ISSN 1990-4789, Journal of Applied and Industrial Mathematics, 2022, Vol. 16, No. 1, pp. 22–44. © Pleiades Publishing, Ltd., 2022.
Russian Text © The Author(s), 2022, published in Sibirskii Zhurnal Industrial'noi Matematiki, 2022, Vol. 25, No. 1, pp. 14–38.

Problem of Determining the Two-Dimensional Kernel of the Viscoelasticity Equation with a Weakly Horizontal Inhomogeneity

D. K. Durdiev[1*] and J. Sh. Safarov[1,2]**

[1] *Romanovskii Institute of Mathematics, Uzbekistan Academy of Sciences, Tashkent, 100170 Uzbekistan*
[2] *Tashkent University of Information Technologies, Tashkent, 100084 Uzbekistan*
*e-mail: *durdiev65@mail.ru, **j.safarov65@mail.ru*

Received August 11, 2021; revised October 1, 2021; accepted October 21, 2021

Abstract—In a domain bounded with respect to the variable z and having a weakly horizontal inhomogeneity, we consider the problem of determining the convolution kernel $k(t,x)$, $t\in[0,T]$, $x\in\mathbb{R}$, occurring in the viscoelasticity equation. It is assumed that this kernel weakly depends on the variable x and has a power series expansion in a small parameter ε. A method is constructed for finding the first two coefficients $k_0(t)$ and $k_1(t)$ of this expansion. Theorems on the global unique solvability of the problem are obtained.

Keywords: *inverse problem, viscoelasticity, integral equation, integral kernel, Banach theorem*

DOI: 10.1134/S1990478922010033

1. STATEMENT OF THE PROBLEM

Consider the two-dimensional hyperbolic integro-differential equation

$$u_{tt}=\Delta u+\int_0^t k(\tau,x)\Delta u(t-\tau,x,z)\,d\tau \tag{1}$$

in the domain $D:=\{(t,x,z)\mid(t,x)\in\mathbb{R}^2,\ 0<z<l\}$ bounded with respect to the variable z with the initial and boundary conditions

$$u|_{t<0}\equiv 0, \tag{2}$$

$$\left.\left(\frac{\partial u}{\partial z}+\int_0^t k(\tau,x)\frac{\partial u}{\partial z}(t-\tau,x,z)\,d\tau\right)\right|_{z=0}=\delta(x)\delta'(t), \tag{3}$$

$$\left.\left(\frac{\partial u}{\partial z}+\int_0^t k(\tau,x)\frac{\partial u}{\partial z}(t-\tau,x,z)d\tau\right)\right|_{z=l}=0, \tag{4}$$

where $\Delta=\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial z^2}$ is the Laplace operator, $\delta(\cdot)$ is the Dirac delta function, and $l>0$ is some number.

Equation (1) arises in the theory of viscoelastic bodies with constant density and Lamé coefficients. Here the function $u(t,x,z)$ has the physical meaning of the y-component of the body particle displacement vector. The integral operator in this equation describes the influence of prehistory on the process of propagation of elastic waves caused by the lumped force (3) applied at the boundary of the domain D. The boundary condition at the left end of the domain in question means that one of the stress tensor components has an instantaneous directional force. At the same time, there is no such force at the right end.

We state the inverse problem as follows: find the kernel $k(t,x)$, $t>0$, $x\in\mathbb{R}$, of the integral term in (1) given the values of the solution of problem (1)–(4) for $z=0$, i.e., the function

$$u(t,x,0)=g(t,x),\quad t>0,\quad x\in\mathbb{R}. \tag{5}$$

Nowadays, the study of one- and multidimensional inverse problems of determining the kernel of the integral term in integro–differential equations of hyperbolic type has become the object of research by many scientists. The papers [1, 2] considered one-dimensional problems of finding the kernel of hyperbolic integro–differential equations with distributed perturbation sources, and the papers [3–12] deal with problems of finding the kernel of an integro-differential equation with a delta function on the right-hand side or in the boundary condition. For the problems posed in these papers, existence and uniqueness theorems are proved, and stability estimates are obtained based on the contraction mapping principle.

The papers [13–17] studied the problems of determining multidimensional memory from Maxwell's and viscoelasticity integro-differential equations. Estimates of the conditional stability of the solution of the considered inverse problems were obtained. In the papers [18–21], for multidimensional inverse problems of finding the kernel in second-order hyperbolic integro-differential equations, theorems on the unique local solvability in the class of functions analytic in the space variables and continuous in time were proved.

The papers [22–24] deal with the numerical solution of direct problems for a system of viscoelasticity equations, and [25–35], with the numerical solution of direct and inverse problems for hyperbolic integro–differential equations and systems. In particular, they construct a numerical method for determining the parameters of the memory function for a horizontally layered medium.

Problem (1)–(5) belongs to multidimensional inverse problems for differential equations. In this paper, developing the methods for solving inverse problems used in [37], we study the problem of reconstructing the convolution kernel of the integral term of Eq. (1). It is assumed that the kernel $k(t,x)$ weakly depends on the horizontal variable x,

$$k(t,x)=k_0(t)+\varepsilon x k_1(t)+\cdots, \tag{6}$$

where ε is a small parameter.

The main result of this paper is that it proposes a method for finding one-dimensional functions $k_0(t)$ and $k_1(t)$ up to a quantity of the order $O(\varepsilon^2)$. To this end, as we will see below, it suffices to specify the Fourier transform of the function $g(t,x)$ with respect to x for one fixed value of the parameter.

2. REDUCTION OF THE PROBLEM TO A SERIES OF ONE-DIMENSIONAL INVERSE PROBLEMS

We seek a solution of the direct problem (1)–(4) in the form of a series in powers of ε; i.e.,

$$u(t,z,x)=u_0(t,z,x)+\varepsilon u_1(t,z,x)+\cdots. \tag{7}$$

Substituting (7) into Eq. (1) and matching the coefficients of like powers of ε, as a result, we obtain a recursive system of direct problems from which we can find u_0, u_1, and so on. Then, obviously, according to formula (7), the function $g(t,x)$ will have the same structure

$$g(t,x)=g_0(t,x)+\varepsilon g_1(t,x)+\cdots. \tag{8}$$

It can readily be verified that the functions $u_n(t,z,x)$ (and hence also $g_n(x,t)$) are even in x for even n and odd for odd n. This can be seen from the direct problems given below: u_n with even n (odd n) is a solution of the problem with data even (odd) in x. Thereby, given a known function $g(t,x)$ we can find $g_0(t,x)$ and $g_1(t,x)$ with accuracy $O(\varepsilon^2)$,

$$g_0(t,x) = \big(g(t,x) + g(t,-x)\big)/2,$$
$$g_1(t,x) = \big(g(t,x) - g(t,-x)\big)/2.$$

