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ONE-DIMENSIONAL INVERSE PROBLEMS OF FINDING THE KERNEL OF INTEGRODIFFERENTIAL HEAT EQUATION IN A BOUNDED DOMAIN

D. K. Durdiev[1,2] **and Zh. Zh. Zhumaev**[3]

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We consider an integrodifferential heat equation with time-convolution integral on the right-hand side. The direct problem is an initial boundary-value problem for this equation. We consider two inverse problems for this direct problem, namely, the problems of finding the kernel of the integral term for given two additional conditions imposed on the solution of the direct problem. These problems are replaced with equivalent systems of integral equations for the unknown functions, and the unique solvability of the inverse problems is proved by using the principle of contracting mapping,

1. Introduction

The problems of determination of the coefficients appearing on right-hand sides or some other physical parameters of differential or integrodifferential equations with given additional "experimental" data about their solutions are frequently encountered in various applications. These problems are inverse to the "direct" problems in which differential equations and the corresponding initial and boundary conditions are given [1].

Inverse problems for parabolic and hyperbolic equations with partial derivatives naturally appear in geophysics, in the problems of oil finding, in the design of optical instruments, and in numerous other fields where the inner structure of an object can be described by the results of measuring the fields in available domains. The problems of determination of the memory kernels in equations of this kind are extensively studied since the end of the last century (see [2–5]).

At present, there are numerous works devoted to the investigation of the inverse problems for parabolic integrodifferential equations (see, e.g., [6–10]).

Consider an initial-boundary-value problem of determination of a function $u(x,t),\ x \in (0,l),\ t \in (0,T]$, from the equation

$$u_t - a^2 u_{xx} = \int_0^t k(\tau) u(x, t-\tau) d\tau + h(x,t), \quad x \in (0,l), \quad 0 < t \le T, \tag{1.1}$$

$$u|_{t=0} = \varphi(x), \quad x \in [0,l], \tag{1.2}$$

$$u|_{x=0} = \mu_1(t), \qquad u|_{x=l} = \mu_2(t), \quad 0 \le t \le T, \qquad \varphi(0) = \mu_1(0), \qquad \varphi(l) = \mu_2(0), \tag{1.3}$$

where a is a positive constant and l and T are arbitrary positive numbers. For given functions $k(t),\ h(x,t),\ \varphi(x)$, $\mu_1(t)$, and $\mu_2(t)$, this problem is called the direct problem.

[1] Bukhara Division of the Institute of Mathematics, Academy of Sciences of the Uzbekistan Republic, Bukhara, Uzbekistan; e-mail: durdimurod@inbox.ru.
[2] Corresponding author.
[3] Bukhara State University, Bukhara, Uzbekistan; e-mail: jonibekjj@mail.ru.

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In the inverse problem, it is assumed that the kernel $k(t),\ t > 0,$ of the integral term in (1.1) is unknown. The problem is to find this kernel by using additional information about the solution of the direct problem

$$\int_0^l u(x,t)dx = f(t), \quad t \in (0,T], \tag{1.4}$$

or

$$u(x_0,t) = f(t), \qquad x_0 \in (0,l), \quad t \in (0,T]. \tag{1.5}$$

In this case, $\varphi(x),\ x \in [0,l],\ \mu_1(t),\ \mu_2(t),\ t \in [0,T],$ are given functions. In what follows, we say that the problem of determination of the functions $u(x,t),\ k(t),\ x \in (0,l),$ and $t \in (0,T],$ from Eqs. (1.1)–(1.4) is the ***inverse problem* 1**, while the problem of determination of these functions from Eqs. (1.1)–(1.3) and (1.5) is the ***inverse problem* 2**.

For the sake of simplicity, we denote the function u_t by ϑ, i.e., $u_t = \vartheta$. Differentiating Eq. (1.1) with respect to t and using condition (1.2), we get

$$\vartheta_t - a^2\vartheta_{xx} = k(t)\varphi(x) + \int_0^t k(\tau)\vartheta(x,t-\tau)d\tau + h_t(x,t). \tag{1.6}$$

Setting $t = 0$ in Eq. (1.1) and using equality (1.2), we get the initial condition for ϑ:

$$\vartheta|_{t=0} = a^2\varphi''(x) + h(x,0). \tag{1.7}$$

In order to obtain boundary conditions for the function ϑ, we differentiate conditions (1.3) with respect to t and find

$$\begin{gathered}\vartheta|_{x=0} = \mu_1'(t), \qquad \vartheta|_{x=l} = \mu_2'(t), \quad 0 < t \le T, \\ a^2\varphi''(0) + h(0,0) = \mu_1'(0), \qquad a^2\varphi''(l) + h(l,0) = \mu_2'(0).\end{gathered} \tag{1.8}$$

Differentiating the additional conditions (1.4) and (1.5) with respect to $t,$ we get these conditions for the function ϑ in the inverse problem 1:

$$\int_0^l \vartheta(x,t)dx = f'(t), \quad t \in (0,T], \tag{1.9}$$

and in the inverse problem 2:

$$\vartheta(x_0,t) = f'(t), \qquad x_0 \in (0,l), \quad t \in (0,T]. \tag{1.10}$$

We replace the initial-boundary-value problem (1.6)–(1.8) by an equivalent integral Volterra-type equation. To this end, we derive the following equation for $\vartheta(x,t)$ from Eqs. (1.6)–(1.8) (see [11, pp. 180–219]):

$$\vartheta(x,t) = \psi(x,t) + \int_0^t\int_0^l G(x,\xi,t-\tau)\left[k(\tau)\varphi(\xi) + \int_0^\tau k(\alpha)\vartheta(\xi,\tau-\alpha)d\alpha\right]d\xi d\tau, \tag{1.11}$$

where

$$\psi(x,t)=\int_0^t\int_0^l G(x,\xi,t-\tau)h_\tau(\xi,\tau)d\xi d\tau$$

$$+\sum_{n=1}^{\infty}\left[\frac{2}{l}\int_0^l (a^2\varphi''(x)+h(x,0))\sin\frac{\pi n}{l}xdx\right.$$

$$\left.+\frac{2\pi a^2 n}{l^2}\int_0^t(\mu_1'(\tau)-(-1)^n\mu_2'(\tau))e^{(\frac{\pi an}{l})^2\tau}d\tau\right]e^{-(\frac{\pi an}{l})^2 t}\sin\frac{\pi n}{l}x,$$

$$G(x,\xi,t-\tau)=\frac{2}{l}\sum_{n=1}^{\infty}e^{-(\frac{\pi an}{l})^2(t-\tau)}\sin\frac{\pi an}{l}\xi\sin\frac{\pi an}{l}x$$

is the Green function of the first initial-boundary-value problem for the one-dimensional heat-conduction equation.

