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Inverse Problem for an Integro-Differential Wave Equation in a Cylindrical Domain

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Abstract—In this paper, for an integro-differential wave equation in a cylindrical domain it is studied an inverse problem of searching the unknown kernel in the integral term. By the method of separation of variables, the problem is reduced to determine the same kernel from ordinary differential equations with respect to coefficients of Fourier-Bessel series of the solution of the direct problem. Orthonormal Bessel functions of the first kind of zero order is used. An additional information obtained in the form of Volterra integral equation of the second is used. It is proved the global unique solvability of the inverse problem by the method of contraction mappings in the space of continuous functions with weighted norms.

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1. INTRODUCTION

The theory and application of integro-differential equations with convolution integral play an important role in the mathematical models of physical, biological, engineering systems and in other areas, where it is necessary to take into account the prehistory of processes. Constitutive relations in the linear- inhomogeneous diffusion and wave propagation processes with memory contain time- and space-dependent memory kernel. Often in practice these kernels are unknown functions. Problems of identification for memory kernels in parabolic and hyperbolic integro-differential equations have been intensively studied since the end of the eighties of the last century [1—3].

Various formulations and methods of studying inverse problems for partial differential and integro-differential equations can be founded in the works [4—12]. We note the works [13—21], which close to considered problem in present work. In particular, in these works the one-dimensional problems of finding a kernel in a hyperbolic integro-differential equation with a delta function [13—21]. In works [22—24], the memory identification problems were solved for Maxwell and viscoelasticity integro-differential equations. In [25, 26], for multidimensional inverse problems of finding the kernel in hyperbolic integro-differential equations of the second order are proved a unique local solvability theorems in the class of analytic functions. In the work [27], it is proved that a space- and time-dependent kernel occurring in a hyperbolic integro-differential equation in three space dimensions can be uniquely reconstructed from the restriction of the Dirichlet-to-Neumann operator of the equation.

Note that the interest on studying of integro-differential equations of the convolution type also increased thanks to their discovered connection with fractional-order equations. In the works [28, 29], it was noticed that if the kernel in the equations is taken as the Mittag-Leffler function type [30], then the equations are equivalent to fractional differential equations.

In present paper, we study an inverse problem of determining kernel in the integral term for the inhomogeneous integro-differential equations in a cylindrical domain. The method of separation of variables is applied.

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2. STATEMENT OF PROBLEM AND PRELIMINARIES

Consider in a cylinder $G := \{(x,y,t) :\ 0 < r < 1,\ 0 < t < T\}$, $r = \sqrt{x^2+y^2}$, the integro-differential equation

$$u_{tt} - \Delta u = \int\limits_0^t k(\alpha) u(x,y,t-\alpha)\, d\alpha + f(r,t) \tag{1}$$

with initial

$$u|_{t=0} = \varphi(r), \quad u_t|_{t=0} = 0, \quad 0 \le x^2 + y^2 \le 1 \tag{2}$$

and boundary conditions

$$((x,y), \nabla u)\,|_{r=0} = 0, \quad u|_{r=1} = 0, \quad 0 \le t \le T, \tag{3}$$

where $\triangle$ is the Laplace operator in variables x and y, $((x,y),\nabla u)$ is scalar product of vectors (x,y) and ∇u, $f(r,t)$, $\varphi(r)$ are given sufficiently smooth functions.

The problem of finding the function $u(x,y,t)$ from relations (1)–(3) with a known kernel $k(t)$ will be called a direct problem. The inverse problem is to determine the unknown kernel $k(t)$, $t > 0$ in the equation (1), according to an additional condition

$$u(x_0,y_0,t) = h(t), \quad 0 < x_0^2 + y_0^2 < 1, \quad 0 \le t \le T, \tag{4}$$

where $h(t)$ is given sufficiently smooth function.

Since the source function in equation (1) and the initial condition in (2) depend on the distance r, then $u(x,y,t) = u(r,t)$, i.e. we have an axisymmetric case. Then, the Laplace operator of the function $u(x,y,t)$ in polar coordinate system will not depend on the angle and has the form

$$\triangle u(x,y,t) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}.$$

Therefore, the problem (1)–(3) in this coordinate system can be written as follows

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{r}\frac{\partial u}{\partial r} - \frac{\partial^2 u}{\partial r^2} = \int\limits_0^t k(\alpha) u(r,t-\alpha)\, d\alpha + f(r,t), \quad (r,t) \in G, \tag{5}$$

$$u|_{t=0} = \varphi(r), \quad u_t|_{t=0} = 0, \quad 0 \le r \le 1, \tag{6}$$

$$\lim_{r\to 0}(r u_r) = 0, \quad u|_{r=1} = 0, \quad 0 \le t \le T. \tag{7}$$

Condition (4) takes the following form

$$u(r_0,t) = h(t), \quad r_0 = \sqrt{x_0^2 + y_0^2}, \quad 0 < r_0 < 1, \quad 0 \le t \le T. \tag{8}$$