Let us proceed to solving the problem. Using the expansions (7) of u and (6) of k and matching the coefficients of like powers of ε, we conclude that the inverse problem (1)–(5) splits into the following problem of successively determining $k_0, k_1, \ldots$:

$$u_{0tt} = \Delta u_0 + \int\limits_0^t k_0(t-\tau)\Delta u_0(\tau,x,z)\,d\tau, \quad (t,x,z) \in D, \tag{9}$$

$$u_0|_{t<0} \equiv 0, \tag{10}$$

$$\frac{\partial}{\partial z}\left(u_0 + \int\limits_0^t k_0(t-\tau)u_0(\tau,x,z)\,d\tau\right)\bigg|_{z=0} = \delta(x)\delta'(t), \tag{11}$$

$$\frac{\partial}{\partial z}\left(u_0 + \int\limits_0^t k_0(t-\tau)u_0(\tau,x,z)\,d\tau\right)\bigg|_{z=l} = 0, \tag{12}$$

$$u_0|_{z=0} = g_0(t,x), \quad (t,x) \in \mathbb{R}^2, \tag{13}$$

$$u_{ntt} = \Delta u_n + \int\limits_0^t \sum_{j=0}^n x^j k_j(t-\tau)\Delta u_{n-j}(\tau,x,z)\,d\tau, \quad (t,x,z) \in D, \tag{14}$$

$$u_n|_{t<0} \equiv 0, \tag{15}$$

$$\frac{\partial}{\partial z}\left(u_n + \int\limits_0^t \sum_{j=0}^n x^j k_j(t-\tau)u_{n-j}(\tau,x,z)\,d\tau\right)\bigg|_{z=0} = 0, \tag{16}$$

$$\frac{\partial}{\partial z}\left(u_n + \int\limits_0^t \sum_{j=0}^n x^j k_j(t-\tau)u_{n-j}(\tau,x,z)\,d\tau\right)\bigg|_{z=l} = 0, \tag{17}$$

$$u_n|_{z=0} = g_n(t,x), \quad (t,x) \in \mathbb{R}^2, \quad n = 1, 2, \ldots. \tag{18}$$

In what follows, we will be interested in the problems for the functions $k_0(t)$ and $k_1(t)$. To this end, it suffices to consider problems (9)–(13) and (14)–(18) for $n = 1$.

Let us proceed from the functions $u_j(t,x,z)$, $j = 1, 2, \ldots$, to their exponential Fourier transforms in the variable x,

$$\tilde{u}_i(t,\lambda,z) = \int\limits_{\mathbb{R}} u_i(t,x,z)e^{-i\lambda x}\,dx, \quad \lambda \in \mathbb{R}.$$

The Fourier transform of the functions $u_j(t,x,z)$, $j = 1, 2, \ldots$, exists for any finite t, since each u_j is the sum of some singular generalized function of finite order and a regular function, the supports of the functions u_j being bounded.

In terms of the functions $\tilde{u}_j$, the inverse problems (9)–(13) and (14)–(18) look like problems of finding $k_0(t)$ and $k_1(t)$ from the following problems:

$$\tilde{u}_{0tt} = \left[\frac{\partial^2}{\partial z^2} - \lambda^2\right]\left(\tilde{u}_0 + \int_0^t k_0(t-\tau)\tilde{u}_0(\tau,\lambda,z)\,d\tau\right), \quad (t,\lambda)\in\mathbb{R}^2, \quad z\in(0,l), \tag{19}$$

$$\tilde{u}_0|_{t<0} \equiv 0, \tag{20}$$

$$\frac{\partial}{\partial z}\left(\tilde{u}_0 + \int_0^t k_0(t-\tau)\tilde{u}_0(\tau,\lambda,z)\,d\tau\right)\Bigg|_{z=0} = \delta'(t), \tag{21}$$

$$\frac{\partial}{\partial z}\left(\tilde{u}_0 + \int_0^t k_0(t-\tau)\tilde{u}_0(\tau,\lambda,z)\,d\tau\right)\Bigg|_{z=l} = 0, \tag{22}$$

$$\tilde{u}_0|_{z=0} = \tilde{g}_0(\lambda,t), \quad (\lambda,t)\in\mathbb{R}^2, \tag{23}$$

$$\begin{aligned}\tilde{u}_{1tt} &= \left[\frac{\partial^2}{\partial z^2} - \lambda^2\right]\left(\tilde{u}_1 + \int_0^t k_0(t-\tau)\tilde{u}_1(\tau,\lambda,z)\,d\tau\right)\\ &- i\int_0^t k_1(t-\tau)\left[2\lambda\tilde{u}_0(\tau,\lambda,z) + \lambda^2\tilde{u}_{0\lambda}(\tau,\lambda,z) - \frac{\partial^2\tilde{u}_{0\lambda}}{\partial z^2}\right]d\tau, \quad (t,\lambda)\in\mathbb{R}^2, \quad z\in(0,l),\end{aligned} \tag{24}$$

$$\tilde{u}_1|_{t<0} \equiv 0, \tag{25}$$

$$\frac{\partial}{\partial z}\left(\tilde{u}_1 + \int_0^t \left(k_0(t-\tau)\tilde{u}_1(\tau,\lambda,z)d\tau - ik_1(t-\tau)\frac{\partial\tilde{u}_0}{\partial\lambda}(t-\tau,\lambda,z)\right)d\tau\right)\Bigg|_{z=0} = 0, \tag{26}$$

$$\frac{\partial}{\partial z}\left(\tilde{u}_1 + \int_0^t \left(k_0(t-\tau)\tilde{u}_1(\tau,\lambda,z)d\tau - ik_1(t-\tau)\frac{\partial\tilde{u}_0}{\partial\lambda}(t-\tau,\lambda,z)\right)d\tau\right)\Bigg|_{z=l} = 0, \tag{27}$$

$$\tilde{u}_1|_{z=0} = \tilde{g}_1(t,\lambda), \quad (t,\lambda)\in\mathbb{R}^2, \tag{28}$$

respectively, where $\tilde{g}_m(t,\lambda) = \int_{\mathbb{R}} g_m(t,x)e^{-i\lambda x}\,dx$, $m = 0,1,\dots$.

In the sections to follow, we study the inverse problems (19)–(23) and (24)–(28).

3. PROBLEM OF DETERMINING THE FUNCTIONS k_0 AND $\tilde{u}_0$

The inverse problem (19)–(23) is overdetermined, because a function of two variables is specified (condition (23)) to determine one function $k_0(t)$. Below we will see that for the unique solvability of the inverse problem, it suffices to specify the Fourier transform of the function $g_0(t,x)$ for one fixed value of the transform parameter. In what follows, without stipulating each time, we will assume that the parameter λ in equalities (19)–(28) is fixed and $\lambda \neq 0$ everywhere.

