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Muassis: Buxoro davlat universiteti
Tahririyat manzili: 200117, O‘zbekiston Respublikasi,Buxoro shahri Muhammad Iqbol ko‘chasi, 11-uy.
Elektron manzil: nashriyot_buxdu@buxdu.uz

MUNDARIJA *** СОДЕРЖАНИЕ *** CONTENTS

UO‘K 519.6

CHEKLI ELEMENTLAR USULLARI

Imomova Shafoat Mahmudovna,
Buxoro davlat universiteti
Amaliy matematika va dasturlash texnologiyalari kafedrasi dotsenti
s.m.imomova@buxdu.uz
Amonova Nilufar Avaz qizi,
Buxoro davlat universiteti
Amaliy matematika va dasturlash texnologiyalari kafedrasi o‘qituvchisi
n.a.amonova@buxdu.uz

Annotatsiya. *Oddiy differensial tenglamalar uchun qo‘yilgan chegaraviy masalalarni taqribiy yechishning ko‘plab usullari mavjud. Masalan: Koshi masalasiga ya’ni boshlang‘ich masalaga keltirib yechiladigan usullar(o‘q otish usuli, reduksiya usuli, differensial progonka usuli), chekli ayirmalar usuli, kollokatsiya usuli, kichik kvadratlar usuli, proyeksion usullar (momentlar usuli), balanslar usuli yoki integro-interpolyatsion usul, Fredgolm integral tenglamalariga keltiriladigan usullar va hokazo. Biz ushbu maqolada chegaraviy masalaning taqribiy qiymatini analitik ifoda va diskret shaklda topishga imkon beradigan chekli elementlar usuli bilan tanishamiz.*

Kalit so‘zlar: *oddiy differensial tenglamalar, chegaraviy masala, chekli elementlar usuli, bazis funksiya, Galerkin usuli, Rits usuli, xatolik, to‘r, ortogonal, tenglama, kvadratur formula, chiziqli forma, bichiziqli forma, taqribiy yechim, aniq yechim.*

МЕТОДЫ КОНЕЧНЫХ ЭЛЕМЕНТОВ

Аннотация. *Существует несколько методов приближённого решения краевых задач для обыкновенных дифференциальных уравнений. Например: задача Коши, т.е. методы, которые решаются на основе исходной задачи (метод стрельбы, метод редукции, дифференциальный метод), метод конечных разностей, метод коллокации, метод наименьших квадратов, методы проекции (метод моментов), метод баланса или интегральный метод, метод интерполяции, методы интегральных уравнений Фредгольма и т.д. В этой статье мы познакомимся с методом конечных элементов, который позволяет получить приближённое значение краевой задачи в аналитическом выражении и дискретном виде.*

Key words: *обыкновенные дифференциальные уравнения, краевая задача, метод конечных элементов, базисная функция, метод Галеркина, метод Ритца, погрешность, сетка, ортогональное, уравнение, квадратурная формула, линейная форма, билинейная форма, приближённое решение, точное решение.*

FINITE ELEMENT METHODS

Abstract. *There are several methods of approximate solution of boundary value problems for ordinary differential equations. For example: Cauchy's problem, that is, methods that are solved based on the initial problem (shooting method, reduction method, differential rrogonka method), finite difference method, collocation method, least squares method, projection methods (moment method), balance method or integro-interrolation method, methods of bringing Fredholm integral equations, etc. In this article, we will get acquainted with the finite element method, which allows us to approximate the boundary value problem in an analytical expression and in a discrete form.*

Keywords: *ordinary differential equations, boundary value problem, finite element method, basis function, Galerkin method, Ritz method, error, grid, orthogonal, equation, quadrature formula, linear form, bilinear form, approximate solution, exact solution.*

Kirish. Hozirgi kunda chekli elementlar usuli amaliy masalalarni yechishning eng keng tarqalgan usullaridan biri, masalan, issiqlik jarayonlarini, mustahkamlik hisoblarini, suyuqlik dinamikasi muammolarini o‘rganish. Usulning qo‘llanilishi qulayligi, uning har qanday murakkab shaklli soha uchun ham qo‘llanilishi soddaligi sababli, bu usul amaliyotchi va ayniqsa, muhandislar orasida keng qo‘llanilib kelinmoqda. Bu usul asosida ishlab chiqarish tizimining bir qator hisoblari muvaffaqiyatli hal qilinmoqda. Bu esa chekli elementlar usulining amaliy ahamiyati naqadar yuqori ekanligini bildiradi.

