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On a non-linear p-adic dynamical system

Rozikov U.A., Sayitova M.

Abstract. In this paper, we study p-adic dynamical system of the function $f(x) = \frac{a}{x-2b}$ on the set of complex p-adic numbers. For each trajectory of the dynamical system we construct the set of limit points and for each indifferent fixed point we give its Siegel disk.

Keywords: Rational dynamical systems; fixed point; invariant set; Siegel disk; complex p-adic field.

MSC (2010): 37P05, 46S10.

1 Introduction

It is known that the theory of p-adic numbers has numerous applications in many branches of mathematics, biology, physics and other sciences (see for example [4], [8], [13] and the references therein).

Let us recall the main definitions. Denote by (n, m) the greatest common divisor of the positive integers n and m and let $\mathbb{Q}$ be the field of rational numbers.

For each fixed prime number p, every rational number $x \neq 0$ can be represented in the form $x = p^r \frac{n}{m}$, where $r, n \in \mathbb{Z}$, m is a positive integer, $(p, n) = 1$, $(p, m) = 1$. The p-adic norm of this x is $|x|_p = p^{-r}$ and $|0|_p = 0$.

This norm has the following properties:

1) $|x|_p \geq 0$ and $|x|_p = 0$ if and only if $x = 0$,

2) $|xy|_p = |x|_p|y|_p$,

3) the strong triangle inequality

$$|x + y|_p \leq \max\{|x|_p, |y|_p\},$$

3.1) if $|x|_p \neq |y|_p$ then $|x + y|_p = \max\{|x|_p, |y|_p\}$,

3.2) if $|x|_p = |y|_p$ then for $p = 2$ we have $|x + y|_p \leq \frac{1}{2}|x|_p$ (see [13]).

The completion of $\mathbb{Q}$ with respect to p-adic norm defines the p-adic field which is denoted by $\mathbb{Q}_p$ (see [5]).

The algebraic completion of $\mathbb{Q}_p$ is denoted by $\mathbb{C}_p$ and it is called the set of *complex p-adic numbers*.

For any $a \in \mathbb{C}_p$ and $r > 0$ denote

$$U_r(a) = \{x \in \mathbb{C}_p : |x - a|_p < r\}, \quad V_r(a) = \{x \in \mathbb{C}_p : |x - a|_p \leq r\},$$

$$S_r(a) = \{x \in \mathbb{C}_p : |x - a|_p = r\}.$$

To define a dynamical system we consider a function $f : x \in U \to f(x) \in U$, (in our case $U = U_r(a)$ or $\mathbb{C}_p$) (see for example [7]).

For $x \in U$ denote by $f^n(x)$ the n-fold composition of f with itself (i.e. n time iteration of f to x):

$$f^n(x) = \underbrace{f(f(f \dots (f}_{n \text{ times}}(x))) \dots).$$

For arbitrary given $x_0 \in U$ and $f : U \to U$ the discrete-time dynamical system (also called the trajectory) of x_0 is the sequence of points

$$x_0, x_1 = f(x_0), x_2 = f^2(x_0), x_3 = f^3(x_0), \dots \tag{1.1}$$

The main problem: Given a function f and initial point x_0 what ultimately happens with the sequence (1.1). Does the limit $\lim_{n\to\infty} x_n$ exist? If not what is the set of limit points of the sequence?

A point $x \in U$ is called a fixed point for f if $f(x) = x$. The set of all fixed points denoted by $\mathrm{Fix}(f)$. A point x is a periodic point of period m if $f^m(x) = x$. The least positive m for which $f^m(x) = x$ is called the prime period of x.

A fixed point x_0 is called an *attractor* if there exists a neighborhood $U(x_0)$ of x_0 such that for all points $x \in U(x_0)$ it holds $\lim\limits_{n\to\infty} f^n(x) = x_0$. If x_0 is an attractor then its *basin of attraction* is

$$\mathcal{A}(x_0) = \{x \in \mathbb{C}_p : \ f^n(x) \to x_0, \ n \to \infty\}.$$

A fixed point x_0 is called *repeller* if there exists a neighborhood $U(x_0)$ of x_0 such that $|f(x) - x_0|_p > |x - x_0|_p$ for $x \in U(x_0)$, $x \neq x_0$.