Thus, the inverse problem (1)–(4) is reduced to the problem (5)–(8) of redefinition for the unknown functions $u(r,t)$, $k(t)$ from equalities. We study the properties of Bessel function and find conditions for the convergence of the Fourier–Bessel series. We recall that the linear Bessel differential equation (or equation of cylindrical functions) with parameter λ of index $\nu \ge 0$ with respect to the function z of the real variable x has the form [31]

$$z'' + \frac{1}{x} z' + \left(\lambda^2 - \frac{\nu^2}{x^2}\right) z = 0. \tag{9}$$

Solution of the equation (9), except for very particular values ν, not expressed in terms of elementary functions (in the final form) and leads to the so-called Bessel functions, which have large applications in the natural sciences [31]. When, ν is an integer number, then equation (9) has the following solution: $z(x) = C_1 J_\nu(\lambda x) + C_2 Y_\nu(\lambda x)$, where J_ν and Y_ν are the Bessel functions of the first and second kind of order ν, respectively. Bessel functions of the second kind are not bounded near the point $x = 0$. So, for a boundness of solution near zero it is necessary to be $C_2 = 0$, i.e., solution (9) has the following form: $z(x) = CJ_\nu(\lambda x)$.

Moreover, if the boundary condition $z(1) = 0$ is imposed, then the parameter λ must satisfy the condition: $J_\nu(\lambda) = 0$, i.e., the values of λ are zeros of the Bessel function $J_\nu(x)$, which has the following asymptotic representation [31]:

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\nu\pi}{2} + \frac{\pi}{4}\right) + \frac{r_\nu(x)}{x\sqrt{x}},$$

where the function $r_\nu(x)$ is bounded as $x \to \infty$. Therefore, for any large k, the zeros of $J_\nu(x)$ are given by the expression [31]: $k\pi + \dfrac{\nu\pi}{2} - \frac{\pi}{4}$.

We define the Fourier–Bessel expansion of the given function $g(x)$ as follows: for any function $g(x)$, absolutely integrable on $[0, 1]$, one can compose a Fourier series in the system $J_\nu(\lambda_k x), k = 1, 2, \ldots$

$$g(x) = \sum_{k=1}^{\infty} c_k J_\nu(\lambda_k x), \tag{10}$$

where $c_k = \frac{2}{J^2_{\nu+1}(\lambda_k)} \int\limits_0^1 x g(x) J_\nu(\lambda_k x) dx,\ k = 1, 2, \ldots$ are Fourier–Bessel coefficients.

Let us give without proof the most important criteria for the convergence of the Fourier–Bessel series to the desired function.

Theorem 1. ([31, p. 282]). *If $\nu \geq 0$ and for all sufficiently large k there holds the estimate*

$$|c_k| \leq \frac{M}{\lambda_k^{1+\varepsilon}},$$

then series (10) converges absolutely and uniformly on $[0, 1]$, where $\varepsilon > 0$ and $M > 0$ are constants.

Theorem 2. ([31, pp. 289–291]). *Let the function $g(x)$ is defined and $2s$ times continuously differentiable on the interval $[0, 1]$ $(s \geq 1)$ and*

1. $g(0) = g'(0) = \ldots = g^{(2s-1)}(0) = 0,$
2. $g^{(2s)}(x)$ *is bounded (this derivative may not exist at some points),*
3. $g(1) = g'(1) = \ldots = g^{(2s-2)}(1) = 0.$

Then, for the Fourier–Bessel coefficients of the function $g(x)$, the inequality is true

$$|c_k| \leq \frac{M}{\lambda_k^{2s-(1/2)}} \quad (M = \text{const}). \tag{11}$$

According to the Fourier method, we present function $u(r, t)$ in form

$$u(r, t) = \Re(r) T(t). \tag{12}$$

Substituting (12) into the equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{r}\frac{\partial u}{\partial r} - \frac{\partial^2 u}{\partial r^2} = 0,$$

we get

$$T''(t)\Re(r) = \frac{1}{r} T(t)\Re'(r) + T(t)\Re''(r).$$

Hence, separating the variables to find the function $\Re(r)$, we obtain the problem

$$\Re''(r) + \frac{1}{r}\Re'(r) + \lambda^2 \Re(r) = 0, \tag{13}$$

$$\lim_{r\to 0}\left(r\Re'(r)\right) = 0, \quad \Re(r)|_{r=1} = 0, \tag{14}$$

which is a self-adjoint problem, where λ is an arbitrary real parameter. The solutions of equation (13) are the following zero-order Bessel functions of the first kind $R_k(r) = J_0(\lambda_k r),\ k = 1, 2, 3, \ldots$ They also are eigenfunctions. We find the eigenvalues, using the second boundary condition (14) (the validity of the first boundary condition in (14) is obvious). Positive roots of the equation $J_0(\lambda_k) = 0$ look like $\lambda_k = (4k-1)\frac{\pi}{4}$.