Asosiy qism. *Chekli elementlar usuli orqali masalani yechishning bosqichlari:*

Chekli elementlarni tanlash. Bir o‘lchovli masalada bu to‘g‘ri chiziq kesmasi. Ikki o‘lchovli masalada bu uchburchakli (simpleks) element, to‘g‘ri to‘rtburchakli element, umumiy holda ixtiyoriy shakl bo‘lib, ular yordamida tadqiqot sohasi kesishmaydigan qism sohalarga ajratiladi, bunda element qancha murakkab bo‘lsa integrallarni hisoblashda katta qiyinchiliklar bilan to‘qnashiladi. Shuning uchun eng ko‘p qo‘llaniladiganlari uchburchakli elementlar va tomonlari koordinat o‘qlariga parallel bo‘lgan to‘g‘ri to‘rtburchakli elementlar. Uch o‘lchovli sohalar uchun esa – tetraedr va parallepipeddir.

Sohani chekli elementlarga ajratish (To‘r hosil qilish). Chekli ayirmali usuldan farqli ravishda bo‘linishlarni notekis olish va fazoviy o‘zgaruvchilar gradiyentini aprior hisobga olgan holda bajarish, ya’ni fazoviy o‘zgaruvchilar tez o‘zgarishi kutiladigan sohalarda to‘rlar zich va aksincha. Sohani chekli elementlarga avtomatik bo‘lishning har xil uslublari mavjud, chunki bu bosqichni qo‘lda bajarish mashaqqatli va ko‘pincha xatoliklarga olib keladi.

Bazis funksiyalarni tanlash. Masalani yechishning talab qilingan va hisoblash sistemasining imkoniyatlaridan kelib chiqib, chiziqli, kvadratik yoki yuqoriroq darajadagi bazis funksiyalar tanlanadi, bunda approksimatsiya tugunlari soni eng kami bilan approksimatsiyalovchi funksiyalar tartibidan bir birlikka katta bo‘lishi lozim.

Koeffitsiyentlar matritsasini va yuklanishlar vektorini tuzish.

Chegaraviy shartlarni hisobga olish.

Chekli algebraik tenglamalar sistemasini yechish. Bunda koeffitsiyentlar matritsasining xususiyatlari e`tiborga olinishi mumkin, chunki bu matritsa lenta shaklida bo‘ladi.

1.Galerkin usuli.

Yuqorida berilgan masala yechish bosqichlarini quyidagi masalada qo‘llaymiz:

$$-u''(x) = f(x), \qquad 0 < x < 1, \quad u(0) = 0, \quad u(1) = 0,$$

Bir o‘lchovli bir jinsli chegaraviy qiymat masalasini Galerkin chekli elementlar usulidan foydalanib yechish[1].

Kuchsiz formani hosil qilish. Differensial tenglamaning ikkala tomonini chegaraviy shartlarni $\nu(0) = 0$, $\nu(1) = 0$ qanoatlantiruvchi $\nu(x)$ test funksiyasiga ko‘paytirish orqali *variatsion yoki kuchsiz formulani* tuziladi.

$$-u''v = fv$$

va 0 dan 1 gacha integrallab (bo‘laklab integrallashdan foydalanib) quyidagi tenglik hosil qilinadi:

$$\int_0^1 (-u''v)dx = -u'v\Big|_0^1 + \int_0^1 u'v'dx = \int_0^1 u'v'dx$$

$$\Rightarrow \int_0^1 u'v'dx = \int_0^1 fvdx$$

To‘rni generatsiya qilish, masalan, bir jinsli Dekart to‘ri $x_i = ih$, $i = 0,1,\dots,n$ bunda $h = 1/n$, (x_{i-1}, x_i), $i = 1,2,\dots,n$ intervallarni aniqlash.

To‘rga asoslangan basiz funksiyalar to‘plamini qurish. Eng sodda bazis funksiyalar to‘plami ya’ni bo‘lakli-chiziqli funksiyalar to‘plamidan foydalanish mumkin. ($i = 1,2,\dots n-1$)

$$\varphi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{h}, & agar\ x_{i-1} \le x < x_i, \\ \dfrac{x_{i+1} - x}{h}, & agar\ x_i \le x < x_{i+1}, \\ 0, & boshqa\ hollarda \end{cases}$$

bo‘lakli-chiziqli funksiyalar to‘plami grafigidan kelib chiqib, *shapkacha funksiyalar to‘plami* deb ham yuritiladi.

Taqribiy yechimni bazis funksiyalarning chiziqli kombinatsiyasi bilan ifodalash. Taqribiy yechim quyidagicha, ko‘phadlar yig‘indisi ko‘rinishida ifodanadi:

$$u_h(x) = \sum_{j=1}^{n-1} c_j \varphi_j(x)$$

bu yerda c_j koeffitsientlar aniqlanishi kerak bo‘lgan noma’lumlar. Shapkacha bazis funksiyalarini tanlangani tufayli, $u_h(x)$ ham bo‘lakli-chiziqli funksiya bo‘ladi. $\int_0^1 u'v'dx = \int_0^1 fvdx$ kuchsiz formadagi $u(x)$ aniq yechimni $u_h(x)$ taqribiy yechim bilan almashtirish orqali, koeffitsiyentlar uchun chiziqli tenglamalar sistemasini hosil bo‘ladi ya`ni

$$\int_0^1 u_h'v'dx = \int_0^1 fvdx$$

$$\Rightarrow \int_0^1 \sum_{j=1}^{n-1} c_j\varphi_j'v'dx = \sum_{j=1}^{n-1} c_j \int_0^1 \varphi_j'v'dx = \int_0^1 fvdx.$$

