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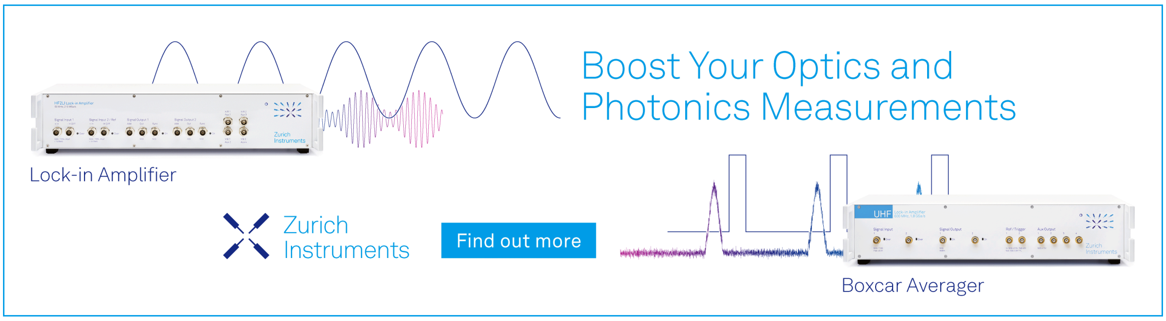

To Construct Basis Functions in $W_2^{(1,0)}$ Space to Finite Element Method for 1D Two-Point Boundary-Value Problems

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Abstract. This paper uses the Ritz method to find approximate solutions to boundary-value problems for ordinary differential equations. We present the approximate finite element solution by a linear combination of the basis functions. Here we get coefficients of the optimal interpolation formula in $W_2^{(1,0)}$ space as basis functions. The difference obtained in numerical results and exact solutions illustrates graphically.

INTRODUCTION

Differential equations and variational problems growth conditions have significantly been studied in recent years. The massive interest in this type of non-standard growth conditions is mainly motivated by their physical applications. The actual cause lay in the fact that some physical phenomena can be modelled by such kinds of equations, such as elastic mechanics [1], electrorheological fluids sometimes referred to as "smart fluids" [2], image restoration [3, 4], flow in porous media [5].

Even if a differential equation can be solved analytically, considerable effort and sound mathematical theory are often needed, and the closed form of the solution may even turn out to be too messy to be valid. Suppose the analytic solution of the differential equation is unavailable or too difficult to obtain or takes some complicated form that could be more helpful. In that case, we may find an approximate solution. Discrete numerical values represent the solution to a sure accuracy. Nowadays, these numerical results and associated tables or plots are obtained using computers to provide practical solutions to many problems that were impossible to obtain.

Numerical solutions to differential equations can be achieved using finite difference and finite element methods. Other methods include finite volume, collocation, and spectral methods.

The finite element method was developed to solve complex problems in engineering involving elliptic partial differential equations and complicated geometries, notably in elasticity and structural mechanics modelling. However, its range of applications nowadays is extensive [6]. One of the essential parts of the finite element method is the construction of basis functions. Therefore, in this work, we construct basis functions in $W_2^{(1,0)}$ space for the finite element methods, and we use them for numerical solutions of boundary-value problems.

The paper is structured as follows. In Section 2, the Ritz method is represented for the second-order boundary-value problems; in Section 3, we present the optimal interpolation formula in $W_2^{(1,0)}$ space; finally, in Section 4, we solve several problems by the Ritz method taking coefficients of optimal interpolation formula in $W_2^{(1,0)}$ space as basis functions.