3. INVESTIGATION OF DIRECT PROBLEM

When, we studied the direct problem, we assumed that the function $k(t)$ is also known. Let $C^{m,n}(G)$ be the class of m times continuously differentiable with respect to r and n times continuously differentiable in t functions in the domain G.

Theorem 3. *Assume that* $k(t) \in C[0,T],\ f(r,t) \in C^{4,0}(\overline{G}),\ \varphi(r) \in C^4[0,1]$ *and the equalities*

$$\varphi^{(m)}(0) = 0, \quad m = \overline{0,3}, \quad \varphi^{(m)}(1) = 0, \quad m = \overline{0,2},$$

$$\frac{\partial^m f}{\partial r^m}(0,t) = 0, \quad m = \overline{0,3}, \quad \frac{\partial^m f}{\partial r^m}(1,t) = 0, \quad m = \overline{0,2}, \quad t \in [0,T]$$

are satisfied. Then, there is a unique classical solution to direct problem (1)–(3).

Proof. We expand all functions in (5)–(7) in a Fourier–Bessel series with respect to eigenfunctions $J_0(\lambda_k r)$ i.e.,

$$u(r,t) = \sum_{n=1}^{\infty} u_n(t) J_0(\lambda_n r), \quad u_n(t) = \frac{2}{J_1^2(\lambda_n)} \int_0^1 r u(r,t) J_0(\lambda_n r) dr, \tag{15}$$

$$f(r,t) = \sum_{n=1}^{\infty} f_n(t) J_0(\lambda_n r), \quad f_n(t) = \frac{2}{J_1^2(\lambda_n)} \int_0^1 r f(r,t) J_0(\lambda_n r) dr, \tag{16}$$

$$\varphi(r) = \sum_{n=1}^{\infty} \varphi_n J_0(\lambda_n r), \quad \varphi_n = \frac{2}{J_1^2(\lambda_n)} \int_0^1 r \varphi(r) J_0(\lambda_n r) dr. \tag{17}$$

Substituting expansions (15)–(17) into equations (5), (6), we obtain the initial problems for ordinary integro-differential equations

$$u_n''(t) + \lambda_n^2 u_n(t) = \int_0^t k(\alpha) u_n(t-\alpha)\, d\alpha + f_n(t), \quad t \in (0,T), \tag{18}$$

$$u_n|_{t=0} = \varphi_n, \quad u_n'|_{t=0} = 0, \quad n = 1, 2, \ldots. \tag{19}$$

From relations (18) and (19) we arrive at the following integral equation

$$u_n(t) = \frac{1}{\lambda_n} \int_0^t \sin \lambda_n (t-\alpha) \int_0^{\alpha} k(\tau) u_n(\alpha - \tau) d\tau\, d\alpha + F_n(t), \tag{20}$$

where

$$F_n(t) := \varphi_n \cos \lambda_n t + \frac{1}{\lambda_n} \int_0^t \sin \lambda_n (t-\alpha) f_n(\alpha) d\alpha, \quad n = 1, 2, \ldots$$

For every fixed n, the equation (20) is a Volterra integral equation of the second kind with respect to u_n. According to the general theory of integral equations, it has a unique solution. The solution can be founded, by the method of successive approximations. It is easy to see that $u_n(t) \in C^2[0,T]$, if $f_n(t) \in C[0,T]$ and $k(t) \in C[0,T]$. Moreover, from (20) one can obtain estimate for $u_n(t)$:

$$|u_n(t)| = \left| \frac{1}{\lambda_n} \int_0^t \sin \lambda_n(t-\alpha) \int_0^\alpha k(\tau) u_n(\alpha-\tau) d\tau \, d\alpha + F_n(t) \right|$$

$$\leq \frac{1}{\lambda_n} \left| \int_0^t \sin \lambda_n(t-\alpha) \int_0^\alpha k(\tau) u_n(\alpha-\tau) d\tau \, d\alpha \right| + |F_n(t)| \leq ||F_n(t)|| + \frac{||k||}{\lambda_n} \int_0^t (t-\tau) \, |u_n(\tau)| \, d\tau.$$