Keyin, $v(x)$ test funksiyani ketma-ket $\varphi_1, \varphi_2, \dots, \varphi_{n-1}$ dek tanlab, chiziqli tenglamalar sistemasi tuziladi (qo‘shimcha xatoliklarga yo‘l qo‘yilganligini ta`kidlab).

$$\begin{gathered}
\left(\int_0^1 \varphi_1'\varphi_1'dx\right)c_1 + \cdots + \left(\int_0^1 \varphi_1'\varphi_{n-1}'dx\right)c_{n-1} = \int_0^1 f\varphi_1 dx \\
\left(\int_0^1 \varphi_2'\varphi_1'dx\right)c_1 + \cdots + \left(\int_0^1 \varphi_2'\varphi_{n-1}'dx\right)c_{n-1} = \int_0^1 f\varphi_2 dx \\
\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\
\left(\int_0^1 \varphi_i'\varphi_1'dx\right)c_1 + \cdots + \left(\int_0^1 \varphi_i'\varphi_{n-1}'dx\right)c_{n-1} = \int_0^1 f\varphi_i dx \\
\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\
\left(\int_0^1 \varphi_{n-1}'\varphi_1'dx\right)c_1 + \cdots + \left(\int_0^1 \varphi_{n-1}'\varphi_{n-1}'dx\right)c_{n-1} = \int_0^1 f\varphi_{n-1} dx
\end{gathered}$$

matritsa-vektor shaklida yozsak:

$$\begin{bmatrix} a(\varphi_1\varphi_1) & a(\varphi_1\varphi_2) & \dots & a(\varphi_1\varphi_{n-1}) \\ a(\varphi_2\varphi_1) & a(\varphi_2\varphi_2) & \dots & a(\varphi_2\varphi_{n-1}) \\ \vdots & \vdots & & \vdots \\ a(\varphi_{n-1}\varphi_1) & a(\varphi_{n-1}\varphi_2) & \dots & a(\varphi_{n-1}\varphi_{n-1}) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} f, \varphi_1 \\ f, \varphi_2 \\ \vdots \\ f, \varphi_{n-1} \end{bmatrix}$$

bunda

$$a(\varphi_i\varphi_j) = \int_0^1 \varphi_i'\varphi_j'dx, \qquad (f, \varphi_i) = \int_0^1 f\varphi_i dx.$$

$a(u, v)$ *bichiziqli forma* deb ataladi, chunki u har bir o‘zgaruvchiga (funksiyaga) chiziqli va (f, v) – *chiziqli forma* deyiladi. Agar φ_i bazis funksiyalar sifatida shapkacha funksiyalar to‘plami tanlangan bo‘lsa, u holda

$$\begin{bmatrix} \frac{2}{h} & -\frac{1}{h} & & & & \\ -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & & \\ & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} \\ & & & & -\frac{1}{h} & \frac{2}{h} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_{n-2} \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} \int_0^1 f\varphi_1 dx. \\ \int_0^1 f\varphi_2 dx. \\ \int_0^1 f\varphi_3 dx. \\ \vdots \\ \int_0^1 f\varphi_{n-2} dx. \\ \int_0^1 f\varphi_{n-1} dx. \end{bmatrix}$$

kelib chiqadi.

Koeffitsientlar uchun chiziqli tenglamalar sistemasini yechish. Hosil qilingan tenglamalar sistemasini yechib, $u_h(x) = \sum_i c_i\varphi_i(x)$ taqribiy yechim hosil qilinadi.

Xatoliklar tahlilini o‘tkazish. Har bir masalada bo‘lgani kabi xatolik darajasi baholanadi. Buni jadval qiymatlari yoki grafiklardan foydalanib amalga oshirish mumkin.

Galerkin usulining soddaroq ko‘rinishini taqdim etamiz. Faraz qilaylik, quyidagi chiziqli chegaraviy masala berilgan bo‘lsin:

$$L[y] = f(x), \tag{1.1}$$

bunda

$$L[y] = y'' + p(x)y' + q(x)y$$

hamda

$$\Gamma_a[y] \equiv \alpha_0 y(a) + \alpha_1 y'(a) = A; \tag{1.2}$$
$$\Gamma_b[y] \equiv \beta_0 y(b) + \beta_1 y'(b) = B$$
$$|\alpha_0| + |\alpha_1| \neq 0\,, \qquad |\beta_0| + |\beta_1| \neq 0$$

chegaraviy shartlar mavjud.