We now present two properties of the Green function [11, pp. 200–221] necessary in what follows.

Remark 1. The integral of the Green function does not exceed 1:

$$\int_0^l G(x,\xi,t)d\xi\le 1,\qquad x\in(0,l),\quad t\in(0,T].$$

Remark 2. The function $G(x,\xi,t)$ is continuously differentiable with respect to $x,\ \xi,\ t$ infinitely many times and, moreover, $G_t(x,\xi,t)$ is a bounded function for $0<x<l,\ 0<\xi<l,\ 0<t\le T$, i.e.,

$$\left|G_t(x,\xi,t-\tau)\right|\le\frac{2}{l}.$$

2. Direct Problem

Lemma 1. *Suppose that*

$$(\varphi(x),\varphi'(x),\varphi''(x))\in C(0,l),\quad (h(x,t),h_t(x,t))\in C(D_{lT}),$$

$$(\mu_1(t),\mu_1'(t),\mu_2(t),\mu_2'(t))\in C(0,T),\quad k(t)\in C(0,T),$$

and that the consistency conditions in (1.3) and (1.8) are satisfied. Then there exists a unique classical solution $\vartheta(x,t)$ of problem (1.6)–(1.8) that belongs to the class $C^{2,1}(D_{lT})$ [$C^{2,1}(D_{lT})$ is the class of functions twice continuously differentiable with respect to x and continuously differentiable with respect to t in the domain $D_{lT}=\{0<x<l,0<t\le T\}$].

In what follows, we also use the ordinary class $C(D_{lT})$ of functions continuous in the domain D_{lT}.

To prove Lemma 1, we rewrite Eq. (1.11) in the form

$$\vartheta(x,t)=\int_0^t\int_0^l G(x,\xi,t-\tau)h_\tau(\xi,\tau)d\xi d\tau$$

$$+\sum_{n=1}^{\infty}\left[\frac{2}{l}\int_0^l (a^2\varphi''(x)+h(x,0))\sin\frac{\pi n}{l}xdx\right.$$

$$\left.+\frac{2\pi a^2 n}{l^2}\int_0^t(\mu_1'(\tau)-(-1)^n\mu_2'(\tau))e^{(\frac{\pi a n}{l})^2\tau}d\tau\right]e^{-(\frac{\pi a n}{l})^2 t}\sin\frac{\pi n}{l}x$$

$$+\int_0^t\int_0^l G(x,\xi,t-\tau)k(\tau)\varphi(\xi)d\xi d\tau$$

$$+\int_0^t\int_0^l G(x,\xi,t-\tau)\int_0^\tau k(\alpha)\vartheta(\xi,\tau-\alpha)d\alpha d\xi d\tau. \tag{2.1}$$

By $\Phi(x,t)$ we denote the sum of the first three terms on the right-hand side of (2.1). For this equation, in the domain $D_{lT},$ we consider a sequence of functions

$$\vartheta_n(x,t)=\Phi(x,t)+\int_0^t\int_0^l G(x,\xi,t-\tau)\int_0^\tau k(\alpha)\vartheta_{n-1}(\xi,\tau-\alpha)d\alpha d\xi d\tau,\quad n=1,2,\ldots, \tag{2.2}$$

where $\vartheta_0(x,t)=0$ for $(x,t)\in D_{lT}$. Under the conditions of Lemma 1, $\Phi(x,t)\in C^{2,1}(D_{lT})$. Thus, it follows from (2.2) that all $\vartheta_n(x,t)$ in the domain D_{lT} have the following properties:

Denote

$$Z_n(x,t):=\vartheta_n(x,t)-\vartheta_{n-1}(x,t)$$

and $\Phi_0=\|\Phi\|_{C(D_{lT})}$. According to relation (2.2), we estimate $Z_n(x,t)$ in the domain D_{lT}:

$$|Z_1(x,t)|\le\Phi_0,$$

$$|Z_2(x,t)|\le\int_0^t\int_0^l G(x,\xi,t-\tau)\int_0^\tau|k(\alpha)|\,|Z_1(\xi,\tau-\alpha)|\,d\alpha d\xi d\tau\le\Phi_0 k_0\frac{t^2}{2!},$$

$$k_0=\max_{t\in[0,T]}|k(t)|,$$

$$|Z_3(x,t)|\le\int_0^t\int_0^l G(x,\xi,t-\tau)\int_0^\tau|k(\alpha)|\,|Z_2(\xi,\tau-\alpha)|\,d\alpha d\xi d\tau\le\Phi_0 k_0^2\frac{t^4}{4!}.$$

Thus, for any $n = k,$ we get

$$|Z_k(x,t)| \le \Phi_0 k_0^{k-1} \frac{t^{2(k-1)}}{(2k-2)!}.$$

It follows from the estimates presented above that the series

$$\sum_{n=1}^{\infty} [\vartheta_n(x,t) - \vartheta_{n-1}(x,t)]$$

is uniformly convergent in D_{lT} and its sum $u(x,t)$ belongs to the function space $C^{2,1}(D_T)$. Hence, the functional sequence $\vartheta_n(x,t)$ given by equality (2.2) uniformly converges to $\vartheta(x,t)$ in D_{lT}. Thus, $\vartheta(x,t)$ is a solution of Eq. (1.11).