We introduce a new function $v(t,\lambda,z)$ defining it by the relation

$$v(t,\lambda,z) := \tilde{u}_0(t,\lambda,z) + \int_0^t k_0(t-\tau)\tilde{u}_0(\tau,\lambda,z)\,d\tau.$$

It can readily be verified that the function $\tilde{u}_0(t,\lambda,z)$ is expressed via $v(t,\lambda,z)$ by the formula

$$\tilde{u}_0(t,\lambda,z) = v(t,\lambda,z) + \int_0^t r(t-\tau)v(\tau,\lambda,z)\,d\tau, \tag{29}$$

where

$$r(t) = -k_0(t) - \int_0^t k_0(t-\tau)r(\tau)\,d\tau. \tag{30}$$

For simplicity, we assume that $k_0(0) = k_0'(0) = 0$. Consequently, as follows from (30), $r(0) = r'(0) = 0$. It can readily be seen in the sequel that this can be achieved by appropriately choosing $f(t)$ for $t = 0$. For the new functions $v(t,\lambda,z)$ and $r(t)$, Eqs. (19)–(22), in view of (23), acquire the form

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial z^2} - \lambda^2 v - \int_0^t h(t-\tau)v(\tau,\lambda,z)d\tau,\ (t,z)\in D, \tag{31}$$

$$v|_{t<0} \equiv 0, \tag{32}$$

$$\left.\frac{\partial v}{\partial z}\right|_{z=0} = \delta'(t),\ \left.\frac{\partial v}{\partial z}\right|_{z=l} = 0, \tag{33}$$

where we have introduced the notation $h(t) := r''(t)$. The additional condition (23) is as follows:

$$v(t,\lambda,0) = \tilde{g}_0(t,\lambda) + \int_0^t k_0(t-\tau)\tilde{g}_0(\tau,\lambda)\,d\tau. \tag{34}$$

Lemma 1. *The following equalities hold*:

$$v(t,\lambda,z) \equiv 0, \quad (z,t)\in D_1 := \big\{(z,t) \mid 0<z<l,\ 0<t<z\big\}, \tag{35}$$

$$\begin{aligned} v(t,\lambda,z) = -\delta(t-z) &+ \int_0^{t-z}\int_0^{\tau/2}\left[\lambda^2 v(\tau-\xi,\lambda,\xi) + \int_0^{\tau-2\xi} h(\alpha)v(\tau-\xi-\alpha,\lambda,\xi)\,d\alpha\right]d\xi d\tau \\ &+ \int_{t-z}^{t}\int_{\tau-t+z}^{\frac{2\tau-t+z}{2}}\left[\lambda^2 v(2\tau-t+z-\xi,\lambda,\xi) + \int_0^{2\tau-t+z-2\xi} h(\alpha)v(2\tau-t+z-\xi-\alpha,\lambda,\xi)\,d\alpha\right]d\xi d\tau \end{aligned} \tag{36}$$

for $(z,t)\in D_2 := \{(z,t) \mid 0<z<l,\ z<t<2l-z\}$.

Proof. In the domain $D_0 := \{(z,t) \mid 0<z<l,\ 0<t<l/2-|z-l/2|\} \subset D_1$, by d'Alembert's formula we obtain a homogeneous integral equation of the Volterra type for $v(t,\lambda,z)$; it follows that $v(t,\lambda,z)\equiv 0$.

In the domain $D_1 \setminus D_0$, we represent the wave operator in the form the product

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right),$$

integrate Eq. (31) along a segment of the fixed characteristic of the pencil $\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial z}$, and use condition (32) to find

$$\left.\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right)v\right|_{z=l} = -\int_{t/2}^{t}\left[\lambda^2 v(\tau,\lambda,\tau-t+l,) + \int_0^{\tau} h(\tau-\alpha)v(\alpha,\lambda,\tau-t+l)\,d\alpha\right]d\tau, \quad t\in(0,l).$$

In view of the boundary condition (33) for $z = l$, from this equality we obtain

$$v(t,\lambda,l) = -\int_0^t \int_{\tau/2}^{\tau} \left[\lambda^2 v(\tau_1, \lambda\tau_1 + \tau + l) + \int_0^{\tau_1} h(\tau_1 - \alpha) v(\alpha, \lambda, \tau_1 - \tau + l)\, d\alpha\right] d\tau_1 d\tau, \quad t \in (0, l).$$

Making the change of variables τ_1 for ξ by the formula $\tau_1 - \tau + l = \xi$ in the inner integral, we rewrite the last equation in the form

$$v(t,\lambda,l) = -\int_0^t \int_{l-\tau/2}^{l} \left[\lambda^2 v(\tau - l + \xi, \lambda, \xi) + \int_0^{\tau - l + \xi} h(\tau - l + \xi - \alpha) v(\alpha, \lambda, \xi)\, d\alpha\right] d\xi d\tau, \quad t \in (0, l). \tag{37}$$

Integrating Eq. (31) along the characteristic $\dfrac{dz}{dt} = 1$, we obtain

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right) v(t,\lambda,z) = -\int_{\frac{l+z-t}{2}}^{z} \left[\lambda^2 v(\xi + t - z, \lambda, \xi) + \int_0^{\xi + t - z} h(\xi + t - z - \alpha) v(\alpha, \lambda, \xi)\, d\alpha\right] d\xi, \quad (t,z) \in D_1 \backslash D_0.$$

Further, using formula (37), we find an equation for $v(t,\lambda,z)$ in the domain $D_1 \backslash D_0$,

$$v(t,\lambda,z) = -\int_0^{t+z-l} \int_{l-\tau/2}^{l} \left[\lambda^2 v(\tau - l + \xi, \lambda, \xi) + \int_0^{\tau - l + \xi} h(\tau - l + \xi - \alpha) v(\alpha, \lambda, \xi)\, d\alpha\right] d\xi d\tau$$
$$- \int_{t+z-l}^{t} \int_{\frac{l+t+z-2\tau}{2}}^{t+z-\tau} \left[\lambda^2 v(\xi, \xi + 2\tau - t - z) + \int_0^{\xi + 2\tau - t - z} h(\xi + 2\tau - t - z - \alpha) v(\alpha, \lambda, \xi)\, d\alpha\right] d\xi d\tau.$$

The resulting equation is a homogeneous equation of the Volterra type with continuous kernel. Consequently, $v(t,\lambda,z) \equiv 0$ in the domain $D_1 \backslash D_0$.

Now consider the domain $D_2 := \{(t,z) \mid 0 < z < l,\ z < t < 2l - z\}$. Integrating (31) along the corresponding characteristic, we find

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial z}\right) v|_{x=0} = \int_0^{t/2} \left[\lambda^2 v(t - \xi, \lambda, \xi) + \int_0^{t-2\xi} h(\alpha) v(t - \xi - \alpha, \lambda, \xi)\, d\alpha\right] d\xi,\ t \in (0, 2l).$$

Combining this with the boundary condition (33), we find $v(t,\lambda,z)$ for $z = 0$,

$$v|_{z=0} = -\delta(t) + \int_0^t \int_0^{\tau/2} \left[\lambda^2 v(\tau - \xi, \lambda, \xi) + \int_0^{\tau - 2\xi} h(\alpha) v(\tau - \xi - \alpha, \lambda, \xi)\, d\alpha\right] d\xi d\tau, \quad t \in (0, 2l).$$

Using the last equality and integrating (31) along the characteristics $\dfrac{dz}{dt} = -1$ and $\dfrac{dz}{dt} = 1$, we obtain the integral equation (36) for $v(t,\lambda,z)$ in the domain D_2. The proof of Lemma 1 is complete. □