Hence, by the Gronwall inequality and the relations $\lambda_1 < \lambda_2 < \ldots$, we get the estimate

$$|u_n(t)| \leq \left(|\varphi_n| + \frac{1}{\lambda_n} ||f_n|| T \right) e^{\frac{||k|| T^2}{2\lambda_1}}, \quad t \in [0,T], \quad n = 1, 2, \ldots, \tag{21}$$

where $||f_n|| = \max\limits_{t \in [0,T]} |f_n(t)|$, $||k|| = \max\limits_{t \in [0,T]} |k(t)|$.

Calculating the derivative $u_n'(t)$ in (20), from the estimate (21) we easily obtain

$$|u_n'(t)| \leq \left(|\varphi_n| + \frac{1}{\lambda_n} ||f_n|| T \right) \left(\lambda_n + ||k|| e^{\frac{||k|| T^2}{2\lambda_1}} \right), \quad t \in [0,T], \quad n = 1, 2, \ldots \tag{22}$$

Equation (18) and inequality (21) also imply the estimate

$$|u_n''(t)| \leq \left(\lambda_n^2 + ||k|| T \right) \left(|\varphi_n| + \frac{1}{\lambda_n} ||f_n|| T \right) e^{\frac{||k|| T^2}{2\lambda_1}} + ||f_n||, \quad t \in [0,T], \quad n = 1, 2, \ldots \tag{23}$$

To prove the existence of a solution to the direct problem (5)–(7), we need to show that the series (15) and the series obtained by differentiating with respect to t and r twice converge uniformly in domain G. To this end, we calculate u_{tt}, u_{rr}, performing differentiation formally under the signs of sums. Using properties of Bessel functions (see [31]) $J_0'(r) = -J_1(r)$, $2J_1'(r) = J_0(r) - J_2(r)$, from the first formula in (15) we obtain

$$u_{tt}(r,t) = \sum_{n=1}^{\infty} u_n''(t) J_0(\lambda_n r), \tag{24}$$

$$u_{rr}(r,t) = \frac{1}{2} \sum_{n=1}^{\infty} \lambda_n^2 u_n(t) \left(J_2(\lambda_n r) - J_0(\lambda_n r) \right). \tag{25}$$

Let the functions $\varphi(r)$ and $f(r,t)$ in r satisfy conditions of Theorem 2 with some $s \geq 1$ and $s_0 \geq 1$ (we define the numbers s and s_0 later). Then, according to the estimate (11), for the Fourier–Bessel coefficients of these functions are true the following estimates

$$|\varphi_n| \leq \frac{M}{\lambda_n^{2s-(1/2)}}, \quad |f_n(t)| \leq \frac{M_0}{\lambda_n^{2s_0-(1/2)}}, \quad t \in [0,T],$$

where M, M_0 are positive constants.

Now we will evaluate the expressions at Bessel functions in the series (15), (24) and (25). From (21)–(25) we have the following estimates

$$|u_n(t)| \leq \left(\frac{M}{\lambda_n^{2s-(1/2)}} + \frac{M_0 T}{\lambda_n^{2s_0+(1/2)}} \right) e^{\frac{||k|| T^2}{2\lambda_1}},$$

$$\left| u_n''(t) \right| \leq \left(\lambda_n^2 + ||k|| T \right) \left(\frac{M}{\lambda_n^{2s-(1/2)}} + \frac{M_0 T}{\lambda_n^{2s_0+(1/2)}} \right) e^{\frac{||k|| T^2}{2\lambda_1}} + \frac{M_0}{\lambda_n^{2s_0-(1/2)}},$$

$$\lambda_n^2 |u_n(t)| \leq \left(\frac{M\lambda_n^2}{\lambda_n^{2s-(1/2)}} + \frac{M_0 T \lambda_n^2}{\lambda_n^{2s_0+(1/2)}} \right) e^{\frac{||k||T^2}{2\lambda_1}}. \quad (26)$$

It follows from these estimates that if $s = s_0 = 2$, then, according to Theorem 1, the series in (15), (24) and (25) converge uniformly. Thus, Theorem 3 is proved. □

4. INVESTIGATION OF INVERSE PROBLEM

In view of (15), the condition (8) takes the form

$$u(r_0, t) = \sum_{n=1}^{\infty} u_n(t) s_n = h(t), \quad (27)$$

where $s_n := J_0(\lambda_n r_0)$. We will assume everywhere that $x_0,\ y_0$ are such that $J_0(\lambda_n r_0) \neq 0, n = 1, 2, \cdots$.