We now show that this solution is unique. Assume that there are two solutions of Eq. (1.11) $\vartheta^1(x,t)$ and $\vartheta^2(x,t)$ with the same initial data. Then their difference $Z(x,t) = \vartheta^2(x,t) - \vartheta^1(x,t)$ is a solution of the equation

$$Z(x,t) = \int_0^t \int_0^l G(x,\xi,t-\tau) \int_0^\tau k(\alpha) Z(\xi,\alpha) d\alpha d\xi d\tau.$$

Let $\tilde{Z}(t)$ be the supremum of the modulus of $Z(x,t)$ for $x \in [0,l]$ and every fixed $t \in (0,T]$. Therefore,

$$\tilde{Z}(t) \le k_0 T \int_0^t \tilde{Z}(\tau) d\tau, \quad t \in [0,T].$$

By using the Gronwall inequality, we obtain $\tilde{Z}(t) = 0$ for $t \in [0,T]$. This means that $Z(x,t) = 0$ in D_{lT}, i.e.,

$$\vartheta^1(x,t) = \vartheta^2(x,t)$$

in D_{lT}. Hence, Eq. (1.11) has a unique solution in the domain D_{lT}.

The lemma is proved.

3. Inverse Problem 1

By using the additional condition (1.9) for the inverse problem 1 and Eq. (1.11), we get

$$f'(t) = \int_0^l \psi(x,t)dx + \int_0^l \int_0^t \int_0^l G(x,\xi,t-\tau)k(\tau)\varphi(\xi) d\xi d\tau dx$$

$$+ \int_0^l \int_0^t \int_0^l G(x,\xi,t-\tau) \int_0^\tau k(\alpha)\vartheta(\xi,\tau-\alpha) d\alpha d\xi d\tau dx.$$

Differentiating this equality with respect to $t,$ we arrive at the equation

$$f''(t) = \int_0^l \psi_t(x,t)dx + \int_0^l\int_0^l G(x,\xi,0)k(t)\varphi(\xi)d\xi dx$$

$$+ \int_0^l\int_0^t k(\tau)\int_0^l G_t(x,\xi,t-\tau)\varphi(\xi)d\xi d\tau dx$$

$$+ \int_0^l\int_0^t\int_0^l G_t(x,\xi,t-\tau)\int_0^\tau k(\alpha)\vartheta(\xi,\tau-\alpha)d\alpha d\xi d\tau dx$$

$$+ \int_0^l\int_0^l G(x,\xi,0)\int_0^t k(\alpha)\vartheta(\xi,t-\alpha)d\alpha d\xi dx.$$

Since $G(x,\xi,0)=\delta(x-\xi),$ where $\delta(\cdot)$ is the Dirac delta function, by using the relations

$$\int_0^l g(\xi)\delta(x-\xi)d\xi = g(x),$$

$$\int_0^l G(x,\xi,0)\int_0^t k(\alpha)\vartheta(\xi,t-\alpha)d\alpha d\xi = \int_0^t k(\alpha)\vartheta(x,t-\alpha)d\alpha,$$

we rewrite the last equation in the form

$$f''(t) = \int_0^l \psi_t(x,t)dx + k(t)\int_0^l \varphi(x)dx$$

$$+ \int_0^l\int_0^t k(\tau)\int_0^l G_t(x,\xi,t-\tau)\varphi(\xi)d\xi d\tau dx$$

$$+ \int_0^l\int_0^t\int_0^l G_t(x,\xi,t-\tau)\int_0^\tau k(\alpha)\vartheta(\xi,\tau-\alpha)d\alpha d\xi d\tau dx$$

$$+ \int_0^l\int_0^t k(\alpha)\vartheta(x,t-\alpha)d\alpha dx. \qquad (3.1)$$

We introduce the following notation:

$$\varphi_0 = \int\limits_0^l \varphi(x)dx.$$

We now rewrite Eq. (3.1) in the form an integral equation of the second kind for the unknown function $k(t)$:

$$k(t) = \frac{1}{\varphi_0}\Bigg[f''(t) - \int\limits_0^l \psi_t(x,t)dx$$
$$- \int\limits_0^l \int\limits_0^t k(\tau) \int\limits_0^l G_t(x,\xi,t-\tau)\varphi(\xi)d\xi d\tau dx - \int\limits_0^l \int\limits_0^t k(\alpha)\vartheta(x,t-\alpha)d\alpha dx$$
$$- \int\limits_0^l \int\limits_0^t \int\limits_0^l G_t(x,\xi,t-\tau) \int\limits_0^\tau k(\alpha)\vartheta(\xi,\tau-\alpha)d\alpha d\xi d\tau dx\Bigg]. \tag{3.2}$$

Further, the system of equations (1.11) and (3.2) can be represented in the form of an operator equation

$$Ag = g, \tag{3.3}$$

where $g = (g_1, g_2) = (\vartheta(x,t),\ k(t))$ is a vector function and $A = (A_1, A_2)$ are determined with the help of the right-hand sides of the integral equations (1.11) and (3.2):

$$A_1 g = g_{01}(x,t) + \int\limits_0^t \int\limits_0^l G(x,\xi,t-\tau)\left[g_2(\tau)\varphi(\xi) + \int\limits_0^\tau g_2(\alpha)g_1(\xi,\tau-\alpha)d\alpha\right]d\xi d\tau, \tag{3.4}$$

$$A_2 g = g_{02}(t) - \frac{1}{\varphi_0}\Bigg[\int\limits_0^l \int\limits_0^t g_2(\tau) \int\limits_0^l G_t(x,\xi,t-\tau)\varphi(\xi)d\xi d\tau dx$$
$$- \int\limits_0^l \int\limits_0^t g_2(\alpha)g_1(x,t-\alpha)d\alpha dx$$
$$- \int\limits_0^l \int\limits_0^t \int\limits_0^l G_t(x,\xi,t-\tau) \int\limits_0^\tau g_2(\alpha)g_1(\xi,\tau-\alpha)d\alpha d\xi d\tau dx\Bigg]. \tag{3.5}$$

In equalities (3.4) and (3.5), we introduce the following notation:

$$g_0(x,t) = (g_{01}(x,t), g_{02}(t)) = \left(\psi(x,t), \frac{1}{\varphi_0}\left[f''(t) - \int\limits_0^l \psi_t(x,t)dx\right]\right).$$