THE RITS METHOD FOR SECOND-ORDER BOUNDARY-VALUE PROBLEMS

Let us consider the following model problem

$$-\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right)+q(x)y=f(x),\ \text{for } 0\le x\le 1, \tag{1}$$

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with two-point boundary conditions $y(0) = y(1) = 0$, where, $p \in C^1[0,1], q, f \in C[0,1]$, and

$$p(x) \geq \delta > 0,\ q(x) \geq 0,\ \text{for } 0 \leq x \leq 1.$$

The function $y \in C_0^2[0,1]$ is an unique solution to the differential equation (1), in the case that y is the unique function in $C_0^2[0,1]$ that minimizes the integral (see, [7], pp.88-89)

$$I[u] = \int_0^1 \left(p(x)[u'(x)]^2 + q(x)[u(x)]^2 - 2f(x)u(x)\right) dx. \tag{2}$$

There are at least three different formulations to consider for this problem, i.e., the differential form, variational or weak form (corresponding finite element method is often called the Galerkin method), and minimization form (respective finite element method is usually called the Ritz method). From the aspect of mathematical modelling, both the weak form and the minimization form are farther natural than the differential formulation.

Solving (1) by the Ritz method takes on three stages. Initially, it is shown that any solution y to (1) also satisfies the equation

- $$\int_0^1 f(x)u(x)dx = \int_0^1 \left(p(x)\frac{dy}{dx}(x)\frac{du}{dx}(x) + q(x)y(x)u(x)\right) dx, \tag{3}$$

  per all $u \in C_0^2[0,1]$.
- The subsequent stage demonstrates that $y \in C_0^2[0,1]$ is a solution to (2) upon these terms (3) holds for all $u \in C_0^2[0,1]$.
- The ending step illustrates that (3) has a unique solution. This unique solution will also be a solution to (2) and to (1), so the solutions to (1) and (2) are same.

The Ritz method approximates the solution y by minimizing the integral, not upon all the functions in $C_0^2[0,1]$, but over a smaller subspace of functions consisting of linear combinations of certain basis functions $\phi_1, \phi_2, \ldots, \phi_n$. The basis functions are linearly independent and satisfy $\phi_i(0) = \phi_i(1) = 0$ and they satisfy this conditions $\phi_i(x_j) = \delta_i^j$, here δ_i^j is the Kronecker symbol, for each $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, n$.

We present the approximate finite element solution by a linear combination of the basis functions

$$y_h(x) = \sum_{i=1}^{n} c_i \phi_i(x) \tag{4}$$

where the coefficients c_i are the unknowns to be determined.

On assuming the hat basis functions, $y_h(x)$ is also a piecewise linear function, although this is not usually the case for the exact solution $y(x)$ of (1). We then derive a linear system of equations for the coefficients by substituting the approximate solution $y_h(x)$ for the exact solution $y(x)$ to minimize the integral $I[\sum_{i=1}^{n} c_i\phi_i]$. Then from (2), we get

$$I[y_h] = I\left[\sum_{i=1}^{n} c_i\phi_i\right] = \int_0^1 \left(p(x)\left(\sum_{i=1}^{n} c_i\phi_i'(x)\right)^2 + q(x)\left(\sum_{i=1}^{n} c_i\phi_i(x)\right)^2 - 2f(x)\sum_{i=1}^{n} c_i\phi_i(x)\right) dx \tag{5}$$

and, for a minimum to occur, it is necessary, when considering I as a function of $c_1, c_2, \ldots, c_n$, to have

$$\frac{\partial I}{\partial c_j} = 0 \text{ for } j = 1, 2, \ldots, n. \tag{6}$$

Differentiating (5) gives

$$\frac{\partial I}{\partial c_j} = \int_0^1 \left(2p(x)\sum_{i=1}^{n} c_i\phi_i'(x)\phi_j'(x) + 2q(x)\sum_{i=1}^{n} c_i\phi_i(x)\phi_j(x) - 2f(x)\phi_j(x)\right) dx,$$

and substituting into (6) yields

$$\sum_{i=1}^{n}\left(\int_0^1 \left(p(x)\phi_i'(x)\phi_j'(x)+q(x)\phi_i(x)\phi_j(x)\right)dx\right)c_i=\int_0^1 f(x)\phi_j(x)dx, \tag{7}$$

for each $j=1,2,\ldots,n$.