Multiplying equation (20) by s_n and summing over n from 1 to ∞, we get

$$\sum_{n=1}^{\infty} u_n(t) s_n = \sum_{n=1}^{\infty} s_n \frac{1}{\lambda_n} \int_0^t \sin \lambda_n (t-\alpha) \int_0^{\alpha} k(\tau) u_n(\alpha - \tau) d\tau \, d\alpha + \sum_{n=1}^{\infty} s_n F_n(t).$$

Changing the order of integration on the right side of this equation and using condition (27), we obtain

$$h(t) = \int_0^t P[k](t-\tau) k(\tau) d\tau + F_0(t), \quad (28)$$

where

$$P[k](t) := \int_0^t \sum_{n=1}^{\infty} s_n \frac{1}{\lambda_n} u_n(\alpha) \sin \lambda_n (t-\alpha) d\alpha,$$

$$F_0(t) = \sum_{n=1}^{\infty} s_n F_n(t). \quad (29)$$

From (28) it follows that

$$h(0) = F_0(0) = \sum_{i=1}^{\infty} s_n \varphi_n = \varphi(r_0).$$

To obtain an integral equation with respect to the function $k(t)$, we differentiate equation (28) successively three times

$$h'(t) = \int_0^t P'[k](t-\tau) k(\tau) d\tau + F_0'(t), \quad (30)$$

where

$$P'[k](t) = \int_0^t \sum_{n=1}^{\infty} s_n u_n(\alpha) \cos \lambda_n (t-\alpha) d\alpha,$$

$$F_0'(t) = \sum_{n=1}^{\infty} s_n F_n'(t) = \sum_{n=1}^{\infty} s_n \left[-\lambda_n \varphi_n \sin \lambda_n t + \int_0^t \cos \lambda_n (t-\alpha) f_n(\alpha) d\alpha \right].$$

In particular, from (30) it follows $h'(0) = 0$.

Further, differentiating again, we get

$$h''(t) = \int_0^t P''[k](t-\tau)k(\tau)d\tau + F_0''(t), \tag{31}$$

where

$$P''[k](t) = h(t) - \int_0^t \sum_{n=1}^{\infty} \lambda_n s_n u_n(\alpha-\tau)\sin\lambda_n(t-\alpha)d\alpha,$$

$$F_0''(t) = \sum_{n=1}^{\infty} s_n \left[-\lambda_n^2 \varphi_n \cos\lambda_n t + f_n(t) - \lambda_n \int_0^t \sin\lambda_n(t-\alpha) f_n(\alpha)d\alpha \right].$$

From (30), it follows that

$$h''(0) = \sum_{n=1}^{\infty} s_n \left(f_n(0) - \lambda_n^2 \varphi_n \right).$$

Differentiating (31) and resolving the resulting equality with respect to $k(t)$, we have

$$k(t) = \frac{1}{h(0)} \left[h'''(t) - F_0'''(t) - \int_0^t P'''[k](t-\tau)k(\tau)d\tau \right], \quad t \in [0,T], \tag{32}$$

where

$$F_0'''(t) = \sum_{n=1}^{\infty} s_n \left[\lambda_n^3 \varphi_n \sin\lambda_n t + f_n'(t) - \lambda_n^2 \int_0^t \cos\lambda_n(t-\alpha) f_n(\alpha)d\alpha \right],$$

$$P'''[k](t) = h'(t) - \int_0^t \sum_{n=1}^{\infty} \lambda_n^2 s_n u_n(\alpha)\cos\lambda_n(t-\alpha)d\alpha. \tag{33}$$

Now we will prove the lemma that will be used in further.

Lemma 1. *Let $u_n^1(t)$, $u_n^2(t)$ be two solutions of integral equation (20) with kernels $k^1(t)$, $k^2(t)$, respectively, but with the same data φ_n, $f_n(t)$. Then, the following estimate holds*

$$\left| u_n^1(t) - u_n^2(t) \right| \le \frac{T^2}{2\lambda_n} \left(|\varphi_n| + \frac{1}{\lambda_n}||f_n||T \right) e^{\frac{||k^2||T^2 + 2||k^1||T}{2\lambda_1}} ||k^1 - k^2||. \tag{34}$$