In the outer integral in the last term in Eq. (36), we replace the integration variable τ by β using the formula $t-\tau=\beta$ and represent the function $v(t,\lambda,z)$ in the form

$$v(t,\lambda,z)=\tilde{v}(t,\lambda,z)-\delta(t-z), \tag{38}$$

where $\tilde{v}(t,\lambda,z)$ is a regular function. Then Eq. (36) becomes

$$\begin{aligned}\tilde{v}(t,\lambda,z)=-\frac{\lambda^2}{2}t+\int_0^{t-z}\int_0^{\tau/2}\left[\lambda^2\tilde{v}(\tau-\xi,\lambda,\xi)-h(\tau-2\xi)+\int_0^{\tau-2\xi}h(\alpha)\tilde{v}(\tau-\xi-\alpha,\lambda,\xi)\,d\alpha\right]d\xi d\tau\\ +\int_0^{z}\int_{z-\beta}^{\frac{t-2\beta+z}{2}}\Bigg[\lambda^2\tilde{v}(t-2\beta+z-\xi,\lambda,\xi)-h(t-2\beta+z-2\xi)\\ +\int_0^{t-2\beta+z-2\xi}h(\alpha)\tilde{v}(t-2\beta+z-\xi-\alpha,\lambda,\xi)\,d\alpha\Bigg]d\xi d\beta;\end{aligned} \tag{39}$$

consequently, $\tilde{v}(t,\lambda,z)|_{t=z+0}=-z\lambda^2/2$. For the inverse problem to be solvable, note that the function $\tilde{g}_0(t,\lambda)$ must have the structure $\tilde{g}_0(t,\lambda)=\tilde{g}_{00}(t,\lambda)-\delta(t)$. Then from condition (34) we obtain

$$\tilde{v}(t,\lambda,0)=\tilde{g}_{00}(t,\lambda)-k_0(t)+\int_0^t k_0(t-\tau)\tilde{g}_{00}(\tau,\lambda)\,d\tau. \tag{40}$$

Thus, in the domain D_2 the function $\tilde{v}(t,\lambda,z)$ satisfies the equations

$$\frac{\partial^2\tilde{v}}{\partial t^2}=\frac{\partial^2\tilde{v}}{\partial z^2}-\lambda^2\tilde{v}-h(t-z)-\int_0^t h(t-\tau)\tilde{v}(\tau,\lambda,z)\,d\tau,\quad (t,z)\in D_2, \tag{41}$$

$$\tilde{v}|_{t<0}\equiv 0, \tag{42}$$

$$\left.\frac{\partial\tilde{v}}{\partial z}\right|_{z=0}=0,\quad \left.\frac{\partial\tilde{v}}{\partial z}\right|_{z=l}=0. \tag{43}$$

In Eq. (39), we set $z=0$ and use the additional condition (40),

$$\begin{aligned}\tilde{g}_{00}(t,\lambda)-k_0(t)+\int_0^t k_0(t-\tau)\tilde{g}_{00}(\tau,\lambda)\,d\tau=-\frac{\lambda^2}{2}t\\ +\int_0^t\int_0^{\tau/2}\left[\lambda^2\tilde{v}(\tau-\xi,\lambda,\xi)-h(\tau-2\xi)+\int_0^{\tau-2\xi}h(\alpha)\tilde{v}(\tau-\xi-\alpha,\lambda,\xi)\,d\alpha\right]d\xi d\tau,\quad t\in(0,2l).\end{aligned}$$

Hence, in particular, we obtain the necessary condition $\tilde{g}_{00}(0,\lambda)=0$ for the solvability of the inverse problem.

Differentiating the preceding integral equation two times with respect to t and solving it for $h(t)$, we obtain

$$\begin{aligned}h(t)=-\frac{\lambda^4}{4}t-2\tilde{g}''_{00t}(t,\lambda)+2k''_0(t)-2\int_0^t k''_0(t-\tau)\tilde{g}_{00}(\tau,\lambda)\,d\tau\\ +2\int_0^{t/2}\left[\lambda^2\tilde{v}_t(t-\xi,\lambda,\xi)-\frac{\lambda^2}{2}\xi h(t-2\xi)+\int_0^{t-2\xi}h(\alpha)\tilde{v}_t(t-\xi-\alpha,\lambda,\xi)\,d\alpha\right]d\xi.\end{aligned} \tag{44}$$

Hence

$$h'(t) = -\frac{3\lambda^4}{4} + \frac{\lambda^2 t^2}{4}h(0) - 2\tilde{g}'''_{00t}(t,\lambda) + 2k_0'''(t) + 2\int\limits_0^t k_0''(\tau)\tilde{g}'_{00}(t-\tau,\lambda)\,d\tau - \frac{\lambda^2}{4}\int\limits_0^t h(\xi)\,d\xi$$
$$+ 2\int\limits_0^{t/2}\left[\lambda^2\tilde{v}_{tt}(t-\xi,\lambda,\xi) - \left(\frac{\lambda^2}{2} + \frac{1}{2}h(0)\xi\right)h(t-2\xi) + \int\limits_0^{t-2\xi} h(\alpha)\tilde{v}_{tt}(t-\xi-\alpha,\lambda,\xi)\,d\alpha\right]d\xi. \tag{45}$$

For further study, we need to know the derivatives $\tilde{v}_t$ and $\tilde{v}_{tt}$ of the function $\tilde{v}$. Let us calculate them,

$$\tilde{v}_t(t,\lambda,z) = -\frac{\lambda^2}{2} - \frac{\lambda^4}{8}tz - \frac{1}{2}h(t-z)z$$
$$+ \int\limits_0^{\frac{t-z}{2}}\left[\lambda^2\tilde{v}(t-z-\xi,\lambda,\xi) - h(t-z-2\xi) + \int\limits_0^{t-z-2\xi} h(\alpha)\tilde{v}(t-z-\xi-\alpha,\lambda,\xi)\,d\alpha\right]d\xi$$
$$+ \int\limits_0^{z}\int\limits_{z-\beta}^{\frac{t+z-2\beta}{2}}\left[\lambda^2\tilde{v}_t(t+z-2\beta-\xi,\lambda,\xi) - \frac{\lambda^2}{2}\xi h(t-2\beta+z-2\xi)\right.$$
$$\left. + \int\limits_0^{t-2\beta+z-2\xi} h(\alpha)\tilde{v}_t(t-2\beta+z-\xi-\alpha,\lambda,\xi)\,d\alpha\right]d\xi d\beta, \tag{46}$$

$$\tilde{v}_{tt}(t,\lambda,z) = -\frac{\lambda^4}{8}t(1+z) - \frac{z}{2}h'(t-z) + \frac{1}{2}m(z)h(t-z)$$
$$+ \int\limits_0^{\frac{t-z}{2}}\left[\lambda^2\tilde{v}_t(t-z-\xi,\lambda,\xi) - \frac{\lambda^2}{2}\xi h(t-z-2\xi) + \int\limits_0^{t-z-2\xi} h(\alpha)\tilde{v}_t(t-z-\xi-\alpha,\lambda,\xi)\,d\alpha\right]d\xi$$
$$- \int\limits_0^{z}\int\limits_{z-\beta}^{\frac{t+z-2\beta}{2}}\left[\lambda^2\tilde{v}_{tt}(t+z-2\beta-\xi,\lambda,\xi) - 1\left(\frac{\lambda^4}{8}\xi^2 - \frac{h(0)}{2}\xi + \frac{\lambda^2}{2}\right)h(t+z-2\beta-2\xi)\right.$$
$$\left. - \int\limits_0^{t+z-2\beta-2\xi} h(\alpha)\tilde{v}_{tt}(t+x-2\beta-\xi-\alpha,\lambda,\xi)\,d\alpha\right]d\xi d\beta, \tag{47}$$