Biror bir to‘la sistemaning qismidan iborat bo‘lgan $\{u_i(x)\}$ $(i = 0, 1, \ldots, n)$ bazis funksiyalarning chekli sistemasini tanlanadi, bunda $u_0(x)$ funksiya

$$\Gamma_a[u_0] = A, \qquad \Gamma_b[u_0] = B$$

bir jinsli bo‘lmagan chegaraviy shartlarni qanoatlantiradigan bo‘lishi va $u_i(x)$ $(i = 1, 2, \ldots, n)$ funksiyalar

$$\Gamma_a[u_i] = \Gamma_b[u_i] = 0 \qquad (i = 1,2, \ldots, n)$$

bir jinsli chegaraviy shartlarini qanoatlantiradigan bo‘lishi kerak. (1.1)-(2.2) chegaraviy masalaning yechimini

$$y(x) = u_0(x) + \sum_{i=1}^{n} C_i u_i(x) \tag{1.3}$$

ko‘rinishda izlanadi.

Tanlangan $u_i(x)$ bazis funksiyalarda (1.3) formula bilan aniqlangan y funksiya, ko‘rinib turibdiki, C_i koeffitsiyentlarning ixtiyoriy tanlanishida (1.2) chegaraviy shartlarni qanoatlantiradi. (1.3) ifodani (1.1) tenglamaga qo‘yilganda, bu quyidagi xatolikni beradi:

$$R(x, C_1, C_2, \ldots, C_n) = \mathrm{L}[u_0] + \sum_{i=1}^{n} C_i L[u_i] - f(x).$$

Chegaraviy masalaning aniq yechimi y da $R \equiv 0$ bo‘ladi; shuning uchun aniq yechimga yaqin taqribiy yechimni olish uchun, C_i koeffitsiyentlarni R funksiya qandaydir ma`noda kichik bo‘ladigan qilib tanlash o‘rinli. Galerkin usuliga asosan $R$ xatolik $u_i(x)$ $(i = 1, 2, \ldots, n)$ bazis funksiyalarga ortogonal bo‘lishini talab qilinadi, bunday funksiyalar sonining yetarlicha kattaligi xatolikni o‘rtacha ma`noda kichikligini ta`minlaydi. Bu holatda taqribiy yechimning aniq yechimga qanchalik yaqinligi umumiy holda ochiq qolayotgan muammolardan.

Shunday qilib, $C_i(i = 1, 2, \ldots, n)$ koeffitsiyentlarni topish uchun

$$\left.\begin{aligned} &\int_a^b u_1(x) R(x, C_1, C_2, \ldots, C_n)dx = 0, \\ &\int_a^b u_2(x) R(x, C_1, C_2, \ldots, C_n)dx = 0, \\ &\qquad\vdots \\ &\int_a^b u_n(x) R(x, C_1, C_2, \ldots, C_n)dx = 0 \end{aligned}\right\}$$

chiziqli tenglamalar sistemasiga kelinadi, yoki, yanada to‘laroq,

$$\sum_{i=1}^{n} C_i \int_a^b u_i(x) L[u_i]dx = \int_a^b u_i(x)\{f(x) - L[u_0]\}dx$$
$$i = 1,2, \ldots, n.$$

2.Rits usuli

Rits usulini tavsiflash uchun quyidagi chiziqli chegaraviy masala yechimini ko‘rib chiqamiz:

$$-\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right)+q(x)y=f(x),\quad 0\leq x\leq 1,\qquad (2.1)$$

chegaraviy shartlar

$$y(0)=y(1)=0.$$

Faraz qilamiz, $p(x)\in C^1[0,1]$ va $(x), f(x)\in C[0,1]$, $\delta > 0$ — o‘zgarmas son, $p(x)\geq\delta,\ q(x)\geq 0,\ 0\leq x\leq 1$ bu shartlar berilgan chegaraviy masala yagona yechimga ega bo‘lishi uchun zarur va yetarli.

Fizik hodisalarni tavsiflovchi ko‘plab chegaraviy masalalarda bo‘lgani kabi, nur tenglamasining yechimi integral minimallashtirish variatsion xossasini qanoatlantiradi. Nur tenglamasining variatsion prinsipi Rits usulini ishlab chiqishda asos bo‘lib, nur tenglamasining yechimini $C_0^2[0,1]$ dagi barcha funksiyalar bo‘yicha integralni minimallashtiradigan funksiya sifatida tavsiflaydigan u funksiyalar to‘plami $C^2[0,1]$ da $u(0) = u(1) = 0$ shartlarni qanoatlantirsin.

y(x) ∈ $C^2[0,1]$ funksiya

$$-\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right)+q(x)y=f(x),\ \ 0\leq x\leq 1,$$

differensial tenglamaning yagona yechimidir, agar $y(x)$ funksiya $C^2[0,1]$ da

$$I[u]=\int_0^1\{p(x)[u'(x)]^2+q(x)\,[u(x)]^2-2f(x)u(x)\}dx\qquad (2.2)$$

integralni minimallashtiruvchi yagona funksiyasi bo‘lsa.