Proof. By the condition of the lemma $u_n^1(t)$, $u_n^2(t)$ are two solutions of equation (20) corresponding to the functions $k^1(t)$, $k^2(t)$. Estimate the modulus of the difference of these functions:

$$|u_n^1(t) - u_n^2(t)| \le \left| \frac{1}{\lambda_n} \int_0^t \sin\lambda_n(t-\alpha) \int_0^\alpha \left[k^1(\tau)u_n^1(\alpha-\tau) - k^2(\tau)u_n^2(\alpha-\tau) \right] d\tau\, d\alpha \right|$$

$$\le \frac{1}{\lambda_n} \int_0^t |\sin\lambda_n(t-\alpha)| \int_0^\alpha \left[\left|k^1(\tau)\right| \left|u_n^1(\alpha-\tau) - u_n^2(\alpha-\tau)\right| + \left|u_n^2(\alpha-\tau)\right| \left|k^1(\tau) - k^2(\tau)\right| \right] d\tau\, d\alpha$$

$$\le \frac{1}{\lambda_n} \int_0^t |\sin\lambda_n(t-\alpha)| \int_0^\alpha \left|u_n^2(\alpha-\tau)\right| \left|k^1(\tau) - k^2(\tau)\right| d\tau\, d\alpha$$

$$+\frac{1}{\lambda_n}\int\limits_0^t \left|\sin\lambda_n(t-\alpha)\right| \int\limits_0^\alpha \left|k^1(\tau)\right| \left|u_n^1(\alpha-\tau)-u_n^2(\alpha-\tau)\right| d\tau\, d\alpha. \tag{35}$$

We estimate each term separately. To evaluate the first term, we use formula (21)

$$\frac{1}{\lambda_n}\int\limits_0^t \left|\sin\lambda_n(t-\alpha)\right| \int\limits_0^\alpha \left|u_n^2(\alpha-\tau)\right| \left|k^1(\tau)-k^2(\tau)\right| d\tau\, d\alpha$$
$$\le \frac{1}{\lambda_n}\frac{T^2}{2}\left(|\varphi_n|+\frac{1}{\lambda_n}||f_n||T\right) e^{\frac{||k^2||T^2}{2\lambda_1}} ||k^1-k^2||, \tag{36}$$

where $||k^1-k^2|| = \max\limits_{t\in[0,T]} |k^1(t)-k^2(t)|$.

To estimate the second term in (35), changing the order of integration, we have

$$\frac{1}{\lambda_n}\int\limits_0^t |\sin\lambda_n(t-\alpha)| \int\limits_0^\alpha \left|k^1(\tau)\right| \left|u_n^1(\alpha-\tau)-u_n^2(\alpha-\tau)\right| d\tau\, d\alpha$$
$$\le \frac{1}{\lambda_n}||k^1||T\int\limits_0^t \left|u_n^1(\tau)-u_n^2(\tau)\right| d\tau. \tag{37}$$

Substituting (36) and (37) into (34), we get

$$\left|u_n^1(t)-u_n^2(t)\right| \le \frac{1}{\lambda_n}\left[\frac{T^2}{2}\left(|\varphi_n|+\frac{1}{\lambda_n}||f_n||T\right) e^{\frac{||k^2||T^2}{2\lambda_1}} ||k^1-k^2|| + ||k^1||T\int\limits_0^t \left|u_n^1(\tau)-u_n^2(\tau)\right| d\tau\right].$$

Now, applying the Gronwall lemma to this inequality, we obtain the estimate (34). □

Theorem 4. *Assume that* $h(t)\in C^3[0,T]$, $f(r,t)\in C^{4,1}(\overline{G})$, $\varphi(r)\in C^6[0,1]$, $h(0)=\varphi(r_0)\neq 0$, $h'(0)=0$,

$$h''(0)=\sum_{n=1}^{\infty} s_n\left(f_n(0)-\lambda_n^2\varphi_n\right)$$

and the equalities

$$\varphi^{(m)}(0)=0, \quad m=\overline{0,5}, \quad \varphi^{(m)}(1)=0, \quad m=\overline{0,4}, \tag{38}$$

$$\frac{\partial^m f}{\partial r^m}(0,t)=0, \qquad m=\overline{0,3}, \quad \frac{\partial^m f}{\partial r^m}(1,t)=0, \quad m=\overline{0,2}, \quad t\in[0,T] \tag{39}$$

are satisfied. Then, there is a unique solution of the inverse problem (1)–(4).

Proof. Note that $|s_n|\le 1$. If conditions (38) and (39) are satisfied, then all series of the form $\sum_{n=1}^{\infty}|s_n|\lambda_n^j|\varphi_n|$, $j=0,1,2,3$, $\sum_{n=1}^{\infty}|s_n|\lambda_n^j||f_n||$, $j=0,1,2$, $\sum\limits_{n=1}^{\infty}\lambda_n^2||f_n'||$ included in formula (32) and used to obtain it will converge, where $||f_n'||=\max\limits_{t\in[0,T]}|f_n'(t)|$. We represent equation (32) as an operator equation

$$k=Ak, \tag{40}$$

where A has the form

$$Ak=k_0-\frac{1}{|h(0)|}\int\limits_0^t P'''[k](t-\tau)k(\tau)d\tau,$$

$k_0(t) := \frac{1}{|h(0)|}\left[h'''(t) - F_0'''(t)\right],\ P'''[k](t-\tau)$ defines from the formula (33)).