The **normal equations** described in (7) produce an $n\times n$ linear system $A\mathbf{c}=\mathbf{b}$ in the variables $c_1,c_2,\ldots,c_n$, where the symmetric matrix A has the following elements

$$a_{ij}=\int_0^1 \left(p(x)\phi_i'(x)\phi_j'(x)+q(x)\phi_i(x)\phi_j(x)\right)dx,\ \ i,j=1,2,\ldots,n$$

and the vector $\mathbf{b}$ is defined by

$$b_i=\int_0^1 f(x)\phi_i(x)dx, i=1,2,\ldots,n.$$

Now we need analytical forms of basis functions ϕ_i.

It should be noted that in [6] it was used the hat functions when finding an approximate solution in the Sobolev space $H_0^1(0,1)$.

In the present work, we use piecewise continuous exponential functions in the space $W_2^{(1,0)}$ instead of the hat functions. Moreover, in certain examples, we show convergence of approximate solution to exact solution of problem (1) graphically. To do this, we use the coefficients of the optimal interpolation formula constructed in the $W_2^{(1,0)}$ space as the basis functions.

THE OPTIMAL INTERPOLATION FORMULA IN $W_2^{(1,0)}$ SPACE

In [8], the optimal interpolation formula in $W_2^{(1,0)}$ space was constructed, and it was found the explicit form of the norm of the error functional ℓ. We note that the coefficients of the optimal interpolation formulas constructed in the works [9, 10, 11, 12, 13, 14, 15, 16, 17] and [18] can also be used as basis functions.

Assume we have the table of the values $\varphi(x_i)$, $i=0,1,\ldots,n$ of functions φ at points $x_i\in[0,1]$. It is required approximate functions φ by another more simple function P_φ, i.e.

$$\varphi(x)\cong P_\varphi(x)=\sum_{i=0}^{n}\phi_i(x)\cdot\varphi(x_i), \tag{8}$$

which satisfies the following equalities

$$\varphi(x_i)=P_\varphi(x_i),\ i=0,1,\ldots,n. \tag{9}$$

Here $\phi_i(x)$ and x_i ($\in[0,1]$) are *the coefficients* and *the nodes* of the interpolation formula (8), respectively. The difference $\varphi-P_\varphi$ is called *the error* of the interpolation formula (8). The following theorem was proved in [8].

Theorem 1. Coefficients of the optimal interpolation formula (8) with equal spaced nodes in the space $W_2^{(1,0)}$ have the the following form

$$\begin{aligned}\phi_i(x)= & \tfrac{1}{2(1-e^{2h})}\Big[\mathrm{sign}(x-hi-h)\cdot(e^{hi+2h-x}-e^{x-hi})+\mathrm{sign}(x-hi+h)\cdot(e^{hi-x}-e^{x-hi+2h})\\ & +(1+e^{2h})\cdot\mathrm{sign}(x-hi)\cdot(e^{x-hi}-e^{hi-x})\Big],\ \ i=0,1,\ldots,n.\end{aligned} \tag{10}$$

Doing some simplification in equation (10)

$$\phi_i(x)=\begin{cases}0, & 0\le x\le h(i-1),\\ \frac{e^{hi-x}-e^{x-hi}\cdot e^{2h}}{1-e^{2h}}, & h(i-1)<x\le hi,\\ \frac{e^{x-hi}-e^{hi-x}\cdot e^{2h}}{1-e^{2h}}, & hi<x\le h(i+1),\\ 0, & h(i+1)<x\le 1.\end{cases} \tag{11}$$

OPTIMAL COEFFICIENTS AS BASIS FUNCTIONS

The primary purpose of this section is to solve several boundary value problems using the Ritz method and analyze the differences between the exact solution and the approximate solution in graphics.

Now, we consider the boundary-value problem (1).