The ball $U_r(x_0)$ is called a *Siegel disk* if each sphere $S_\rho(x_0)$, $\rho < r$ is an invariant sphere of $f(x)$, i.e. if $x \in S_\rho(x_0)$ then all iterated points $f^n(x) \in S_\rho(x_0)$ for all $n = 1, 2 \dots$. The union of all Siegel desks with the center at x_0 is called *a maximum Siegel disk* and is denoted by $SI(x_0)$.

In this paper we continue our study of p-adic dynamical systems generated by rational functions (see [1]-[12] and references therein for motivations and history of p-adic dynamical systems). We consider the function $f(x) = \frac{a}{x-2b}$ and study the dynamical systems generated by this function in $\mathbb{C}_p$. We give fixed points, periodic points, basin of attraction and Siegel disk of each fixed point.

2 Main results

Consider the dynamical system associated with the function $f : \mathbb{C}_p \to \mathbb{C}_p$ defined by

$$f(x) = \frac{a}{x - 2b}, \quad a \neq 0, \quad a, b \in \mathbb{C}_p, \tag{2.1}$$

where $x \neq 2b$.

Our goal here is to investigate the behavior of trajectories of (2.1) in the complex p-adic filed $\mathbb{C}_p$.

Remark 2.1. The case $b = 0$ is simple: in this case any point $x \in \mathbb{C}_p \setminus \{-b\}$ is two periodic. That is $f(f(x)) = x$. Indeed,

$$f(f(x)) = \frac{a}{\frac{a}{x}} = a \cdot \frac{x}{a} = x.$$

Therefore, below we consider the case $b \neq 0$.

Since $\mathbb{C}_p$ is algebraic closed, this function (for $ab \neq 0$) has two fixed points:

$$f(x) = x \quad \Rightarrow \quad x^2 - 2bx - a = 0 \quad \Rightarrow \quad x_1 = b - \sqrt{b^2 + a}, \quad x_2 = b + \sqrt{b^2 + a}. \quad (2.2)$$

The following proposition says that f may have periodic (except fixed points) iff $b = 0$.

Property 2.2. If $b(b^2 + a) \neq 0$ then $f^n(x) = x$, $n \geq 2$ does not have any solution (except solutions of $f(x) = x$).

Proof. Using induction over $n \geq 1$ one can show that f^n has the following form

$$f^n(x) = \frac{a_n x + b_n}{c_n x + d_n}, \quad \text{for some} \quad a_n, b_n, c_n, d_n \in \mathbb{C}_p.$$

Indeed, for $n = 1$ the formula is true with

$$a_1 = 0, \quad b_1 = a, \quad c_1 = 1, \quad d_1 = -2b. \quad (2.3)$$

Assuming that the formula is true for n we get it for $n + 1$ with

$$\begin{aligned} a_{n+1} &= b_n \\ b_{n+1} &= aa_n - 2bb_n \\ c_{n+1} &= d_n \\ d_{n+1} &= ac_n - 2bd_n. \end{aligned} \quad (2.4)$$

Thus we have reduced the dynamical system $\{f^n(x)\}_{n\geq 1}$ to the dynamical system (2.4) with initial point (2.3). Since in (2.4) the vectors (a_n, b_n) and (c_n, d_n) are independent, it suffices to study only one of them.

Denote

$$M = \begin{pmatrix} 0 & 1 \\ a & -2b \end{pmatrix}.$$

Let $\lambda_1 = -x_1, \lambda_2 = -x_2$ (see (2.2)) be the distinct eigenvalues of M (because by condition of the proposition we have $b^2 + a \neq 0$). By (2.2) we get

$$\lambda_1 + 2b = x_2 = -\lambda_2, \quad \lambda_2 + 2b = x_1 = -\lambda_1. \quad (2.5)$$

From (2.4) we get $(a_{n+1}, b_{n+1}) = M\,(a_n, b_n)^T$ and $(c_{n+1}, d_{n+1}) = M\,(c_n, d_n)^T$. Thus

$$(a_{n+1}, b_{n+1}) = M^n\,(a_1, b_1)^T, \quad (c_{n+1}, d_{n+1}) = M^n\,(c_1, d_1)^T. \tag{2.6}$$