Theorem 1. *Suppose that $f(t) \in C^2[0,T]$, $\varphi_0 \neq 0$, and that all conditions of Lemma 1 are satisfied. Then, for any fixed $l > 0$ and $T > 0$, the operator equation (3.3) possesses a unique solution in the domain D_{lT}.*

Proof. For the unknown vector function $g(x,t) \in C(D_{lT})$, we define its weighted norm as follows:

$$\|g\|_\sigma = \max\left\{ \sup_{(x,t)\in\overline{D}_T} \left|g_1(x,t)e^{-\sigma t}\right|, \sup_{t\in[0,T]} \left|g_2(t)e^{-\sigma t}\right| \right\}$$

$$= \max\left\{\|g_1\|_\sigma, \|g_2\|_\sigma\right\}, \quad \sigma \geq 0.$$

For $\sigma = 0$, this norm coincides with the ordinary norm

$$\|g\| = \max\left\{ \sup_{(x,t)\in\overline{D}_T} |g_1(x,t)|, \sup_{t\in[0,T]} |g_2(t)| \right\}.$$

The number $\sigma \geq 0$ is chosen in what follows. By $B(g_0, \rho)$ we denote a ball of radius $\rho > 0$ centered at the point g_0 in the space $C(D_{lT})$, i.e.,

$$B(g_0, \rho) = \{g \in C(D_{lT})\colon \|g - g_0\|_\sigma \leq \rho\}.$$

The number $\rho > 0$ is found in what follows.

It is clear that

$$\|g\| \leq \rho + \|g_0\| \quad \text{for} \quad g(x,t) \in B(g_0, \rho).$$

We show that A is a contraction operator in the Banach space $C(D_{lT})$ with the weighted norm introduced above if the numbers σ and ρ are properly chosen. Recall that A is a contraction operator if the following conditions are satisfied:

(1) if $g(x,t) \in B(g_0, \rho)$, then $Ag \in B(g_0, \rho)$;

(2) if g^1 and g^2 are any two elements from $B(g_0, \rho)$, then the inequality

$$\|Ag^1 - Ag^2\|_\sigma \leq \mu\|g^1 - g^2\|_\sigma$$

is true with $\mu \in (0,1)$.

Note that the weighted norm $\|\cdot\|_\sigma$ is equivalent to the ordinary norm $\|\cdot\|$:

$$\|\cdot\|_\sigma \leq \|\cdot\| \leq e^{\sigma T}\|\cdot\|_\sigma, \quad \sigma > 0. \tag{3.6}$$

The convolution operator is commutative and invariant under multiplications by $e^{-\sigma t}$:

$$(h_1 * h_2)(t) = \int_0^t h_1(t-s)h_2(s)ds = \int_0^t h_1(s)h_2(t-s)ds = (h_2 * h_1)(t), \tag{3.7}$$

$$e^{-\sigma t}(h_1 * h_2)(t) = (e^{-\sigma t}h_1(t)) * (e^{-\sigma t}h_2(t)). \tag{3.8}$$

By using the last relation, we arrive at the estimate

$$\|h_1 * h_2\|_\sigma \leq \|h_1\|_\sigma \|h_2\|_\sigma T. \tag{3.9}$$

Moreover, since

$$\int_0^t e^{-\sigma s} ds = \int_0^t e^{-\sigma(t-s)} ds \leq \frac{1}{\sigma}, \quad \sigma > 0, \tag{3.10}$$

by using (3.6) and the results presented in [10], we arrive at the inequalities

$$\|h_1 * h_2\|_\sigma \leq \frac{1}{\sigma}\|h_1\|\|h_2\|_\sigma \leq \frac{1}{\sigma}\|h_1\|\|h_2\|, \quad \sigma > 0. \tag{3.11}$$

We now check the validity of the first condition for contracting mapping. For the sake of simplicity, we denote

$$\varphi_1 = \max_{x\in[0,l]} |\varphi(x)|.$$

Let $g(x,t)$ be an element of the ball $B(g_0,\rho)$, i.e., $g \in B(g_0,\rho)$. Thus, for $(x,t) \in D_{lT}$, we obtain

$$\begin{aligned}
\|A_1 g - g_{01}\|_\sigma &= \sup_{(x,t)\in D_{lT}} |(A_1 g - g_{01})e^{-\sigma t}| \\
&= \sup_{(x,t)\in D_{lT}} e^{-\sigma t} \left| \int_0^t \int_0^l G(x,\xi,t-\tau) g_2(\tau)\varphi(\xi) d\xi d\tau \right. \\
&\quad \left. + \int_0^t \int_0^l G(x,\xi,t-\tau) \int_0^\tau g_2(\alpha) g_1(\xi,\tau-\alpha) d\alpha d\xi d\tau \right| \\
&\leq \sup_{(x,t)\in D_{lT}} \left| \int_0^t \int_0^l G(x,\xi,t-\tau) g_2(\tau) e^{-\sigma\tau} \varphi(\xi) e^{-\sigma(t-\tau)} d\xi d\tau \right| \\
&\quad + \sup_{(x,t)\in D_{lT}} \left| \int_0^t e^{-\sigma(t-\tau)} \int_0^l G(x,\xi,t-\tau) \int_0^\tau g_2(\alpha) e^{-\sigma\alpha} g_1(\xi,\tau-\alpha) e^{-\sigma(\tau-\alpha)} d\alpha d\xi d\tau \right| \\
&\leq \frac{\rho + \|g_0\|}{\sigma} (\varphi_1 + (\rho + \|g_0\|)T).
\end{aligned}$$

If we now choose σ as follows:

$$\sigma \geq \sigma_1 = \frac{\rho}{(\rho + g_0)(\varphi_1 + (\rho + \|g_0\|)T)},$$

then we get

$$\|A_1 g - g_{01}\|_\sigma \le \rho,$$

i.e., the first condition of contracting mapping for the operator A_1 is satisfied.