Here we use the optimal coefficients (11) as the basis functions. The first step is to form a partition of $[0,1]$ by choosing points $x_0, x_1, \ldots, x_{n+1}$ with

$$0 = x_0 < x_1 < \cdots < x_n < x_{n+1} = 1.$$

We define the basis functions ϕ_1, ϕ_2,..., ϕ_n by

$$\phi_i(x) = \begin{cases} 0, & 0 \leq x \leq h(i-1), \\ \frac{e^{-(x-hi)} - e^{2h+(x-hi)}}{1-e^{2h}}, & h(i-1) < x \leq hi, \\ \frac{e^{x-hi} - e^{2h-(x-hi)}}{1-e^{2h}}, & hi < x \leq h(i+1), \\ 0 & h(i+1) < x \leq 1 \end{cases} \tag{12}$$

for each $i = 1, 2, \ldots, n$.

The function ϕ_i is a piecewise continuous exponential function, then the first derivative ϕ_i' has the form

$$\phi_i'(x) = \begin{cases} 0, & 0 \leq x \leq h(i-1), \\ \frac{-e^{-(x-hi)} - e^{2h+(x-hi)}}{1-e^{2h}}, & h(i-1) < x \leq hi, \\ \frac{e^{x-hi} + e^{2h-(x-hi)}}{1-e^{2h}}, & hi < x \leq h(i+1), \\ 0, & h(i+1) < x \leq 1 \end{cases} \tag{13}$$

for each $i = 1, 2, \ldots, n$.

$\phi_i(x)$ and $\phi_i'(x)$ are nonzero only on $(h(i-1), h(i+1))$, so $\phi_i(x)\phi_j(x) \equiv 0$ and $\phi_i'(x)\phi_j'(x) \equiv 0$, except when j is $i-1$, i, or $i+1$. As a consequence, the linear system given by (7) reduces to an $n \times n$ tridiagonal linear system. The nonzero entries in A are

$$a_{ii} = \int_0^1 \left(p(x)[\phi_i'(x)]^2 + q(x)[\phi_i(x)]^2 \right) dx,$$

for each $i = 1, 2, \ldots, n$,

$$a_{i,i+1} = \int_0^1 \left(p(x)\phi_i'(x)\phi_{i+1}'(x) + q(x)\phi_i(x)\phi_{i+1}(x) \right) dx$$

for each $i = 1, 2, \ldots, n-1$, and

$$a_{i,i-1} = \int_0^1 \left(p(x)\phi_i'(x)\phi_{i-1}'(x) + q(x)\phi_i(x)\phi_{i-1}(x) \right) dx,$$

for each $i = 2, 3, \ldots, n$.

We calculate the above integrals using quadrature formulas (see, [4, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31]). After that, we can calculate coefficients c_i of the approximate solution (4) from the system of equations (7).

NUMERICAL RESULTS

We solve several boundary value problems in this subsection using the Ritz method. We divide an interval into equal meshes with $n = 10$ and $n = 100$. As mentioned above, we approximate the solution of the problem by the Ritz method using optimal coefficients as the basis functions. Furthermore, by comparing the approximate solutions $y_h(x)$ with the corresponding exact solutions $y(x)$, we show the graphics of $lg|y_h(x) - y(x)|$.

Example 1. ([32], page 719). Examine the two-point boundary-value problem

$$-y'' + \pi^2 y = 2\pi^2 \sin(\pi x)$$

for $0 \leq x \leq 1$, with $y(0) = y(1) = 0$.

We see that $p(x) = 1$, $q(x) = \pi^2$, $f(x) = 2\pi^2 \sin(\pi \mathrm{x})$. The exact solution to the boundary-value problem is $y(x) = \sin \pi x$. Now, we show the following graphics demonstrating the ten-basis logarithm, the absolute value of the difference between the exact and approximate solutions. The figures illustrates $lg\left|\sum_{i=1}^n c_i \phi_i(x) - y(x)\right|$ with respective values of n.