where we have introduced the notation $m(z) = \lambda^2 z^2/4 - 1$. To close the system of integral equations (39) and (44)–(47), we use the following obvious relations:

$$k_0(t) = \int\limits_0^t (t-\tau)k_0''(\tau)\,d\tau, \tag{48}$$

$$k_0'(t) = \int\limits_0^t k_0''(\tau)\,d\tau, \tag{49}$$

$$k_0''(t) = -h(t) - \lambda^2 k_0(t) - \int\limits_0^t h(t-\tau)k_0(\tau)\,d\tau, \tag{50}$$

$$k_0'''(t) = -h'(t) - \lambda^2 k_0'(t) - \int_0^t h(\tau) k_0'(t-\tau)\, d\tau. \tag{51}$$

Theorem 1. *Assume that* $\tilde{g}_{00}(t,\lambda) \in C^3[0,2l]$, $\tilde{g}_{00}(+0,\lambda) = 0$, *and* $\tilde{g}'_{00t}(+0,\lambda) = -\lambda^2/2$. *Then there exists a unique solution* $k_0(t) \in C^3[0,2l]$ *of the inverse problem* (9)–(13) *for each* $l > 0$.

Proof. Equations (39) and (44)–(51) determine a closed system of integral equations for the nine unknown functions $\tilde{v}(t,\lambda,z)$, $\tilde{v}_t(t,\lambda,z)$, $\tilde{v}_{tt}(t,\lambda,z)$, $h(t)$, $h'(t)$, $k_0(t)$, $k_0'(t)$, $k_0''(t)$, and $k_0'''(t)$ in D_2. Note that it would have been sufficient to consider the six functions $\tilde{v}(t,\lambda,z)$, $\tilde{v}_t(t,\lambda,z)$, $h(t)$, $k_0(t)$, $k_0'(t)$, and $k_0''(t)$, which also form a closed system in D_2, but when proving Theorem 3, there arises a need for the functions $\tilde{v}_{tt}(t,\lambda,z)$, $h'(t)$, and $k_0'''(t)$; therefore, we will consider the system of nine functions from the very beginning. This system can be represented in the form of the operator equation

$$A\varphi = \varphi, \tag{52}$$

where

$$\begin{aligned}\varphi &= \big[\varphi_1(t,\lambda,z), \varphi_2(t,\lambda,z), \varphi_3(t,\lambda,z), \varphi_4(t), \varphi_5(t), \varphi_6(t), \varphi_7(t), \varphi_8(t), \varphi_9(t)\big] \\ &= \Big[\tilde{v}(t,\lambda,z), \tilde{v}_t(t,\lambda,z) + \frac{z}{2} h(t-z), \tilde{v}_{tt}(t,\lambda,z) + \frac{z}{2} h'(t-z) \\ &\qquad + \frac{1}{2} m(z) h(t-z), h(t) - 2k_0''(t), h'(t) - 2k_0'''(t), k_0(t), k_0'(t), k_0''(t) \\ &\qquad + h(t) + \lambda^2 k_0(t), k_0'''(t) + h'(t) + \lambda^2 k_0'(t)\Big]\end{aligned}$$

is a vector function with components φ_i, $i = 1,\dots,9$, and the operator A is defined on the set of functions $\varphi \in C[D_2]$ and, in accordance with (39) and (44)–(51), has the form

$$A = (A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9),$$

where

$$\begin{aligned} A_1\varphi &= \varphi_{01} + \int_0^{t-z}\int_0^{\tau/2}\Bigg[\lambda^2 \varphi_1(t-\xi,\lambda,\xi) - \frac{1}{3}(2\varphi_8(t-2\xi) + \varphi_4(t-2\xi) - 2\lambda^2\varphi_6(t-2\xi)) \\ &\qquad + \frac{1}{3}\int_0^{t-2\xi} [2\varphi_8(\alpha) + \varphi_4(\alpha) - 2\lambda^2\varphi_6(\alpha)]\varphi_1(\xi,\lambda,t-\xi-\alpha)\, d\alpha\Bigg] d\xi d\tau \\ &\quad + \int_{t-z}^{t}\int_{\tau-t+z}^{\frac{2\tau-t+z}{2}}\Bigg[\lambda^2\varphi_1(t-2\beta+z-\xi,\lambda,\xi) \\ &\qquad - \frac{1}{3}\big(2\varphi_8(t-2\beta+z-2\xi) + \varphi_4(t-2\beta+z-2\xi) - 2\lambda^2\varphi_6(t-2\beta+z-2\xi)\big) \\ &\qquad + \frac{1}{3}\int_0^{2\tau-t+z-2\xi} \big[2\varphi_8(\alpha) + \varphi_4(\alpha) - 2\lambda^2\varphi_6(\alpha)\big] \\ &\qquad\qquad \times \varphi_1(2\tau-t+z-\xi-\alpha,\lambda,\xi)\, d\alpha\Bigg] d\xi d\tau, \end{aligned} \tag{53}$$

$$
\begin{aligned}
A_2\varphi = \varphi_{02} + \int\limits_0^{\frac{t-z}{2}} \Bigg[& \varphi_1(t-z-\xi,\lambda,\xi) \\
& - \frac{1}{3}\big(2\varphi_8(t-z-2\xi) + \varphi_4(t-z-2\xi) - 2\lambda^2\varphi_6(t-z-2\xi)\big) \\
& - \frac{1}{3}\int\limits_0^{t-z-2\xi} \big[2\varphi_8(\alpha) + \varphi_4(\alpha) - 2\lambda^2\varphi_6(\alpha)\big]\varphi_1(t-z-\xi-\alpha,\lambda,\xi)\,d\alpha\Bigg]\,d\xi \\
+ \int\limits_0^{z}\int\limits_{x-\beta}^{\frac{t+z-2\beta}{2}} \Bigg[& \lambda^2\varphi_2(t+z-2\beta-\xi,\lambda,\xi) \\
& - \frac{\lambda^2}{6}\xi\big[2\varphi_8(t+z-2\beta-2\xi) + \varphi_4(t+z-2\beta-\xi) - 2\lambda^2\varphi_6(t+z-2\beta-2\xi)\big] \\
& + \frac{1}{3}\int\limits_0^{t+z-2\beta-2\xi} \big[2\varphi_8(\alpha) + \varphi_4(\alpha) - 2\lambda^2\varphi_6(\alpha)\big] \\
& \quad \times\Big(\lambda^2\varphi_2(t+z-2\beta-\xi,\lambda,\xi) - \frac{\lambda^2}{6}\xi\big[2\varphi_8(t+z-2\beta-2\xi) \\
& \qquad + \varphi_4(t+z-2\beta-\xi) - 2\lambda^2\varphi_6(t+z-2\beta-2\xi)\big]\Big)d\alpha\Bigg]\,d\xi d\beta,
\end{aligned}
\tag{54}
$$