We now establish the estimate for A_2:

$$\|A_2 g - g_{02}\|_\sigma = \sup_{t\in(0,T)} |(A_2 g - g_{02})e^{-\sigma t}|$$

$$= \sup_{t\in(0,T)} \frac{1}{\varphi_0} e^{-\sigma t} \Bigg| \int_0^l \int_0^t g_2(\tau) \int_0^l G_t(x,\xi,t-\tau)\varphi(\xi)d\xi d\tau dx$$

$$+ \int_0^l \int_0^t g_2(\alpha) g_1(x,t-\alpha) d\alpha dx$$

$$+ \int_0^l \int_0^t \int_0^l G_t(x,\xi,t-\tau) \int_0^\tau g_2(\alpha) g_1(\xi,\tau-\alpha) d\alpha d\xi d\tau dx) \Bigg|$$

$$\le \sup_{t\in(0,T)} \frac{1}{\varphi_0} \left| \int_0^l \int_0^t g_2(\tau) e^{-\sigma\tau} e^{-\sigma(t-\tau)} \int_0^l G_t(x,\xi,t-\tau)\varphi(\xi) d\xi d\tau dx \right|$$

$$+ \sup_{t\in(0,T)} \frac{1}{\varphi_0} \left| \int_0^l \int_0^t g_2(\alpha) e^{-\sigma\alpha} g_1(x,t-\alpha) e^{-\sigma(t-\alpha)} d\alpha dx \right|$$

$$+ \sup_{t\in(0,T)} \frac{1}{\varphi_0} \Bigg| \int_0^l \int_0^t e^{-\sigma(t-\tau)} \int_0^l G_t(x,\xi,t-\tau)$$

$$\times \int_0^\tau g_2(\alpha) e^{-\sigma\alpha} g_1(\xi,\tau-\alpha) e^{-\sigma(\tau-\alpha)} d\alpha d\xi d\tau dx \Bigg|.$$

We denote terms in the last relation by $I_i,\ i = 1,2,3,$ and estimate each of these terms in turn. Thus, for I_1, we get

$$I_1 = \sup_{t\in(0,T)} \frac{1}{\varphi_0} \left| \int_0^l \int_0^t g_2(\tau) e^{-\sigma\tau} e^{-\sigma(t-\tau)} \int_0^l G_t(x,\xi,t-\tau)\varphi(\xi) d\xi d\tau dx \right|$$

$$\le \frac{\varphi_1}{\varphi_0} \|g_2\|_\sigma \sup_{t\in(0,T)} \left| \int_0^t e^{-\sigma(t-\tau)} \int_0^l \int_0^l G_t(x,\xi,t-\tau) d\xi d\tau dx \right|$$

$$\le \frac{2l\varphi_1(\rho + \|g_0\|)}{\varphi_0} \frac{1}{\sigma}.$$

By using relations (3.6)–(3.11), we estimate I_2 as follows:

$$I_2 = \sup_{t\in(0,T)} \frac{1}{\varphi_0} \left| \int_0^l \int_0^t g_2(\alpha) g_1(x, t-\alpha) e^{-\sigma t} d\alpha dx \right|$$

$$= \sup_{t\in(0,T)} \frac{1}{\varphi_0} \left| \int_0^l (g_2 * g_1)(t) e^{-\sigma t} dx \right|$$

$$= \frac{1}{\varphi_0} \sup_{t\in(0,T)} \left| \int_0^l \Big\{ \left[(g_2 - g_{02}) * (g_1 - g_{01})\right](t) + (g_2 * g_{01})(t) \right.$$

$$\left. + (g_1 * g_{02})(t) - (g_{02} * g_{01})(t) \Big\} e^{-\sigma t} dx \right|$$

$$\leq \frac{1}{\varphi_0} \int_0^l \Big(\|g_2 - g_{02}\|_\sigma \|g_1 - g_{01}\|_\sigma T + \frac{1}{\sigma} \|g_2\|_\sigma \|g_{01}\|$$

$$+ \frac{1}{\sigma} \|g_1\|_\sigma \|g_{01}\| + \frac{1}{\sigma} \|g_{01}\|_\sigma \|g_{02}\| \Big) dx$$

$$\leq \frac{l}{\varphi_0} \left(\rho^2 T + \frac{2}{\sigma} (\rho + \|g_0\|) \|g_0\| + \frac{1}{\sigma} \|g_0\|^2 \right).$$

As in the case of I_1, for I_3, we obtain

$$I_3 = \sup_{t\in(0,T)} \frac{1}{\varphi_0} \left| \int_0^l \int_0^t e^{-\sigma(t-\tau)} \int_0^l G_t(x,\xi,t-\tau) \int_0^\tau g_2(\alpha) e^{-\sigma\alpha} g_1(\xi, \tau-\alpha) e^{-\sigma(\tau-\alpha)} d\alpha d\xi d\tau dx \right|$$

$$\leq \frac{1}{\varphi_0} \|g_1\|_\sigma \|g_2\|_\sigma \sup_{t\in(0,T)} \left| \int_0^l \int_0^t e^{-\sigma(t-\tau)} \int_0^l G_t(x,\xi,t-\tau) d\xi d\tau dx \right| \leq \frac{2lT(\rho + \|g_0\|)^2}{\varphi_0} \frac{1}{\sigma}.$$

Further, we get

$$\|A_2 g - g_{02}\|_\sigma \leq I_1 + I_2 + I_3$$

$$\leq \frac{2l\varphi_1(\rho + \|g_0\|)}{\varphi_0} \frac{1}{\sigma} + \frac{\rho^2 lT}{\varphi_0}$$

$$+ \frac{2l(\rho + \|g_0\|)\|g_0\|}{\varphi_0} \frac{1}{\sigma} + \frac{l\|g_0\|^2}{\varphi_0} \frac{1}{\sigma} + \frac{2lT(\rho + \|g_0\|)^2}{\varphi_0} \frac{1}{\sigma}. \tag{3.12}$$

We can now choose ρ and σ such that the inequalities

$$\frac{\rho^2 lT}{\varphi_0} < \frac{1}{3}\rho,$$

$$\frac{l\|g_0\|^2}{\varphi_0\sigma} < \frac{1}{3}\rho,$$

$$\frac{2l(\rho+\|g_0\|)(\varphi_1+\|g_0\|+T(\rho+\|g_0\|))}{\varphi_0\sigma} < \frac{1}{3}\rho$$

are true. By using these relations, we deduce the inequalities

$$\rho < \frac{\varphi_0}{3Tl} =: \rho_1,$$

$$\beta_1 := \frac{9l^2\|g_0\|^2 T}{\varphi_0^2} < \sigma,$$

$$\beta_2 := \frac{18Tl^2}{\varphi_0^2}\left(\frac{\varphi_0}{3Tl}+\|g_0\|\right)\left(\varphi_1+\|g_0\|+T\left(\frac{\varphi_0}{3Tl}+\|g_0\|\right)\right) < \sigma.$$

Hence, we get $A_2 g \in B(g_0,\rho)$.