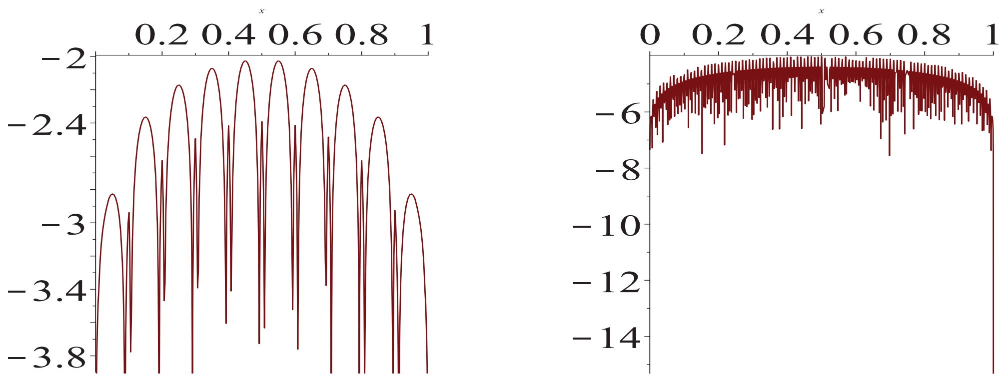

FIGURE 1. $lg|y_h(x)-y(x)|$, for (Left) $n=10$ and (Right) $n=100$

Example 2.(Exercise 1, page 726 in [32]). Use the Ritz method to approximate the solution to the two-point boundary-value problem

$$y'' + \frac{\pi^2}{4}y = \frac{\pi^2}{16}\cos(\frac{\pi}{4}x)$$

for $0 \le x \le 1$, with $y(0) = y(1) = 0$. Compare your outcomes to the true solution $y(x) = -\frac{1}{3}\cos(\frac{\pi}{2}x) - \frac{\sqrt{2}}{6}\sin(\frac{\pi}{2}x) + \frac{1}{3}\cos(\frac{\pi}{4}x)$.

Here we have the following results

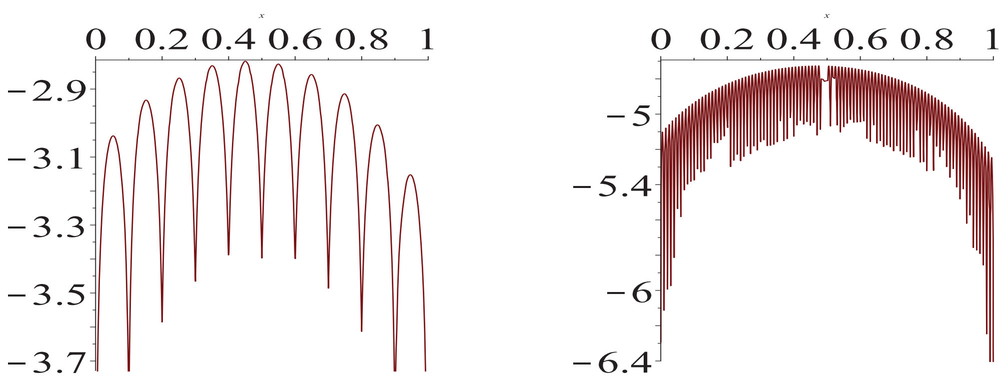

FIGURE 2. $lg|y_h(x)-y(x)|$, for (Left) $n=10$ and (Right) $n=100$

Example 3.(Exercise 2, page 726 in [32]). Use the Ritz method to approximate the solution to the two-point boundary-value problem

$$-\frac{d}{dx}(xy') + 4y = 4x^2 - 8x + 1,$$

for $0 \le x \le 1$, with $y(0) = y(1) = 0$. Compare numerical results to the exact solution $y(x) = x^2 - x$.