Therefore we need to find M^n. To find it we use a little Cayley-Hamilton Theorem[c] and (2.5) to obtain the following formula

$$M^n = \frac{\lambda_2\,\lambda_1^n - \lambda_1\,\lambda_2^n}{\lambda_2 - \lambda_1} \cdot I_2 + \frac{\lambda_2^n - \lambda_1^n}{\lambda_2 - \lambda_1} \cdot M$$

$$= \frac{1}{\lambda_2 - \lambda_1}\begin{pmatrix} \lambda_2\,\lambda_1^n - \lambda_1\,\lambda_2^n & \lambda_2^n - \lambda_1^n \\ a\,(\lambda_2^n - \lambda_1^n) & \lambda_2^{n+1} - \lambda_1^{n+1} \end{pmatrix}. \tag{2.7}$$

By this formula and (2.3) from (2.6) we get

$$\begin{aligned}
a_{n+1} &= a \cdot (\lambda_2 - \lambda_1)^{-1}(\lambda_2^n - \lambda_1^n) \\
b_{n+1} &= a \cdot (\lambda_2 - \lambda_1)^{-1}(\lambda_2^{n+1} - \lambda_1^{n+1}) \\
c_{n+1} &= (\lambda_2 - \lambda_1)^{-1}(\lambda_2^{n+1} - \lambda_1^{n+1}) \\
d_{n+1} &= (\lambda_2 - \lambda_1)^{-1}((a - 2b\lambda_2)\lambda_2^n - (a - 2b\lambda_1)\lambda_1^n).
\end{aligned} \tag{2.8}$$

Consequently,

$$f^n(x) = x \quad \Leftrightarrow \quad \hat{c}_n x^2 + (\hat{d}_n - \hat{a}_n)x - \hat{b}_n = 0, \tag{2.9}$$

where

$$\begin{aligned}
\hat{a}_{n+1} &= a \cdot (\lambda_2^n - \lambda_1^n) \\
\hat{b}_{n+1} &= a \cdot (\lambda_2^{n+1} - \lambda_1^{n+1}) \\
\hat{c}_{n+1} &= \lambda_2^{n+1} - \lambda_1^{n+1} \\
\hat{d}_{n+1} &= (a - 2b\lambda_2)\lambda_2^n - (a - 2b\lambda_1)\lambda_1^n.
\end{aligned} \tag{2.10}$$

For each $n \geq 2$, from $\lambda_1 \neq \lambda_2$ it follows that $\hat{a}_n, \hat{b}_n, \hat{c}_n, \hat{d}_n$ can not be simultaneously zero.

Since each solution of $f(x) = x$ is solution to the quadratic equation (2.9), we conclude that (2.9) does not have solutions different from the fixed points. □

Denote:

$$\mathcal{P} = \{x \in \mathbb{C}_p : \exists n \in \mathbb{N} \cup \{0\}, f^n(x) = 2b\}. \tag{2.11}$$

For example, $x = \hat{x} = 2b + \frac{a}{2b} \in \mathcal{P}$, because $f(\hat{x}) = 2b$.

The following proposition describes the set $\mathcal{P}$

[c] https://www.freemathhelp.com/forum/threads/formula-for-matrix-raised-to-power-n.55028/

Property 2.3. If $b(b^2+a)\neq 0$ then the set $\mathcal{P}$ is the following

$$\mathcal{P}=\{2b\}\cup\left\{2b-\frac{\hat{b}_n}{\hat{d}_n}\,:\,\hat{d}_n\neq 0,\,n\geq 1\right\},$$

where $\hat{b}_n$ and $\hat{d}_n$ are defined in (2.10).

Proof. For each fixed $n\geq 1$ the corresponding element of $\mathcal{P}$ is solution of the equation

$$f^n(x)=\frac{\hat{a}_nx+\hat{b}_n}{\hat{c}_nx+\hat{d}_n}=2b.$$

That is

$$(2b\hat{c}_n-\hat{a}_n)x=\hat{b}_n-2b\hat{d}_n.$$

Note that $\hat{d}_n=-(2b\hat{c}_n-\hat{a}_n)$. It is easy to see that if $\lambda_1\neq\lambda_2$ (i.e. $b^2+a\neq 0$) then $\hat{b}_n$ and $\hat{d}_n$ can not be zero simultaneously. This completes the proof. ☐

Let x_0 be a fixed point of a function $f(x)$. Put $\lambda=f'(x_0)$. The point x_0 is attractive if $0<|\lambda|_p<1$, *indifferent* if $|\lambda|_p=1$, and repelling if $|\lambda|_p>1$.