$$
\begin{aligned}
A_3\varphi = \varphi_{03} + \int\limits_0^{\frac{t-z}{2}} \Bigg[& \lambda^2\varphi_2(t-z-\xi,\lambda,\xi) \\
& - \frac{\lambda^2}{6}\xi\big[2\varphi_8(t-z-2\xi) + \varphi_4(t-z-2\xi) - 2\lambda^2\varphi_6(t-z-2\xi)\big] \\
& + \frac{1}{3}\int\limits_0^{t-z-2\xi} \big[2\varphi_8(\alpha) + \varphi_4(\alpha) - 2\lambda^2\varphi_6(\alpha)\big]\Big(\lambda^2\varphi_2(t-z-\xi-\alpha,\lambda,\xi) \\
& - \frac{\lambda^2}{6}\xi\big[2\varphi_8(t-z-2\xi-\alpha) + \varphi_4(t-z-2\xi-\alpha) - 2\lambda^2\varphi_6(t-z-2\xi-\alpha)\big]\Big)d\alpha\Bigg]\,d\xi \\
+ \int\limits_0^{z}\int\limits_{z-\beta}^{\frac{t+z-2\beta}{2}} \Bigg[& \lambda^2\varphi_2(t+z-2\beta-\xi,\lambda,\xi) - \frac{\lambda^2}{6}\big[2\varphi_8(t+z-2\beta-2\xi) \\
& \qquad + \varphi_4(t+z-2\beta-2\xi) - 2\lambda^2\varphi_6(t+z-2\beta-2\xi)\big] \\
& - \frac{\lambda^2\xi}{6}\big[2\varphi_9(t+x-2\beta-\xi) + \varphi_5(t+x-2\beta-\xi) - 2\lambda^2\varphi_7(t+x-2\beta-\xi)\big] \\
& - \Big(\frac{h(0)}{6}\xi - \frac{\lambda^2}{6}\Big)\big[2\varphi_8(t+z-2\beta-2\xi) \\
& \qquad + \varphi_4(t+z-2\beta-2\xi) - 2\lambda^2\varphi_6(t+z-2\beta-2\xi)\big] \\
& + \frac{1}{3}\int\limits_0^{t+z-2\beta-2\xi} \big[2\varphi_8(\alpha) + \varphi_4(\alpha) - 2\lambda^2\varphi_6(\alpha)\big]
\end{aligned}
$$

$$\times\Big[\varphi_3(t+x-2\beta-\xi-\alpha,\lambda,\xi)$$
$$-\frac{\xi}{6}\big[2\varphi_9(t+x-2\beta-2\xi-\alpha)+\varphi_5(t+x-2\beta-2\xi-\alpha)$$
$$-2\lambda^2\varphi_7(t+x-2\beta-2\xi-\alpha)\big]+\frac{1}{6}\big[2\varphi_8(t+z-2\beta-2\xi-\alpha)$$
$$+\varphi_4(t+z-2\beta-2\xi-\alpha)-2\lambda^2\varphi_6(t+z-2\beta-2\xi-\alpha)\big]\Big]d\alpha\Bigg]d\xi d\beta, \qquad (55)$$

$$A_4\varphi=\varphi_{04}-\frac{1}{3}\int\limits_0^t\big[2\varphi_8(\tau)-\varphi_4(\tau)-2\lambda^2\varphi_6(\tau)\big]\tilde g_{00}(t-\tau,\lambda)\,d\tau$$
$$-2\int\limits_0^{t/2}\Bigg[\lambda^2\Big[\varphi_2(t-\xi,\lambda,\xi)$$
$$-\frac{\xi}{6}\big[2\varphi_8(t-2\xi)+\varphi_4(t-2\xi)-2\lambda^2\varphi_6(t-2\xi)\big]-\lambda^2\varphi_5(t-2\xi)\Big] \qquad (56)$$
$$-\frac{1}{3}\int\limits_0^{t-2\xi}\big[2\varphi_8(\alpha)+\varphi_4(\alpha)-2\lambda^2\varphi_6(\alpha)\big]\Big[\varphi_2(t-\xi-\alpha,\lambda,\xi)$$
$$-\frac{\xi}{6}\big[2\varphi_8(t-2\xi-\alpha)-\varphi_4(t-2\xi-\alpha)-2\lambda^2\varphi_6(t-2\xi\big]\Big]d\alpha\Bigg]d\xi,$$

$$A_5\varphi=\varphi_{05}-\frac{1}{3}\int\limits_0^t\big[\varphi_8(\tau)-2\varphi_4(\tau)-\lambda^2\varphi_6(\tau)\big]\tilde g'_{00}(t-\tau,\lambda)\,d\tau$$
$$-2\int\limits_0^{t/2}\Bigg[\lambda^2\Big[\varphi_3(t-\xi,\lambda,\xi)-\frac{\xi}{6}\big[2\varphi_9(t-2\xi)+\varphi_5(t-2\xi)-2\lambda^2\varphi_7(t-2\xi)\big]$$
$$-\frac{1}{6}\big[2\varphi_8(t-2\xi)+\varphi_4(t-2\xi)-2\lambda^2\varphi_6(t-2\xi)\big]\Big]$$
$$-\left(\frac{h(0)}{6}\xi-\frac{\lambda^2}{6}\right)\big[2\varphi_8(t-2\xi)+\varphi_4(t-2\xi)-2\lambda^2\varphi_6(t-2\xi)\big] \qquad (57)$$
$$+\frac{1}{3}\int\limits_0^{t-2\xi}\big[2\varphi_8(\alpha)+\varphi_4(\alpha)-2\lambda^2\varphi_6(\alpha)\big]$$
$$\times\Big(\varphi_3(t-\xi-\alpha,\lambda,\xi)$$
$$+\frac{\xi}{6}\big[2\varphi_8(t-2\xi-\alpha)+\varphi_4(t-2\xi-\alpha)-2\lambda^2\varphi_6(t-2\xi-\alpha)\big]$$
$$-\frac{1}{6}\big[2\varphi_8(t-2\xi-\alpha)+\varphi_4(t-2\xi-\alpha)-\lambda^2\varphi_6(t-2\xi-\alpha)\big]\Big)d\alpha\Bigg]d\xi,$$

$$A_6\varphi=\varphi_{06}+\frac{1}{3}\int\limits_0^t(t-\tau)\big[\varphi_8(\tau)-\varphi_4(\tau)-\lambda^2\varphi_6(\tau)\big]\,d\tau, \qquad (58)$$

$$A_7\varphi = \varphi_{07} + \frac{1}{3}\int_0^t \left[\varphi_8(\tau) - \varphi_4(\tau) - \lambda^2\varphi_6(\tau)\right] d\tau, \tag{59}$$

$$A_8\varphi = \varphi_{08} + \frac{1}{3}\int_0^t \left[2\varphi_8(\tau) + \varphi_4(\tau) - 2\lambda^2\varphi_6(\tau)\right]\varphi_6(t-\tau)\, d\tau, \tag{60}$$

$$A_9\varphi = \varphi_{09} + \frac{1}{3}\int_0^t \left[2\varphi_8(\tau) + \varphi_4(\tau) - 2\lambda^2\varphi_6(\tau)\right]\varphi_7(t-\tau)\, d\tau, \tag{61}$$

and we have introduced the notation

$$\begin{aligned}\varphi_0(t,\lambda,z) &= (\varphi_{01},\varphi_{02},\varphi_{03},\varphi_{04},\varphi_{05},\varphi_{06},\varphi_{07},\varphi_{08},\varphi_{09})\\ &:= \left[-\frac{\lambda^2}{2}(t-z), -\frac{\lambda^2}{2}, -\frac{\lambda^2}{8}h(0)tz - \frac{\lambda^4 z}{4}, -2\tilde{g}''_{00t}(t,\lambda), \frac{\lambda^4}{2} + \frac{1}{4}h(0)t - 2\tilde{g}'''_{00t}(t,\lambda), 0, 0, 0, 0\right].\end{aligned}$$

