1$, i.e. the fixed point is an indifferent point. Denote $B = |b|_p$ and

$$B^*(x) = |f(x) - x_1|_p, \quad \text{if} \quad x \in S_B(x_1).$$

We have the following

Theorem 1. The p-adic dynamical system is generated by the function (1), for $b^2+a = 0$, has the following properties:

1. $SI(x_1) = U_B(x_1)$.
2. $\mathcal{P} \subset \mathcal{C}_p \setminus U_B(x_1)$.
3. If $r > B$ and $x \in S_r(x_1)$, then $f(x) \in S_B(x_1)$ and

$$f^n(x) \in S_{B^*(f^{n-1}(x))}(x_1), \quad n \geq 2,$$

where $B^*(x) = |f(x) - x_1|_B \geq B$.

Case: $b^2 + a \neq 0$. In this case $x_1 \neq x_2$. We denote

$$\alpha = |x_1|_p = |b - \sqrt{b^2+a}|_p, \quad \beta = |x_2|_p = |b + \sqrt{b^2+a}|_p.$$

We obtain the following theorem

Theorem 2. The p-adic dynamical system is generated by the function (1), for $b^2+a \neq 0$ and $\alpha = \beta$, has the following properties:

i. $SI(x_i) = U_\alpha(x_i)$, with

$$SI(x_1) = SI(x_2), \quad \text{if} \quad |x_1 - x_2|_p < \alpha$$

$$SI(x_1) \cap SI(x_2) = \emptyset, \quad \text{if} \quad |x_1 - x_2|_p = \alpha.$$

ii. $\mathcal{P} \subset \mathcal{C}_p \setminus (SI(x_1) \cup SI(x_2))$.

iii. If $r > \alpha$ and $x \in S_r(x_1)$, then $f(x) \in S_\alpha(x_1)$ and

$$f^n(x) \in S_{A^*(f^{n-1}(x))}(x_1), \quad n \geq 2,$$

where $A^*(x) = |f(x) - x_1|_\alpha \geq \alpha$.

References

1. *Albeverio S., Rozikov U.A., Sattarov I.A.,* p-adic $(2,1)$-rational dynamical systems. *Jour. Math. Anal. Appl.* **398**(2) (2013), 553–566.

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