For (2.1) we have

$$f'(x)=-\frac{a}{(x-2b)^2}=-\frac{1}{a}\left(\frac{a}{(x-2b)}\right)^2=-\frac{1}{a}(f(x))^2.$$

Using this formula and $x_1x_2=-a$ we get

$$|f'(x_1)|_p=\frac{|x_1|_p}{|x_2|_p},\quad |f'(x_2)|_p=\frac{|x_2|_p}{|x_1|_p},$$

i.e., if the point x_1 (resp. x_2) is repeller then x_2 (resp. x_1) is attractive. Moreover, x_1 is indifferent iff x_2 is indifferent. Thus we need to compare $|x_1|_p=|b-\sqrt{b^2+a}|_p$ and $|x_2|_p=|b+\sqrt{b^2+a}|_p$.

Case: $b^2+a=0$. In this case $x_1=x_2$, i.e. the function has unique fixed point $x_1=b$. Moreover, $|f'(x_1)|_p=1$, i.e. the fixed point is an indifferent point.

Denote

$$B=|b|_p.$$

Take $x\in S_r(x_1)$, i.e. $r=|x-x_1|_p=|x-b|_p$, then it follows from (2.1) that

$$|f(x)-x_1|_p=\left|\frac{-b^2}{x-2b}-\frac{-b^2}{b-2b}\right|_p=B\cdot\frac{|x-b|_p}{|x-b-b|_p}$$

$$=\varphi(r)\equiv\varphi_{B^*}(r)=\begin{cases} r, & \text{if } r<B\\ B^*, & \text{if } r=B\\ B, & \text{if } r>B,\end{cases} \qquad (2.12)$$

where $B^*\geq B$ is a given number (parameter).

Remark 2.4. Note that the value $B^* = \varphi(B)$ is not concretely defined. We only have its estimation. But in our analysis the estimations given for undefined value will be sufficient.

Let the function $\varphi : [0, +\infty) \to [0, +\infty)$ be defined by (2.12).

The following simple lemma shows that the real dynamical system compiled from φ^n is directly related to the p-adic dynamical system $f^n(x),\ n \geq 1,\ x \in \mathbb{C}_p \setminus \mathcal{P}$.

Lemma 2.5. *If $x \in S_r(x_1)$, then the following holds for the function (2.1):*

$$|f^n(x) - x_1|_p = \varphi^n(r).$$

The following lemma gives properties of this real dynamical system.

Lemma 2.6. *The function φ has the following properties*

1. $\mathrm{Fix}(\varphi) = \{r : 0 \leq r < B\} \cup \{B\ \ : \text{if}\ \ B^* = B\}$
2. *If $r = B$ then $\varphi(B) = B^*$, $\varphi(B^*) = B$.*
3. *If $r > B$ then $\varphi(r) = B,\ \ \varphi(B) = B^*,\ \ \varphi(B^*) = B$.*

Proof. It easily follows from the definition of φ. □

From this lemma it follows that

$$\lim_{n\to\infty} \varphi^n(r) = \begin{cases} r, & \text{if}\ \ 0 \leq r < B \\ B^*, & \text{if}\ \ r = B,\ \ B = B^* \\ B^*, & \text{if}\ \ r \geq B,\ \ n = 2k - 1 \\ B, & \text{if}\ \ r \geq B,\ \ n = 2k,\ \ k = 1, 2, \ldots \end{cases} \tag{2.13}$$

Denote

$$B^*(x) = |f(x) - x_1|_p,\ \ \text{if}\ \ x \in S_B(x_1).$$

By the applying Lemma 2.5, and 2.6, and formula (2.13) we get the following properties of the p-adic dynamical system complied by the function (2.1).

Theorem 2.7. *The p-adic dynamical system generated by the function (2.1), for $b^2 + a = 0$, has the following properties:*

1. *$SI(x_1) = U_B(x_1)$.*
2. *$\mathcal{P} \subset \mathbb{C}_p \setminus U_B(x_1)$.*
3. *If $r > B$ and $x \in S_r(x_1)$, then $f(x) \in S_B(x_1)$ and*

$$f^n(x) \in S_{B^*(f^{n-1}(x))}(x_1),\ \ n \geq 2,$$

where $B^*(x) = |f(x) - x_1|_B \geq B$.

Proof.