Rits usuli integralni $C_0^2[0,1]$ dagi barcha funksiyalar bo‘yicha emas, balki $\varphi_1,\varphi_2,\dots,\varphi_n$ ma`lum bazis funksiyalarining chiziqli birikmalaridan tashkil topgan kichikroq funksiyalar to‘plami bo‘yicha minimallashtirish yo‘li bilan yechimni y ga yaqinlashtiradi. Bazis funksiyalari chiziqli, bog‘liqsiz va

$$\varphi_i(0)=\varphi_i(1)=0,\quad i=1,2,\dots,n$$

shartni qanoatlantiradi.

Bazis funksiyalar birikmasini $\varphi(x)=\sum_{i=1}^{n}c_i\varphi_i(x)$ ko‘rinishda ifodalab, (2.2) integralni minimallashtirish uchun aniq yechim $y(x)$ taqribiy yechim bilan almashtiriladi.

$$I[\varphi]=I\left[\sum_{j=1}^{n-1}c_i\varphi_i\right]=$$

$$=\int_0^1\left\{p(x)\left[\sum_{i=1}^{n}c_i\varphi_i'(x)\right]^2+q(x)\left[\sum_{i=1}^{n}c_i\varphi_i(x)\right]-2f(x)\sum_{i=1}^{n}c_i\varphi_i(x)^2\right\}dx,\quad (2.3)$$

bu yerda $c_1,c_2,\dots,c_n$ koeffitsiyentlar aniqlanishi kerak bo‘lgan noma’lumlardir.

I integral minimal bo‘lishi uchun, uni $c_1,c_2,\dots,c_n$ o‘zgarmaslarning funksiyasi sifatida ko‘rib chiqish zarur.

$$\frac{\partial I}{\partial c_j}=0,\qquad j=1,2,\dots,n.\qquad (2.4)$$

(2.3) ni differensiallaymiz

$$\frac{\partial I}{\partial c_j}=\int_0^1\left\{2p(x)\sum_{i=1}^{n}c_i\varphi_i'(x)\varphi_j'(x)+2q(x)\sum_{i=1}^{n}c_i\varphi_i(x)\varphi_j(x)-2f(x)\varphi_j(x)\right\}dx,$$

Natijani (2.4)ga qo‘yamiz

$$0=\sum_{i=1}^{n}\left[\int_0^1\left\{p(x)\varphi_i'(x)\varphi_j'(x)+q(x)\varphi_i(x)\varphi_j(x)\right\}dx\right]c_i-\int_0^1 f(x)\varphi_j(x)dx,\ \ (2.5)$$

(2.5) tenglama *normal tenglamadir* va undan $c_1,c_2,\dots,c_n$ o‘zgaruvchili $n\times n$ o‘lchamli chiziqli sistema $Ac = b$ hosil bo‘ladi. Bu yerda A simmetrik matritsaga ega.

Quyidagicha belgilash kiritamiz:

$$a_{ij}=\int_0^1\left[p(x)\varphi'_i(x)\varphi'_j(x)+q(x)\varphi_i(x)\varphi_j(x)\right]dx$$

$$b_i=\int_0^1 f(x)\varphi_i(x)dx$$

Bazis funksiyalar sifatida bo‘lakli-chiziqli funksiyalar to‘plamini tanlaymiz. Ular haqida keyinroq batafsil ma`lumot beramiz. Bo‘lakli-chiziqli funksiyalar to‘plamining umumiy ko‘rinishi quyidagicha:

$$\varphi_i(x)=\begin{cases}0, & agar\ \ 0\le x\le x_{i-1}\\ \dfrac{1}{h_{i-1}}(x-x_{i-1}), & agar\ \ x_{i-1}<x\le x_i\\ \dfrac{1}{h_i}(x_{i+1}-x), & agar\ \ x_i<x\le x_{i+1}\\ 0, & agar\ \ x_{i+1}<x\le 1\end{cases}\qquad (2.6)$$

har bir $i=1,2,\dots,n$ uchun $\varphi_i(x)$ funksiyaning hosilasi mavjud:

$$\varphi'_i(x)=\begin{cases}0, & agar\ \ 0\le x\le x_{i-1}\\ \dfrac{1}{h_{i-1}}, & agar\ \ x_{i-1}<x\le x_i\\ -\dfrac{1}{h_i}, & agar\ \ x_i<x\le x_{i+1}\\ 0, & agar\ \ x_{i+1}<x\le 1\end{cases}\qquad (2.7)$$

Tanlangan bazis funksiya $\varphi_i(x)$ va uning hosilasi $\varphi'_i(x)$ faqat (x_{i-1},x_{i+1}) oraliqda noldan farqli, qolgan barcha hollarda nolga teng. Faqatgina $j=i-1,\ i,i+1$ bo‘lgan holdan tashqari barcha j lar uchun quyidagi tengliklar o‘rinli bo‘ladi:

$$\varphi_i(x)\varphi_j(x)\equiv 0 \qquad va \qquad \varphi'_i(x)\varphi'_j(x)\equiv 0$$