By C_σ we denote the Banach space of continuous functions generated by the family of weighted norms

$$\|\varphi\|_\sigma = \max\left\{\sup_{(t,\lambda,z)\in D_2}\left|\varphi_i(t,\lambda,z)e^{-\sigma t}\right|,\ i=1,\dots,3,\ \sup_{t\in[0,2l]}\left|\varphi_j(t)e^{-\sigma t}\right|,\ j=4,\dots,9\right\}, \quad \sigma \ge 0.$$

For $\sigma = 0$, this space coincides with the space of continuous functions with the usual norm. We denote this norm by $\|\varphi\|$ in what follows. The inequality

$$e^{-\sigma t}\|\varphi\| \le \|\varphi\|_\sigma \le \|\varphi\| \tag{62}$$

implies the equivalence of the norms $\|\varphi\|_\sigma$ and $\|\varphi\|$ for each $l \in (0,\infty)$. The number σ will be chosen later. Let $Q_\sigma(\varphi_0, \|\varphi_0\|) := \{\varphi \mid \|\varphi - \varphi_0\| \le \|\varphi_0\|\}$ be the ball of radius $\|\varphi_0\|$ centered at the point φ_0 in some weighted space $C_\sigma(\sigma \ge 0)$ in which

$$\|\varphi_0\| = \max\left(\|\varphi_{01}\|, \|\varphi_{02}\|, \|\varphi_{03}\|, \|\varphi_{04}\|, \|\varphi_{05}\|, \|\varphi_{06}\|, \|\varphi_{07}\|, \|\varphi_{08}\|, \|\varphi_{09}\|\right).$$

It is easily seen that for $Q_\sigma(\varphi_0, \|\varphi_0\|)$ we have the estimate

$$\|\varphi\|_\sigma \le \|\varphi_0\|_\sigma + \|\varphi_0\| \le 2\|\varphi_0\|.$$

Let $\varphi(x,t) \in Q_\sigma(\varphi_0, \|\varphi_0\|)$. Let us show that, for a suitable choice of $\sigma > 0$, the operator A takes the ball to itself; i.e., $A\varphi \in Q_\sigma(\varphi_0, \|\varphi_0\|)$. Indeed, using Eqs. (53)–(61) to compose the norm of differences, for $(t,z) \in D_2$ we have

$$\begin{aligned}\|A_1\varphi - \varphi_{01}\|_\sigma &= \sup_{(t,z)\in D_2}\left|(A_1\varphi - \varphi_{01})e^{-\sigma t}\right|\\ &= \sup_{(t,z)\in D_2}\left|\int_0^{t-z}\int_0^{\tau/2}\left[\lambda^2\varphi_1(t-\xi,\lambda,\xi)e^{-\sigma(t-\xi)}e^{-\sigma\xi}\right.\right.\\ &\quad - \frac{1}{3}\left(2\varphi_8(t-2\xi) + \varphi_4(t-2\xi) - 2\lambda^2\varphi_6(t-2\xi)\right)e^{-\sigma(t-2\xi)}e^{-2\sigma\xi}\\ &\quad + \frac{1}{3}\int_0^{t-2\xi}\left[2\varphi_8(\alpha) + \varphi_4(\alpha) - 2\lambda^2\varphi_6(\alpha)\right]e^{-\sigma\alpha}\end{aligned}$$

$$
\times\ \varphi_1(t-\xi-\alpha,\lambda,\xi)e^{-\sigma(t-\xi-\alpha)}e^{-\sigma\xi}\,d\alpha\Bigg]\,d\xi d\tau
$$

$$
+\int\limits_{t-z}^{t}\int\limits_{\tau-t+z}^{\frac{2\tau-t+z}{2}}\Bigg[\lambda^2\varphi_1(2\tau-t+z-\xi,\lambda,\xi)e^{-\sigma(2\tau-t+z-\xi)}e^{-\sigma(\xi-2\tau-z)}
$$

$$
-\frac{1}{3}\big(2\varphi_8(t-2\beta+z-2\xi)+\varphi_4(t-2\beta+z-2\xi)
$$

$$
-2\lambda^2\varphi_6(t-2\beta+z-2\xi)\big)e^{-\sigma(2\tau-t+z-2\xi)}e^{-\sigma(2\xi-2\tau-z)}
$$

$$
+\frac{1}{3}\int\limits_0^{2\tau-t+z-2\xi}\big[2\varphi_8(\alpha)+\varphi_4(\alpha)-2\lambda^2\varphi_6(\alpha)\big]e^{-\sigma\alpha}\varphi_1(2\tau-t+z-\xi-\alpha,\lambda,\xi)
$$

$$
\times\ e^{-\sigma(2\tau-t+z-\xi-\alpha)}e^{-\sigma(2t+\xi-2\tau-z)}\,d\alpha\Bigg]\,d\xi d\tau\Bigg|
$$

$$
\le\frac{2\|\varphi_0\|}{\sigma}l\Big[3\lambda^2+(3+2\lambda^2)\big(4\|\varphi_0\|l+2/3\big)\Big]=:\frac{\|\varphi_0\|}{\sigma}\alpha_1.
$$

In a similar way, we obtain the estimates

$$
\|A_j\varphi-\varphi_{0j}\|_\sigma\le\frac{\|\varphi_0\|}{\sigma}\alpha_j,\quad j=2,\dots,9,
$$

where

$$
(\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7,\alpha_8,\alpha_9)
$$

$$
=\Bigg\{2\Big[\lambda^2l+1+L\Big(1+\lambda^2l^2+\big(16+2\lambda^2(1+l)\big)\|\varphi_0\|\Big)\Big],
$$

$$
2\Bigg[\lambda^2L\bigg(1+L+\frac{1}{2}L(1+l)\bigg)+\lambda^2(L+1)\big(4\lambda^2L+3(1+lL)\big)\|\varphi_0\|\Bigg],
$$

$$
2\Big[6\lambda^2+2L\big(G_0+l+8(1+l^2L)\|\varphi_0\|\big)\Big],
$$

$$
2\Big[2\big(\lambda^2+L(G_1+l+1+6L_0)+8(2L+l^2)/3\|\varphi_0\|\big)\Big],
$$

$$
\frac{(2+\lambda^2)}{3}l,\frac{(2+\lambda^2)}{3},2\bigg[\frac{2}{9}l^2\|\varphi_0\|\bigg(3+\frac{5\lambda^2}{2}\bigg)\bigg(2+\frac{5\lambda^2}{4}\bigg)\bigg]\Bigg\},
$$

$$
2\bigg[\frac{2}{9}l^3\|\varphi_0\|\bigg(3+\frac{5\lambda^2}{2}\bigg)\bigg(2+\frac{5\lambda^2}{4}\bigg)\bigg].
$$