1. By lemma 2.5 and part 1 of Lemma 2.6, sphere $S_r(x_1)$ is invariant for f if and only if $r < B$.
2. Note that $|2b - x_1|_p = |b|_p = B$, i.e., $2b \in S_B(x_1)$. By part 1 of this theorem if $x \in S_r(x_1), r < B$, then $f(x) \notin S_B(x_1)$. By definition of set $\mathcal{P}$ and Lemma 2.6, we can conclude that $\mathcal{P} \subset \mathbb{C}_p \setminus U_B(x_1)$.
3. The proof of part 3 easily follows from Lemmas 2.5 and Lemma 2.6.

□

Case: $b^2 + a \neq 0$. In this case $x_1 \neq x_2$. We denote

$$\alpha = |x_1|_p = |b - \sqrt{b^2+a}|_p, \quad \beta = |x_2|_p = |b + \sqrt{b^2+a}|_p.$$

For $x \in S_r(x_1)$, i.e. $r = |x - x_1|_p$, from (2.1) using $x_1 x_2 = -a$ and $x_1 + x_2 = 2b$ we get

$$|f(x) - x_1|_p = \left| \frac{a}{x-2b} - \frac{a}{x_1 - 2b} \right|_p = \alpha \cdot \frac{|x - x_1|_p}{|x - x_1 - x_2|_p}$$

$$= \eta(r) \equiv \eta_A(r) = \begin{cases} \frac{\alpha}{\beta} r, & \text{if } \ r < \beta \\ A, & \text{if } \ r = \beta \\ \alpha, & \text{if } \ r > \beta, \end{cases}$$

where $A \geq \alpha$.

Similarly, for $x \in S_r(x_2)$ we get

$$|f(x) - x_2|_p = \left| \frac{a}{x-2b} - \frac{a}{x_2 - 2b} \right|_p = \beta \cdot \frac{|x - x_2|_p}{|x - x_2 - x_1|_p}$$

$$= \zeta(r) \equiv \zeta_D(r) = \begin{cases} \frac{\beta}{\alpha} r, & \text{if } \ r < \alpha \\ D, & \text{if } \ r = \alpha \\ \beta, & \text{if } \ r > \alpha, \end{cases}$$

where $D \geq \beta$.

Subcase: $\alpha = \beta$. In this case we have $|x_1 - x_2| \leq \alpha$ and since $|f'(x_i)|_p = 1$, $i = 1, 2$, both fixed points are indifferent. Moreover, the functions φ, η and ζ have similar graphs, therefore they generate similar dynamical systems. The limit points of which are as in (2.13) replaced parameters of φ by parameters of η.

Using these properties we prove the following theorem

Theorem 2.8. *The p-adic dynamical system is generated by the function (2.1), for $b^2 + a \neq 0$ and $\alpha = \beta$, has the following properties:*

i. $SI(x_i) = U_\alpha(x_i)$, *with*

$$SI(x_1) = SI(x_2), \quad \textit{if} \ \ |x_1 - x_2|_p < \alpha$$

$$SI(x_1) \cap SI(x_2) = \emptyset, \quad \textit{if} \ \ |x_1 - x_2|_p = \alpha.$$

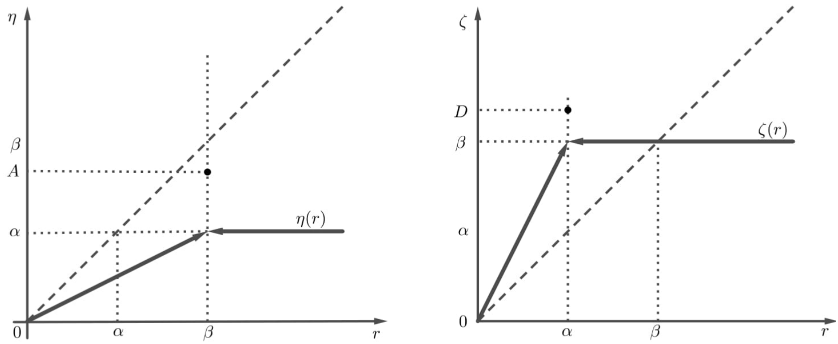

Fig 1: The graph of the function η (left), and ζ (right).

ii. $\mathcal{P} \subset \mathbb{C}_p \setminus (SI(x_1) \cup SI(x_2))$.

iii. If $r > \alpha$ *and* $x \in S_r(x_1)$, *then* $f(x) \in S_\alpha(x_1)$ *and*

$$f^n(x) \in S_{A^*(f^{n-1}(x))}(x_1), \quad n \geq 2,$$

where $A^*(x) = |f(x) - x_1|_\alpha \geq \alpha$.