Natijada (2.6)dan hosil bo‘ladigan chiziqli tenglamalar sistemasi 3x3 uch diagonalli sistemaga o‘tadi. A matritsadagi dagi nolga teng bo‘lmagan ifodalar:

$$a_{ii}=\int_0^1\left\{p(x)\left[\varphi'_i(x)\right]^2+q(x)[\varphi_i(x)]^2\right\}dx=\left(\frac{1}{h_{i-1}}\right)^2\int_{x_i-1}^{x_i}p(x)dx$$

$$+\left(\frac{-1}{h_i}\right)^2\int_{x_i}^{x_{i+1}}p(x)dx+\left(\frac{1}{h_{i-1}}\right)^2\int_{x_i-1}^{x_i}(x-x_{i-1})^2q(x)dx+\left(\frac{1}{h_i}\right)^2\int_{x_i}^{x_{i+1}}(x_{i+1}-x)^2q(x)dx,$$

$$a_{i,i+1}=\int_0^1\{p(x)\varphi'_i(x)\varphi'_{i+1}(x)+q(x)\varphi_i(x)\varphi_{i+1}(x)\}dx$$

$$=-\left(\frac{1}{h_i}\right)^2\int_{x_i}^{x_{i+1}}p(x)dx+\left(\frac{1}{h_i}\right)^2\int_{x_i}^{x_{i+1}}(x_{i+1}-x)(x-x_i)q(x)dx,$$

$$a_{i,i-1}=\int_0^1\{p(x)\varphi'_i(x)\varphi'_{i-1}(x)+q(x)\varphi_i(x)\varphi_{i-1}(x)\}dx$$

$$=-\left(\frac{1}{h_{i-1}}\right)^2\int_{x_i-1}^{x_i}p(x)dx+\left(\frac{1}{h_{i-1}}\right)^2\int_{x_{i-1}}^{x_i}(x_i-x)(x-x_{i-1})q(x)dx,$$

$$b_i=\int_0^1 f(x)\varphi_i(x)dx$$

$$=\frac{1}{h_{i-1}}\int_{x_{i-1}}^{x_i}(x-x_{i-1})f(x)dx+\frac{1}{h_i}\int_{x_i}^{x_{i+1}}(x_{i+1}-x)f(x)dx,$$

Bu yerda 6 xil integralni hisoblash talab etiladi:

$$Q_{1,i}=\left(\frac{1}{h_i}\right)^2\int_{x_i}^{x_{i+1}}(x_{i+1}-x)(x-x_i)q(x)dx,\quad i=1,2,\dots,n-1,$$

$$Q_{2,i}=\left(\frac{1}{h_{i-1}}\right)^2\int_{x_{i-1}}^{x_i}(x-x_{i-1})^2q(x)dx,\quad i=1,2,\dots,n,$$

$$Q_{3,i}=\left(\frac{1}{h_i}\right)^2\int_{x_i}^{x_{i+1}}(x_{i+1}-x)^2q(x)dx,\quad i=1,2,\dots,n,$$

$Q_{4,i}=\left(\frac{1}{h_{i-1}}\right)^2\int_{x_i-1}^{x_i}p(x)dx,\quad i=1,2,\dots,n+1,$

$Q_{5,i}=\left(\frac{1}{h_{i-1}}\right)^2\int_{x_i-1}^{x_i}(x-x_{i-1})f(x)dx,\quad i=1,2,\dots,n,$

$Q_{6,i}=\frac{1}{h_i}\int_{x_i}^{x_{i+1}}(x_{i+1}-x)f(x)dx,\quad i=1,2,\dots,n,$

Ac = b chiziqli sistemasidagi A matritsasi va b vektori quyidagicha:

$a_{i.i}=Q_{4,i}+Q_{4,i+1}+Q_{2,i}+Q_{3,i},\qquad i=1,2,\dots,n,$

$a_{i.i+1}=-Q_{4,i+1}+Q_{1,i},\qquad i=1,2,\dots,n-1,$

$a_{i.i-1}=-Q_{4,i}+Q_{1,i-1},\qquad i=1,2,\dots,n,$

$b_i=Q_{5,i}+Q_{6,i},\qquad i=1,2,\dots,n.$

c lar noma'lum $c_1,c_2,\dots,c_n$ koeffitsientlar, ular orqali $\varphi(x)=\sum_{i=1}^{n}c_i\varphi_i(x)$ bilan berilgan Rits yaqinlashuvi tuziladi.