Denote by C_σ the Banach space of continuous functions generated by the family of weight norms

$$||k||_\sigma = \max\left\{\max_{t\in[0,T]}\left|k(t)e^{-\sigma t}\right|\right\}, \quad \sigma \geq 0.$$

Obviously, for $\sigma = 0$ this space coincides with the space of continuous functions with the usual norm. We will denote this rule by $||k||$, because in the inequality

$$e^{-\sigma T}||k|| \leq ||k||_\sigma \leq ||k||,$$

the norms $||k||_\sigma$ and $||k||$ are equivalent for any fixed $T \in (0,\infty)$. We choose the number σ later. Let $Q_\sigma(k_0, ||k_0||) := \left\{k : ||k-k_0||_\sigma \leq ||k_0||\right\}$ be a ball of radius $||k_0||$ centered at point k_0 of some weighted space $Q_\sigma(\sigma \geq 0)$.

$$||k_0|| = \frac{1}{|h(0)|}||h'''|| + \frac{1}{|h(0)|}\sum_{n=1}^{\infty}|s_n|\left(\lambda_n^3|\varphi_n| + ||f_n'|| + \lambda_n^2||f_n)||T\right),$$

where $||f_n'|| = \max\limits_{t\in[0,T]}|f_n'(t)|$.

It is easy to see that $||k||_\sigma \leq ||k_0||_\sigma + ||k_0|| \leq 2||k_0||$. Let $k(t) \in Q_\sigma(k_0, ||k_0||)$. Let us show that for $\sigma > 0$ the operator A transforms the ball $Q_\sigma(k_0, ||k_0||)$ into itself, i.e. $A \in Q_\sigma(k_0, ||k_0||)$. Recall that an operator is said to be contractive on the set $Q_\sigma(k_0, ||k_0||)$, if the following two conditions are satisfied:

1) if $g \in Q_\sigma(k_0, ||k_0||)$, then $Ag \in Q_\sigma(k_0, ||k_0||)$,

2) if g^1, g^2 are arbitrary elements of $Q_\sigma(k_0, ||k_0||)$, then $||Ag^1 - Ag^2||_\sigma \leq \rho||g^1 - g^2||_\sigma$ and $0 < \rho < 1$.

Let us check the fulfillment of these conditions for A. For this, we have

$$||Ak - k_0||_\sigma = \max_{t\in[0,T]}\left|(Ak-k_0)e^{-\sigma t}\right| = \max_{t\in[0,T]}\left|\frac{1}{|h(0)|}\int_0^t P'''[k](t-\tau)k(\tau)e^{-\sigma\tau}e^{-\sigma(t-\tau)}d\tau\right|$$

$$\leq \frac{2T}{|h(0)|}\left[||h'|| + T\sum_{n=1}^{\infty}s_n\lambda_n\left(|\varphi_n| + \frac{1}{\lambda_n^2}||f_n||T\right)e^{\frac{||k||T^2}{2\lambda_1}}\right]\frac{||k_0||}{\sigma} =: \frac{||k_0||}{\sigma}\alpha_0.$$

Choosing σ as $\sigma \geq \alpha_0$, we obtain that A maps the ball $Q_\sigma(k_0, ||k_0||)$ into itself $Q_\sigma(k_0, ||k_0||)$.

Now let's check the fulfillment of the second condition. For this purpose, we get

$$||(Ak^1 - Ak^2)||_\sigma = \max_{t\in[0,T]}\left|(Ag^1 - Ag^2)e^{-\sigma t}\right|$$

$$\max_{t\in[0,T]}\left|\frac{1}{|h(0)|}\int_0^t\left[P'''[k^1](t-\tau)k^1(\tau) - P'''[k^2](t-\tau)k^2(\tau)\right]e^{-\sigma\tau}e^{-\sigma(t-\tau)}d\tau\right|$$

$$\leq \max_{t\in[0,T]}\frac{1}{|h(0)|}\int_0^t\left|\left[h'(t-\tau) - \int_0^t\sum_{n=1}^{\infty}\lambda_n^2 s_n u_n^1(\alpha-\tau)\cos\lambda_n(t-\alpha)d\alpha\right]k^1(\tau)\right.$$