Thus, if

$$\sigma > \sigma_2 = \max\{\beta_1,\beta_2\} \tag{3.13}$$

and $\rho \in (0,\rho_1)$, then the operator A_2 maps $B(g_0,\rho)$ into itself, i.e., $A_2 g \in B(g_0,\rho)$.

Therefore, if σ and ρ satisfy the conditions $\sigma > \max\{\sigma_1,\sigma_2\}$ and $\rho \in (0,\rho_1)$, then the operator A maps $B(g_0,\rho)$ into itself, i.e., $Ag \in B(g_0,\rho)$.

We now check the second condition of contracting mapping. According to (3.4), for the first component of the operator A, we obtain

$$\begin{aligned}\|(Ag^1-Ag^2)_1\|_\sigma \le \sup_{(x,t)\in D_{lT}} &\left|\int_0^t\int_0^l G(x,\xi,t-\tau)[g_2^1(\tau)-g_2^2(\tau)]\varphi(\xi)d\xi d\tau e^{-\sigma t}\right| \\ &+ \sup_{(x,t)\in D_{lT}}\left|\int_0^t\int_0^l G(x,\xi,t-\tau)\int_0^\tau \left[g_2^1(\alpha)g_1^1(\xi,\tau-\alpha)\right.\right. \\ &\left.\left. - g_2^2(\alpha)g_1^2(\xi,\tau-\alpha)\right]d\alpha d\xi d\tau e^{-\sigma t}\right|.\end{aligned}$$

The integrand of the last integral can be estimated as follows:

$$\left\|g_2^1 g_1^1 - g_2^2 g_1^2\right\|_\sigma = \left\|(g_2^1-g_2^2)g_1^1 + g_2^2(g_1^1-g_1^2)\right\|_\sigma$$

$$\le 2\left\|g^1-g^2\right\|_\sigma \max\left(\left\|g_1^1\right\|_\sigma, \left\|g_2^2\right\|_\sigma\right)$$

$$\le 2(\|g_0\|+\rho)\left\|g^1-g^2\right\|_\sigma .$$

Thus, we find

$$\left\|(Ag^1-Ag^2)_1\right\|_\sigma \le \frac{1}{\sigma}\left(\varphi_1+2(\rho+\|g_0\|)T\right)\|g^1-g^2\|_\sigma.$$

If we choose σ such that

$$\sigma > \sigma_3 = \varphi_1 + 2(\rho+\|g_0\|),$$

then we get

$$\|(Ag^1-Ag^2)_1\|_\sigma \le \frac{\sigma_3}{\sigma}\left\|g^1-g^2\right\|_\sigma ,$$

i.e., the second condition of contracting mapping is satisfied for A_1.

For the second component of the operator A, we obtain similar estimates

$$\|(Ag^1-Ag^2)_2\|_\sigma = \sup_{t\in(0,T)}\frac{1}{\varphi_0}\left|\int_0^l\int_0^t\left[g_2^1-g_2^2\right](\tau)\int_0^l G_t(x,\xi,t-\tau)\varphi(\xi)d\xi d\tau dx e^{-\sigma t}\right|$$

$$+\sup_{t\in(0,T)}\frac{1}{\varphi_0}\left|\int_0^l\int_0^t\left[g_2^1g_1^1-g_2^2g_1^2\right]e^{-\sigma t}d\alpha dx\right|$$

$$+\sup_{t\in(0,T)}\frac{1}{\varphi_0}\left|\int_0^l\int_0^t\int_0^l G_t(x,\xi,t-\tau)\int_0^\tau\left[g_2^1g_1^1-g_2^2g_1^2\right]e^{-\sigma t}d\alpha d\xi d\tau dx\right|.$$

By $J_1,\ J_2$, and J_3 we denote terms on the right-hand side of this equality and estimate each of these terms in turn. For J_1, we get

$$J_1 = \sup_{t\in(0,T)}\frac{1}{\varphi_0}\left|\int_0^l\int_0^t[g_2^1-g_2^2](\tau)e^{-\sigma\tau}e^{-\sigma(t-\tau)}\int_0^l G_t(x,\xi,t-\tau)\varphi(\xi)d\xi d\tau dx\right|$$

$$\le \frac{1}{\sigma}\frac{2l\varphi_1}{\varphi_0}\|g^1-g^2\|_\sigma.$$

By using the equality

$$g_2^1*g_1^1-g_2^2*g_1^2=\left(g_1^2-g_2^2\right)*\left(g_1^1-g_{01}\right)$$

$$+\left(g_1^1-g_1^2\right)*\left(g_2^2-g_{02}\right)+g_{01}*\left(g_2^1-g_2^2\right)+g_{02}*\left(g_1^1-g_1^2\right),$$

we estimate J_2 and J_3 as follows:

$$J_2 = \sup_{t\in(0,T)} \frac{1}{\varphi_0} \left| \int_0^l \int_0^t \left[g_2^1 g_1^1 - g_2^2 g_1^2\right] e^{-\sigma t} d\alpha dx \right|$$

$$= \sup_{t\in(0,T)} \frac{1}{\varphi_0} \left| \int_0^l \left[g_2^1 * g_1^1 - g_2^2 * g_1^2\right] e^{-\sigma t} dx \right|$$