We get the following graphics

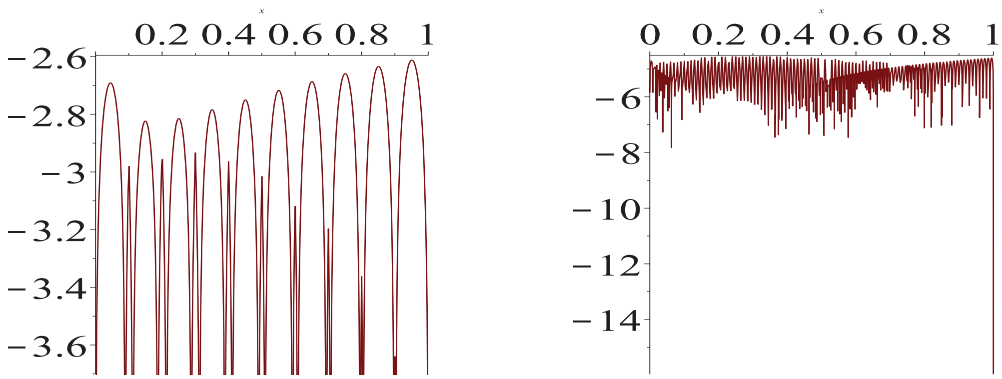

FIGURE 3. $lg|y_h(x) - y(x)|$, for (Left) $n = 10$ and (Right) $n = 100$

Example 4.(Exercise 3b, page 726 in [32]). Use the Ritz method to approximate the solution to the two-point boundary-value problem

$$-\frac{d}{dx}(e^x y') + e^x y = x + (2-x)e^x,$$

for $0 \leq x \leq 1$, with $y(0) = y(1) = 0$. Compare your results to the actual solution $y(x) = (x-1)(e^{-x} - 1)$.

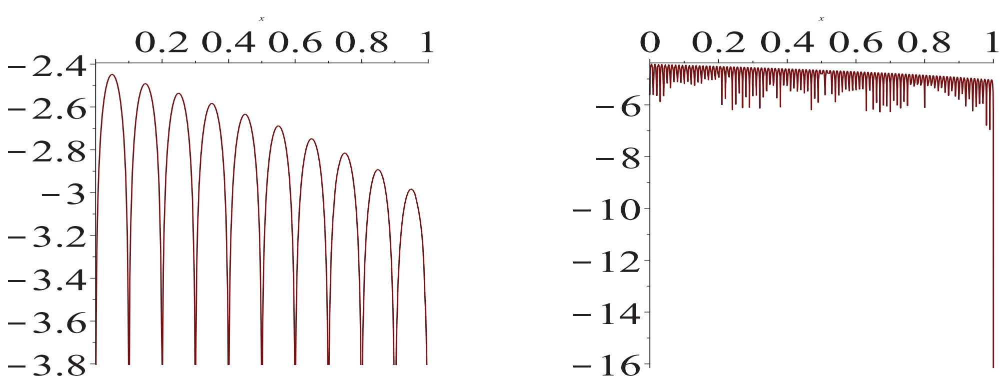

FIGURE 4. $lg|y_h(x) - y(x)|$, for (Left) $n = 10$ and (Right) $n = 100$

So we have used a new basis function in the Ritz method. Moreover, we have applied the coefficients of our optimal interpolation formula in $W_2^{(1,0)}$ space as the basis functions. Finally, we have got new results to approximate solutions of boundary-value problems.

CONCLUSION

Here, the Ritz method approximately solves the boundary-value problems for differential equations. The Ritz method uses a piecewise-linear basis. In this work, we applied coefficients of optimal interpolation formulas in $W_2^{(1,0)}$ space as the basis functions.

Finally, we solved particular examples by this method and compared them with the exact solution numerically. Numerical results show that the coefficients of optimal interpolation formulas in $W_2^{(1,0)}$ space get the same results with a piecewise-linear basis [6]. In our subsequent works, we will construct optimal bases for the approximate solution of boundary value problems by the Ritz or Galerkin method.

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