Ushbu usuldan foydalanish uchun to‘g‘ridan-to‘g‘ri yoki kvadratur formulasi bilan 6n ta integralni hisoblash kerak. Integral hisoblashning muqobil yondashuvi $p(x)$, $q(x)$ va $f(x)$ funksiyalarning har birini bo‘lak-chiziqli interpolyatsiya qiluvchi ko‘phad bilan yaqinlashtirish va keyin yaqinlashuvni integrallashdir. Masalan, $Q_{1,i}$ integralini ko‘rib chiqaylik. $q(x)$ ning bo‘lakli-chiziqli interpolyatsiyasi

$$P_q(x)=\sum_{i=0}^{n+1}q(x_i)\varphi_i(x),$$

bu yerda $\varphi_1,\dots,\varphi_n$ (2.6) da aniqlangan va

$$\varphi_0(x)=\begin{cases}\frac{x_1-x}{x_1}, & agar\ 0\le x\le x_i\\ 0, & boshqa\ hollarda\end{cases}$$

$$\varphi_{n+1}(x)=\begin{cases}\frac{x-x_n}{1-x_n}, & agar\ x_n\le x\le 1\\ 0, & boshqa\ hollarda.\end{cases}$$

Integrallash oralig‘i $[x_i,x_{i+1}]$ ga teng, shuning uchun $P_q(x)$ bo‘lakli ko‘phadga kamayadi.

$$P_q(x)=q(x_i)\varphi_i(x)+q(x_{i+1})\varphi_{i+!}(x)$$

$$\begin{aligned}Q_{1,i}&=\left(\frac{1}{h_i}\right)^2\int_{x_i}^{x_{i+1}}(x_{i+1}-x)(x-x_i)q(x)dx\\&\approx\left(\frac{1}{h_i}\right)^2\int_{x_i}^{x_{i+1}}(x_{i+1}-x)(x-x_i)\left[\frac{q(x_i)(x_{i+1}-x)}{h_i}+\frac{q(x_{i+1})(x-x_i)}{h_i}\right]dx\\&=\frac{h_i}{12}[q(x_i)+(x_{i+1})].\end{aligned}$$

Bundan tashqari, agar $q(x)\in C^2[x_i,x_{i+1}]$ bo‘lsa, u holda

$$\left|Q_{1,i}-\frac{h_i}{12}[q(x_i)+(x_{i+1})]\right|=O(h_i^3).$$

Boshqa integrallarga yaqinlashishlar ham xuddi shunday tarzda chiqariladi:

$$Q_{2,i}\approx\frac{h_{i-1}}{12}[3q(x_i)+q(x_{i-1})],\qquad Q_{3,i}\approx\frac{h_i}{12}[3q(x_i)+q(x_{i+1})],$$

$$Q_{4,i}\approx\frac{h_{i-1}}{2}[p(x_i)+p(x_{i-1})],\qquad Q_{5,i}\approx\frac{h_{i-1}}{6}[2f(x_i)+f(x_{i-1})]$$

$$Q_{6,i}\approx\frac{h_i}{6}[2f(x_i)+f(x_{i+1})].$$

Natijalar.

1-masala:

$$y''+xy'+y=2x$$

differensial tenglamaning

$$y(0)=1,\qquad y(1)=0$$

chegaraviy shartlarni qanoatlantiruvchi taqribiy yechimini Galerkin usulidan foydalanib toping.

Yechish:

$u_i(x)$ $(i = 0, 1, 2, 3)$ bazis funksiyalar sistemasi sifatida quyidagi funksiyalarni tanlaymiz:

$$u_0(x) = 1 - x, \qquad u_1(x) = x(1-x),$$
$$u_3(x) = x^2(1-x), \qquad u_4(x) = x^3(1-x),$$

Masalaning taqribiy yechimini quyidagi ko‘phad ko‘rinishida izlaymiz

$$y = (1-x) + C_1x(1-x) + C_2x^2(1-x) + C_3x^3(1-x).$$

y ni berilgan differensial tenglamaning chap tomoniga qo‘yib

$$R(x, C_1, C_2, C_3) = (1-4x) + C_1(-2+2x+3x^2)$$
$$+C_2(2-6x+3x^2+4x^3) + C_3(6x-12x^2+4x^3-5x^4)$$

xatolikni olamiz.

R funksiyaning $u_1(x), u_2(x), u_3(x)$ funksiyalarga ortogonallik shartlari quyidagi sistemaga olib keladi:

$$\left.\begin{array}{l} \int_0^1 (x-x^2)R(x, C_1, C_2, C_3)dx = 0\,, \\ \int_0^1 (x^2-x^3)R(x, C_1, C_2, C_3)dx = 0\,, \\ \int_0^1 (x^3-x^4)R(x, C_1, C_2, C_3)dx = 0\,, \end{array}\right\}$$

$R(x)$ ning o‘rniga uning qiymatini qo‘yib, mos integrallashdan keyin

$$\left.\begin{array}{r} 133C_1 + 63C_2 + 36C_3 = -70, \\ 140C_1 + 108C_2 + 79C_3 = -98 \\ 264C_1 + 252C_2 + 211C_3 = -210 \end{array}\right\}$$

sistemani olamiz. Bu yerdan $C_1 = -0.2090;$ $C_2 = -0.7894;$ $C_3 = 0.2090$ ni topamiz va demak,

$$\mathrm{y} = (1 - \mathrm{x})(1 - 0.2090\mathrm{x} - 0.7894x^2 + 0.2090x^3).$$

2-masala: Quyidagi chegaraviy masalani Ritz usuli asosida ko‘rib chiqamiz:

$$-y'' + \pi^2 y = 2\pi^2\sin(\pi x), \quad 0 \le x \le 1$$
$$y(0) = y(1) = 0$$

$h_i = h = 0.1$ deb oladigan bo‘lsak, har bir $i = 0,1, \dots, 9$ uchun $x_i = 0.1i$ o‘rinli bo‘ladi.