Proof. i. Follows from the properties of the function η and the fact that in p-adic field any point of a ball is its center. Moreover, two balls are either disjoint, or one is contained in the other.

ii. For $b^2 + a \neq 0$, we have

$$|2b - x_1|_p = |x_2|_p = |2b - x_2|_p = |x_1|_p = \alpha,$$

i.e., $2b \in S_\alpha(x_i)$, $i = 1, 2$. By part i of this theorem if $x \in S_r(x_i), r < \alpha$, then $f(x) \notin S_\alpha(x_i)$. This completes the proof of part ii.

iii. By property of the function η (in case $\alpha = \beta$) we have $f(x) \in S_\alpha(x_1)$ or $f(x) \in S_{A^*(x)}(x_1)$. Therefore, iterating f we get iii. □

Subcase: $\alpha < \beta$. (The case $\alpha > \beta$ is similar). In this case we have

$$|x_1|_p = \alpha < |x_2|_p = \beta, \quad |x_1 - x_2|_p = \beta.$$

Moreover, $|f'(x_1)|_p < 1$, i.e., x_1 is attractive and $|f'(x_2)|_p > 1$, i.e., x_2 is repeller.

Following Fig 1. one can easily prove the following lemmas

Lemma 2.9. *The function η has the following properties*

1. $\mathrm{Fix}(\eta) = \{0\} \cup \{A \;: \text{if} \;\; A = \beta\}$

2. If $\alpha \le A \ne \beta$ then

$$\lim_{n\to\infty} \eta^n(r) = 0, \quad \textit{for all} \;\; r \ge 0.$$

3. If $A = \beta$ then $\eta(\beta) = \beta$ and

$$\lim_{n\to\infty} \eta^n(r) = 0, \quad \textit{for all} \;\; 0 \le r \ne \beta.$$

Lemma 2.10. *The function ζ has the following properties*

1. $\mathrm{Fix}(\zeta) = \{0, \beta\}$

2.

$$\lim_{n\to\infty} \zeta^n(r) = \beta, \quad \textit{for all} \;\; r > 0.$$

Then using Lemmas 2.9 and 2.10, we obtain the following

Theorem 2.11. *If $\alpha < \beta$, then the p-adic dynamical system generated by the function (2.1) has the following properties:*

1. $\mathcal{P} \subset S_\beta(x_1)$.

2. The set $\mathbb{C}_p \setminus S_\beta(x_1)$ is a subset to the basin of attraction for the attractive fixed point x_1, i.e.,

$$\mathbb{C}_p \setminus S_\beta(x_1) \subseteq \mathcal{A}(x_1).$$

Proof. 1. We have

$$|x_1 - 2b|_p = |-b - \sqrt{b^2 + a}|_p = |x_2|_p = \beta,$$

$$|x_2 - 2b|_p = |-b + \sqrt{b^2 + a}|_p = |x_1|_p = \alpha.$$

Thus $2b \in S_\beta(x_1)$. From Lemma 2.9 we get that if $x \notin S_\beta(x_1)$ then $f(x) \notin S_\beta(x_1)$. Consequently, $f^n(x) \notin S_\beta(x_1)$. Hence $\mathcal{P} \subset S_\beta(x_1)$. We also know that $x_2 \in S_\beta(x_1)$. This completes the proof of part 1.

2. Follows from the part 1 and Lemmas 2.9 and 2.10. $\square$

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Rozikov U.A.,
V.I.Romanovskiy Institute of Mathematics of Uzbekistan Academy of Sciences, Tashkent, Uzbekistan.
AKFA University, 1st Deadlock 10, Kukcha Darvoza, 100095, Tashkent, Uzbekistan;
Faculty of Mathematics, National University of Uzbekistan, 100170, Tashkent, Uzbekistan;
e-mail: rozikovu@yandex.ru
Sayitova M.,
Bukhara State University, The department of Mathematics, 11, M.Iqbol, Bukhara city, Bukhara, Uzbekistan.
e-mail: sayitovamehinbonu@gmail.com

Contents

Mathematical life

Компьютерная верстка: *K.K. Abdurasulov*

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