$$\left.-\left[h'(t-\tau) - \int_0^t\sum_{n=1}^{\infty}\lambda_n^2 s_n u_n^2(\alpha-\tau)\cos\lambda_n(t-\alpha)d\alpha\right]k^2(\tau)e^{-\sigma\tau}e^{-\sigma(t-\tau)}\right|d\tau$$

$$\leq \max_{t\in[0,T]}\frac{1}{|h(0)|}\int_0^t\left[\left|h'(t-\tau)\right|\left|k^1(\tau) - k^2(\tau)\right|e^{-\sigma\tau}e^{-\sigma(t-\tau)}\right.$$

$$\left.+\int_0^t\sum_{n=1}^{\infty}\lambda_n^2 s_n\left|\cos\lambda_n(t-\alpha)\left[u_n^1(\alpha-\tau)k^1(\tau) - u_n^2(\alpha-\tau)k^2(\tau)\right]\right|e^{-\sigma\tau}e^{-\sigma(t-\tau)}d\alpha\right]d\tau. \qquad (41)$$

Here, the first term is estimated as follows

$$\max_{t\in[0,T]}\frac{1}{|h(0)|}\int\limits_0^t\Big[\big|h'(t-\tau)\big|\big|k^1(\tau)-k^2(\tau)\big|e^{-\sigma\tau}e^{-\sigma(t-\tau)}\Big]d\tau\le\frac{||h'||T}{|h(0)|}\frac{||k^1-k^2||}{\sigma}. \tag{42}$$

To estimate the second term, we use the obvious inequality

$$\big|g_k^1g_s^1-g_k^2g_s^2\big|\le\big|g_k^1-g_k^2\big|\big|g_s^1\big|+\big|g_k^2\big|\big|g_s^1-g_s^2\big|.$$

Then, we obtain

$$\max_{t\in[0,T]}\frac{1}{h(0)|}\int\limits_0^t\int\limits_0^t\Big[\sum_{n=1}^{\infty}\lambda_n^2s_n\Big|\cos\lambda_n(t-\alpha)\big[u_n^1(\alpha-\tau)k^1(\tau)-u_n^2(\alpha-\tau)k^2(\tau)\big]\Big|e^{-\sigma\tau}e^{-\sigma(t-\tau)}d\alpha\Big]d\tau$$

$$\le\max_{t\in[0,T]}\frac{1}{|h(0)|}\int\limits_0^t\int\limits_0^t\Big[\sum_{n=1}^{\infty}\lambda_n^2s_n\big|\cos\lambda_n(t-\alpha)\big|\Big[\big|u_n^1(\alpha-\tau)-u_n^2(\alpha-\tau)\big|\,\big|k^1(\tau)\big|$$

$$+\big|u_n^2(\alpha-\tau)\big|\,\big|k^1(\tau)-k^2(\tau)\big|\Big]e^{-\sigma\tau}e^{-\sigma(t-\tau)}d\alpha\Big]d\tau$$

$$\le\frac{T^2}{|h(0)|}\sum_{n=1}^{\infty}\lambda_n^2s_n\Big[T^2||k_0||e^{\int_0^t|k_1(\theta)|d\theta}+1\Big]\Big(||\varphi_n||+\frac{1}{\lambda_n}||f_n||T\Big)e^{\frac{||k||T^2}{2\lambda_1}}\frac{||k^1-k^2||}{\sigma}. \tag{43}$$

Substituting (42) and (43) into (41), we get

$$||(Ak^1-Ak^2)||_\sigma=\max_{t\in[0,T]}\big|(Ag^1-Ag^2)e^{-\sigma t}\big|$$

$$\le\frac{T}{|h(0)|}\Bigg[||h'||+T\sum_{n=1}^{\infty}\lambda_n^2s_n\Big[T^2||k_0||e^{\int_0^t|k_1(\theta)|d\theta}+1\Big]\Big(|\varphi_n|+\frac{1}{\lambda_n}||f_n||T\Big)e^{\frac{||k||T^2}{2\lambda_1}}\Bigg]\frac{||k^1-k^2||}{\sigma}$$

$$=:\frac{\alpha_1}{\sigma}||k^1-k^2||.$$

As follows from the obtained estimates, if the number σ is chosen from the condition $\sigma>\max(\alpha_0,\alpha_1)$, then the operator A is contracting on $Q_\sigma(k_0,||k_0||)$. Then, according to Banach principle [32], the equation (40) has the unique solution in $Q_\sigma(k_0,||k_0||)$ for any fixed $T>0$. Since equation (40) is equivalent to the inverse problem (1)–(4). □

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