Mos qiymatlarni qo‘yib integrallarni hisoblaymiz:

$$Q_{1,i} = 100\int_{0.1i}^{0.1i+0.1} (0.1i + 0.1 - x)(x - 0.1i)\pi^2 dx = \frac{\pi^2}{60},$$

$$Q_{2,i} = 100\int_{0.1i-0.1}^{0.1i} (x - 0.1i + 0.1)^2\pi^2 dx = \frac{\pi^2}{30},$$

$$Q_{3,i} = 100\int_{0.1i}^{0.1i+0.1} (0.1i + 0.1 - x)^2\pi^2 dx = \frac{\pi^2}{30},$$

$$Q_{4,i} = 100\int_{0.1i-0.1}^{0.1i} dx = 10,$$

$$Q_{5,i} = 10\int_{0.1i-0.1}^{0.1i} (x - 0.1i + 0.1)2\pi^2 \sin\pi x\, dx = -2\pi\cos 0.1\pi i + 20[\sin(0.1\pi i) - \sin((0.1i - 0.1)\pi)]\,,$$

$$Q_{6,i} = 10\int_{0.1i}^{0.1i+0.1} (0.1i + 0.1 - x)2\pi^2 \sin\pi x dx =$$
$$= 2\pi\cos 0.1\pi i - 20\big[\sin\big((0.1i + 0.1)\pi\big) - \sin(0.1\pi i)\big].$$

Ac = b chiziqli sistema

$$a_{i,i} = 20 + \frac{\pi^2}{15}, \qquad i = 1,2,\dots,9,$$

$$a_{i,i+1} = -10 + \frac{\pi^2}{60}, \qquad i = 1,2,\dots,8,$$

$$a_{i,i-1} = -10 + \frac{\pi^2}{60}, \qquad i = 2,3,\dots,9,$$

va

$$b_i = 40\sin(0.1\pi i)[1 - \cos 0.1\pi], \qquad i = 1,2,\dots,9.$$

Uch diagonalli chiziqli tenglamalar sistemasining yechimi:

$$c_9 = 0.3102866742, \qquad c_8 = 0.5902003271, \qquad c_7 = 0.8123410598,$$
$$c_6 = 0.9549641896, \qquad c_5 = 1.0041087710, \qquad c_4 = 0.9549641893,$$
$$c_3 = 0.8123410598, \qquad c_2 = 0.5902003271, \qquad c_1 = 0.3102866742,$$

Berilgan chegaraviy masalaning taqribiy yechimi $\varphi(x) = \sum_{i=1}^{9} c_i\,\varphi_i(x)$

va haqiqiy yechimi $y(x) = \sin\pi x$. 1-jadvalda har bir $i = 1,2,\dots 9$ dagi x_i lar uchun taqribiy yechim, aniq yechim va ular orasidagi xatolik qiymatlari keltirilgan.

1-jadval.

Taqribiy yechim, aniq yechim va ular orasidagi xatolik qiymatlari.

	x_i	$\varphi(x_i)$	$y(x_i)$	$\lvert\varphi(x_i) - y(x_i)\rvert$
1	0.1	0.3102866742	0.3090169943	0.00127
2	0.2	0.5902003271	0.5877852522	0.00241
3	0.3	0.8123410598	0.8090169943	0.00332
4	0.4	0.9549641896	0.9510565162	0.00390
5	0.5	1.0041087710	1.0000000000	0.00411
6	0.6	0.9549641893	0.9510565162	0.00390
7	0.7	0.8123410598	0.8090169943	0.00332
8	0.8	0.5902003271	0.5877852522	0.00241
9	0.9	0.3102866742	0.3090169943	0.00127

Xulosa. Ko‘p masalalar uchun Rits va Galerkin usullari nazariy jihatdan ekvivalentdir. Rits usuli eng qadimgi usullardan biri bo‘lib, muvaffaqiyatli usullardan biri ekanligi o‘z isbotini topgan. Rits usuli minimallashtirish formasi va optimallashtirishga asoslangan texnikadan foydalanib, masalani yechishi mumkin. Galerkin usuli odatda Rits usuliga qaraganda kuchsizroq talablarga ega. Har bir masalaning minimallashtiruvchi formasi mavjud emas, holbuki, deyarli barcha masalalar qandaydir kuchsiz shaklga ega. Tegishli kuchsiz formani qanday tanlash va har xil usullarning yaqinlashuvi chekli elementlar usullari uchun muhim masaladir.

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