$$\leq \frac{l}{\varphi_0} \Big[\left\|g_1^2 - g_2^2\right\|_\sigma \left\|g_1^1 - g_{01}\right\|_\sigma T + \left\|g_1^1 - g_1^2\right\|_\sigma \left\|g_2^2 - g_{02}\right\|_\sigma T$$

$$+ \|g_{01}\|_\sigma \left\|g_2^1 - g_2^2\right\|_\sigma + \|g_{02}\|_\sigma \left\|g_1^1 - g_1^2\right\|_\sigma \Big]$$

$$\leq \frac{2l}{\varphi_0} \left(\rho T + \frac{1}{\sigma}\|g_0\|\right) \left\|g^1 - g^2\right\|_\sigma ,$$

$$J_3 = \sup_{t\in(0,T)} \frac{1}{\varphi_0} \left| \int_0^l \int_0^t \int_0^l G_t(x,\xi,t-\tau) \int_0^\tau \left(\left(g_2^1 - g_2^2\right) g_1^1 + \left(g_1^1 - g_1^2\right) g_2^2\right) e^{-\rho t} d\alpha d\xi d\tau dx \right|$$

$$\leq \sup_{t\in(0,T)} \frac{1}{\varphi_0} \left| \int_0^l \int_0^t e^{-\sigma(t-\tau)} \int_0^l G_t(x,\xi,t-\tau) \int_0^\tau \left(g_2^1 - g_2^2\right) e^{-\sigma\alpha} g_1^1 e^{-\sigma(\tau-\alpha)} d\alpha d\xi d\tau dx \right|$$

$$+ \sup_{t\in(0,T)} \frac{1}{\varphi_0} \left| \int_0^l \int_0^t e^{-\sigma(t-\tau)} \int_0^l G_t(x,\xi,t-\tau) \int_0^\tau \left(g_1^1 - g_1^2\right) e^{-\sigma\alpha} g_2^2 e^{-\sigma(\tau-\alpha)} d\alpha d\xi d\tau dx \right|$$

$$\leq \frac{1}{\sigma} \frac{4(\rho + \|g_0\|) l T}{\varphi_0} \|g^1 - g^2\|_\sigma .$$

Finding the sum of the obtained estimates for $J_i,\ i = 1,2,3,$ we get

$$\|(Ag^1 - Ag^2)_2\|_\sigma \leq J_1 + J_2 + J_3 \leq \frac{2l}{\varphi_0} \left(\rho T + \frac{\varphi_1}{\sigma} + \frac{\|g_0\|}{\sigma} + \frac{2(\rho + \|g_0\|)T}{\sigma}\right) \left\|g^1 - g^2\right\|_\sigma .$$

We now choose the numbers σ and ρ for which the expression $\left\|g^1 - g^2\right\|_\sigma$ is smaller than 1, i.e., the inequality

$$\frac{2l}{\varphi_0} \left(\frac{\varphi_1}{\sigma} + \rho T + \frac{\|g_0\|}{\sigma} + \frac{2(\rho + \|g_0\|)T}{\sigma}\right) < 1$$

holds.

This inequality is true if the numbers σ and ρ are chosen from the conditions

$$\frac{2\rho Tl}{\varphi_0} < \frac{1}{3},$$

$$\frac{2l}{\varphi_0\sigma}(\varphi_1 + \|g_0\|) < \frac{1}{3},$$

$$\frac{4lT}{\varphi_0\sigma}(\rho + \|g_0\|) < \frac{1}{3}.$$

Solving these inequalities for σ and ρ, we get

$$\rho < \frac{\varphi_0}{6Tl} = \rho_2,$$

$$\sigma_4 = \frac{6l}{\varphi_0}(\varphi_1 + \|g_0\|) < \sigma,$$

$$\sigma_5 = \frac{2\varphi_0 + 12lT\|g_0\|}{\varphi_0} < \sigma.$$

According to these estimates, if σ and ρ are such that $\sigma > \sigma_4$ and $\rho < (0, \rho_2)$, then the operator A_2 satisfies the second condition of contracting mapping.

Thus, we conclude that if σ and ρ satisfy the conditions

$$\sigma > \max(\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5) \qquad \text{and} \qquad \rho \in \big(0, \min(\rho_1, \rho_2)\big) = (0, \rho_2),$$

then the operator A is a contracting mapping of the ball $B(g_0,\ \rho)$ into itself and, according to the Banach theorem [12, pp. 87–97], this ball has a unique fixed point, i.e., there exists a unique solution of the operator equation (3.3).

Theorem 1 is proved.

4. Inverse Problem 2

In the first section, we reduced the inverse problem 2 to the problem of determination of the kernel $k(t)$, $t \in (0, T)$ by using Eqs. (1.6)–(1.8) and (1.10). In the analyzed case, in order to deduce the integral equation for the kernel $k(t)$, we use Eq. (1.11) for the solution of the direct problem and the additional condition (1.10). As a result, we get

$$\begin{aligned} k(t) = {} & \frac{1}{\varphi(x_0)}(f''(t) - \psi_t'(x_0, t)) \\ & - \frac{1}{\varphi(x_0)} \int_0^l G(x_0, \xi, 0) \int_0^t k(\alpha)\vartheta(\xi, t-\alpha)\, d\alpha\, d\xi \\ & - \frac{1}{\varphi(x_0)} \int_0^t \int_0^l G_t(x_0, \xi, t-\tau) \int_0^\tau k(\alpha)\vartheta(\xi, \tau-\alpha)\, d\alpha\, d\xi\, d\tau. \end{aligned} \tag{4.1}$$

We now rewrite the system of integral equations (1.11) and (4.1) in the form of an operator equation

$$Ag = g, \tag{4.2}$$

where $g = (g_1\ g_2) = (\vartheta(x,t),\ k(t))$ is an unknown vector function, $A = (A_1, A_2)$ is defined by the right-hand sides of Eqs. (1.11) and (4.1).

