p-adic dynamical systems of the function $a/(x-2b)$

U.A. Rozikov and M. Sayitova

V.I.Romanovskiy Institute of Mathematics, 4, University str., 100174, Tashkent, Uzbekistan. AND Bukhara State University, The department of Mathematics, 11, M.Iqbol, Bukhara, Uzbekistan.

e-mail: rozikovu@yandex.ru, e-mail: sayitovamehinbonu@gmail.com

2000 Mathematics Subject Classification. 46S10

It is known that the completion of the set of rational numbers $\mathcal{Q}$ with respect to p-adic norm $|\cdot|_p$ defines the p-adic field which is denoted by $\mathcal{Q}_p$ (see [3]).

The algebraic completion of $\mathcal{Q}_p$ is denoted by $\mathcal{C}_p$ and it is called the set of *complex p-adic numbers*.

For any $a \in \mathcal{C}_p$ and $r > 0$ denote

$$U_r(a) = \{x \in \mathcal{C}_p : |x-a|_p < r\}, \quad V_r(a) = \{x \in \mathcal{C}_p : |x-a|_p \leq r\},$$

$$S_r(a) = \{x \in \mathcal{C}_p : |x-a|_p = r\}.$$

Let x_0 be a fixed point of a function $f(x)$, i.e. $f(x_0) = x_0$. A fixed point x_0 is called an *attractor* if there exists a neighborhood $U(x_0)$ of x_0 such that for all points $x \in U(x_0)$ it holds $\lim\limits_{n\to\infty} f^n(x) = x_0$. If x_0 is an attractor then its *basin of attraction* is

$$\mathcal{A}(x_0) = \{x \in mathcal C_p : \ f^n(x) \to x_0, \ n \to \infty\}.$$

Put $\lambda = f'(x_0)$. The point x_0 is attractive if $0 < |\lambda|_p < 1$, *indifferent* if $|\lambda|_p = 1$, and repelling if $|\lambda|_p > 1$.

The ball $U_r(x_0)$ is said to be a *Siegel disk* if each sphere $S_\rho(x_0)$, $\rho < r$ is an invariant sphere of $f(x)$, i.e. if $x \in S_\rho(x_0)$ then all iterated points $f^n(x) \in S_\rho(x_0)$ for all $n = 1, 2\dots$. The union of all Siegel desks with the center at x_0 is said to *a maximum Siegel disk* and is denoted by $SI(x_0)$.

Consider the dynamical system associated with function $f : \mathcal{C}_p \to \mathcal{C}_p$ defined by

$$f(x) = \frac{a}{x-2b}, \quad a, b \in \mathcal{C}_p, \tag{1}$$

where $x \neq 2b$.

Our goal here is to present the behavior of trajectories $\{f^n(x), \ x \in \mathcal{C}_p\}$ of (1) in the complex p-adic filed $\mathcal{C}_p$. For motivation of such investigations see [1], [2] and references therein.

The case $a = 0$ is trivial and $a \neq 0$, $b = 0$ is simple: in this case any point $x \in \mathcal{C}_p \setminus \{-b\}$ is two periodic.

Therefore we consider the case $ab \neq 0$. Since $\mathcal{C}_p$ is algebraic closed, this function has two fixed points:

$$f(x) = x \quad \Rightarrow \quad x^2 - 2bx - a = 0 \quad \Rightarrow \quad x_1 = b - \sqrt{b^2+a}, \quad x_2 = b + \sqrt{b^2+a}.$$

Denote:

$$\mathcal{P} = \{x \in \mathcal{C}_p : \exists n \in \mathcal{N} \cup \{0\}, f^n(x) = 2b\}.$$

For example, $x = \hat{x} = 2b + \frac{a}{2b} \in \mathcal{P}$, because $f(\hat{x}) = 2b$.

Note that $\mathcal{P} = \{2b\} \cup \left\{ \frac{b_n - 2bd_n}{2bc_n - a_n} : n \geq 1 \right\}$, where a_n, b_n, c_n, d_n are coefficients of f^n.

For (1) we have

$$|f'(x_1)|_p = \frac{|x_1|_p}{|x_2|_p}, \quad |f'(x_2)|_p = \frac{|x_2|_p}{|x_1|_p},$$

i.e., if the point x_1 (resp. x_2) is repeller then x_2 (resp. x_1) is attractive. Moreover, x_1 is indifferent iff x_2 is indifferent. Thus we need to compare $|x_1|_p = |b - \sqrt{b^2 + a}|_p$ and $|x_2|_p = |b + \sqrt{b^2 + a}|_p$.

Case: $b^2 + a = 0$. In this case $x_1 = x_2$, i.e. the function has unique fixed point $x_1 = b$. Moreover, $|f'(x_1)|_p = 1$, i.e. the fixed point is an indifferent point. Denote $B = |b|_p$ and

$$B^*(x) = |f(x) - x_1|_p, \quad \text{if} \quad x \in S_B(x_1).$$

We have the following

Theorem 1. The p-adic dynamical system is generated by the function (1), for $b^2 + a = 0$, has the following properties:

1. $SI(x_1) = U_B(x_1)$. 2. $\mathcal{P} \subset \mathcal{C}_p \setminus U_B(x_1)$.
3. If $r > B$ and $x \in S_r(x_1)$, then $f(x) \in S_B(x_1)$ and

$$f^n(x) \in S_{B^*(f^{n-1}(x))}(x_1), \quad n \geq 2,$$

where $B^*(x) = |f(x) - x_1|_B \geq B$.

Case: $b^2 + a \neq 0$. In this case $x_1 \neq x_2$. We denote

$$\alpha = |x_1|_p = |b - \sqrt{b^2 + a}|_p, \quad \beta = |x_2|_p = |b + \sqrt{b^2 + a}|_p.$$

We obtain the following theorem

Theorem 2. The p-adic dynamical system is generated by the function (1), for $b^2 + a \neq 0$ and $\alpha < \beta$, then the p-adic dynamical system generated by the function (1) has the following properties:

1. $\mathcal{P} \subset S_\beta(x_1)$.
2. The set $\mathcal{C}_p \setminus S_\beta(x_1)$ is a subset to the basin of attraction for the attractive fixed point x_1, i.e.,

$$\mathcal{C}_p \setminus S_\beta(x_1) \subseteq \mathcal{A}(x_1).$$

[1] Albeverio S., Rozikov U.A., Sattarov I.A., *p-adic* $(2,1)$*-rational dynamical systems.* Jour. Math. Anal. Appl. **398**(2) (2013), 553–566.

[2] Anashin V., Khrennikov A., *Applied algebraic dynamics*, de Gruyter Expositions in Mathematics vol 49, Walter de Gruyter (Berlin - New York), 2009.

[3] Koblitz N., *p-adic numbers, p-adic analysis and zeta-function.* Springer, Berlin, 1977.