Theorem 2. *Suppose that $f(t) \in C^2[0,T]$, $\varphi(x_0) \neq 0$, and all conditions of Lemma 1 are satisfied. Then, for any fixed $l >$ and $T > 0$, the operator equation (4.2) possesses a unique solution in the domain D_{lT}.*

Proof. We introduce a vector function as follows:

$$g_0(x,t) = (g_{01},\ g_{02})\,(x,t) = \left(\psi(x,t),\ \frac{1}{\varphi(x_0)}\left(f''(t) - \psi_t'(x_0,t)\right)\right).$$

Then, according to equalities (1.11) and (4.1), the components of the operator A have the form

$$Ag_1 = \psi(x,t) + \int_0^t\int_0^l G(x,\xi,t-\tau)\left[g_2(\tau)\varphi(\xi) + \int_0^\tau g_2(\alpha)g_1(\xi,\tau-\alpha)d\alpha\right]d\xi d\tau,$$

$$Ag_2 = \frac{1}{\varphi(x_0)}\left(f''(t) - \psi_t'(x_0,t)\right)$$

$$- \frac{1}{\varphi(x_0)}\int_0^l G(x_0,\xi,0)\int_0^t g_2(\alpha)g_1(\xi,t-\alpha)d\alpha d\xi$$

$$- \frac{1}{\varphi(x_0)}\int_0^t\int_0^l G_t(x_0,\xi,t-\tau)\int_0^\tau g_2(\alpha)g_1(\xi,\tau-\alpha)d\alpha d\xi d\tau.$$

The conditions of contraction for the operator A_1 have been obtained in the previous section. Here, it is sufficient to establish conditions of contraction for A_2. Let $g(x,t) \in B(D_{lT)}$. Then the following relations are true:

$$\|A_2 g - g_{02}\|_\sigma \leq \sup_{t\in(0,T)} \frac{1}{\varphi(x_0)}\left|\int_0^l G(x_0,\xi,0)\int_0^t g_2(\alpha)g_1(\xi,t-\alpha)e^{-\sigma t}d\alpha d\xi\right|$$

$$+ \frac{1}{\varphi(x_0)}\left|\int_0^t\int_0^l G_t(x_0,\xi,t-\tau)\int_0^\tau g_2(\alpha)g_1(\xi,\tau-\alpha)e^{-\sigma t}d\alpha d\xi d\tau\right|$$

$$= P_1 + P_2,$$

$$P_1 = \sup_{t\in(0,T)} \frac{1}{\varphi(x_0)}\left|\int_0^l G(x_0,\xi,0)(g_2 * g_1)(t)e^{-\sigma t}d\xi\right|$$

$$\leq \frac{1}{\varphi_0}\left(\rho^2 T + \frac{2}{\sigma}(\rho + \|g_0\|)\|g_0\| + \frac{1}{\sigma}\|g_0\|^2\right),$$

$$P_2 = \frac{1}{\varphi(x_0)}\left|\int_0^t\int_0^l G_t(x_0,\xi,t-\tau)\int_0^\tau g_2(\alpha)g_1(\xi,\tau-\alpha)e^{-\sigma t}d\alpha d\xi d\tau\right|$$

$$\leq \frac{2T(\rho + \|g_0\|)^2}{\sigma\varphi(x_0)},$$

$$\|A_2 g - g_{02}\|_\sigma \leq \frac{1}{\varphi(x_0)}\left(\rho^2 T + \frac{2}{\sigma}(\rho + \|g_0\|)\|g_0\| + \frac{1}{\sigma}\|g_0\|^2\right) + \frac{2T(\rho + \|g_0\|)^2}{\sigma\varphi(x_0)}.$$

Further, let

$$\rho < \frac{\varphi(x_0)}{3T} = \kappa_3,$$

$$\theta_1 = \frac{6T}{\varphi^2(x_0)}\left(\frac{\varphi(x_0)}{3T} + \|g_0\|\right)\left(\|g_0\| + T\left(\frac{\varphi(x_0)}{3T} + \|g_0\|\right)\right) < \sigma,$$

$$\theta_2 = \frac{9T\|g_0\|^2}{\varphi^2(x_0)} < \sigma.$$

Hence, if $\sigma > \max(\theta_1, \theta_2) = \beta_5$ and $\rho < \kappa_3$, then $A_2 g \in B(g_0, \rho)$.

Thus, if the inequality $\sigma > \sigma_1 = \max(\beta_0, \beta_5)$ is true, then the operator A maps the ball $B(g_0, \rho)$ into itself.

We now check the validity of the second condition of contracting mapping. We have

$$\|(Ag^1 - Ag^2)_1\|_\sigma \leq \frac{1}{\sigma}(\varphi_1 + 2(\rho + \|g_0\|)T)\|g^1 - g^2\|_\sigma.$$

In a similar way, we estimate the second component of the operator Ag:

$$\|(Ag^1 - Ag^2)_2\|_\sigma \left(\frac{2\rho T}{\varphi(x_0)} + \frac{2\|g_0\|}{\sigma\varphi(x_0)} + \frac{4T(\rho + \|g_0\|)}{\varphi(x_0)\sigma}\right)\|g^1 - g^2\|_\sigma.$$

Assume that the following relations are true:

$$\rho < \frac{\varphi(x_0)}{6T} = \kappa_4,$$

$$\theta_3 = \frac{6\|g_0\|}{\varphi(x_0)} < \sigma,$$

$$\theta_4 = \frac{2(\varphi(x_0) + 6T\|g_0\|)}{\varphi(x_0)} < \sigma,$$

$$\theta_5 = 2\varphi_1 < \sigma,$$

$$\theta_6 = \frac{2}{3}\varphi(x_0) + 4\|g_0\|T < \sigma.$$

This implies that, for σ and ρ such that

$$\sigma > \sigma_2 = \max(\theta_3, \theta_4, \theta_5, \theta_6) \qquad \text{and} \qquad \rho < \kappa_4,$$

the operator A is a contracting mapping on the set $B(g_0, \rho)$.

Thus, if the numbers σ and ρ satisfy the conditions $\sigma > \max(\sigma_1, \sigma_2)$ and $\rho < \min(\kappa_3, \kappa_4)$, then, by the principle of contracting mappings, the operator A has a unique fixed point on the set $B(g_0, \rho)$.

Theorem 2 is proved.

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