Here we have introduced the notation

$$
L=(3+2\lambda^2)/6,\quad L_0=h(0)l/6-\lambda^2/6,\quad G_0=\max_{t\in[0,2l]}\big|g_0(t,\lambda)\big|,\quad G_1=\max_{t\in[0,2l]}\big|g_0'(t,\lambda)\big|.
$$

Choosing $\sigma\ge\alpha_0:=\max(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7,\alpha_8,\alpha_9)$, we conclude that A takes the ball $Q_\sigma(\varphi_0,\|\varphi_0\|)$ to the ball $Q_\sigma(\varphi_0,\|\varphi_0\|)$.

Now let φ^1 and φ^2 be two arbitrary elements in $Q_\sigma(\varphi_0,\|\varphi_0\|)$. Then, using auxiliary inequalities of the form

$$
\big|\varphi_i^1\varphi_j^1-\varphi_i^2\varphi_j^2\big|e^{-\sigma t}\le\big|\varphi_i^1\big|\big|\varphi_j^1-\varphi_j^2\big|e^{-\sigma t}+\big|\varphi_j^2\big|\big|\varphi_i^1-\varphi_i^2\big|e^{-\sigma t}\le4\|\varphi_0\|\big\|\varphi^1-\varphi^2\big\|_\sigma
$$

for $(t,z) \in D_2$, we obtain

$$\|(A\varphi^1 - A\varphi^2)_j\|_\sigma = \sup_{(t,z)\in D_2} |(A\varphi^1 - A\varphi^2)_j e^{-\sigma t}| \le \frac{\|\varphi^1 - \varphi^2\|_\sigma}{\sigma}\beta_0,$$

where

$$\begin{aligned}\beta_0 = \max\Bigg\{ & \Big[3\lambda^2 + (3+2\lambda^2)\big(8\|\varphi_0\|l + 2/3\big)\Big],\\ & \Big[\lambda^2 l + 1 + L\Big(1 + \lambda^2 l^2 + 2\big(16 + 2\lambda^2(1+l)\big)\|\varphi_0\|\Big)\Big],\\ & \Big[\lambda^2 L\big(1 + L + L(1+l)/2\big) + 2\lambda^2(L+1)\big(4\lambda^2 L + 3(1+lL)\big)\|\varphi_0\|\Big],\\ & \Big[6\lambda^2 + 2L\big(G_0 + l + 16(1+l^2L)\|\varphi_0\|\big)\Big],\\ & \Big[2(\lambda^2 + L(G_1 + l + 1 + 6L_0) + 16(2L + l^2/3)\|\varphi_0\|)\Big],\\ & \frac{(2+\lambda^2)}{6}\,l, \frac{(2+\lambda^2)}{6}, 2\left[\frac{2}{9}\,l^2\|\varphi_0\|\left(3 + \frac{5\lambda^2}{2}\right)\left(2 + \frac{5\lambda^2}{4}\right)\right]\Bigg\},\\ 2\left[\frac{2}{9}\,l^3\|\varphi_0\|\left(3 + \frac{5\lambda^2}{2}\right)\left(2 + \frac{5\lambda^2}{4}\right)\right]&.\end{aligned}$$

It follows from the resulting estimates that if the number σ is chosen based on the condition $\sigma > \max(\alpha_0, \beta_0)$, then the operator A is contracting on $Q_\sigma(\varphi_0, \|\varphi_0\|)$, and by the Banach principle there exists a unique solution of Eq. (52) in $Q_\sigma(\varphi_0, \|\varphi_0\|)$ for each $l > 0$. The proof of Theorem 1 is complete. □

Let $K(m_0)$ be the set of functions $k_0(t) \in C[0, 2l]$ satisfying the condition $\|k_0\|_{C[0,2l]} \leqslant m_0$ for some $l > 0$ with a constant $m_0 > 0$.

Theorem 2. *Let $k_0^1(t), k_0^2(t) \in K(m_0)$ be two solutions of the inverse problem (9)–(13) with the data g_0^1 and g_0^2, respectively. Then there exists a positive number $C = C(m_0, l)$ such that*

$$\big\|k_0^1(t) - k_0^2(t)\big\|_{C[0,2l]} \leqslant C\big\|g_0^1 - g_0^2\big\|_{C^3[0,2l]}. \tag{63}$$

Proof. Let φ^1 and φ^2 be two vector functions that are solutions of (52) with the data g_0^1 and g_0^2, respectively; i.e., $\varphi^j = A\varphi^j$ for $j = 1, 2$. Passing to the differences $\varphi_i^1 - \varphi_i^2$, $i = 1, \dots, 8$, in the integral equations, just as in the paper [37], from the argument in the proof of Theorem 1 for $\sigma > \sigma_0$ we obtain the estimate

$$\|\varphi^1 - \varphi^2\|_\sigma \le C_0\big\|g_0^1 - g_0^2\big\|_{C^3[0,2l]} + \frac{\sigma}{\sigma_0}\|\varphi^1 - \varphi^2\|_\sigma, \tag{64}$$

where the constant C_0 depends on the same parameters as C. Inequalities (62) and (64) imply the estimate

$$\big\|k_0^1(t) - k_0^2(t)\big\|_{C[0,2l]} \leqslant \frac{C_0\sigma}{|\sigma - \sigma_0|}\big\|g_0^1 - g_0^2\big\|_{C^3[0,2l]}.$$

If we denote

$$\frac{C_0\sigma}{|\sigma - \sigma_0|} =: C$$

in this inequality, then we obtain the estimate (63). The proof of Theorem 2 is complete. □

Now let us establish some facts that will be useful when proving theorems in the next section. It follows from formulas (29) and (38) that u_0 can be expressed via $\tilde{v}$ by the formula

$$\tilde{u}_0(t,\lambda,z) = \tilde{v}(t,\lambda,z) - \delta(t-z) + \int_0^t r(t-\tau)(\tilde{v}(\tau,\lambda,z) - \delta(\tau - z))\,d\tau. \tag{65}$$

Since $\tilde{v}(t,x,z) = v(t,x,z)$ in the domain $t > z > 0$, we obtain, removing the tilde on the function $\tilde{v}$ and differentiating Eq. (65) with respect to λ,

$$\tilde{u}_{0\lambda}(t,\lambda,z) = v_\lambda(t,\lambda,z) + \int_z^t r(t-\tau)v_\lambda(\tau,\lambda,z)\,d\tau. \tag{66}$$

It should be noted that for the function v_λ one has the following problem obtained by differentiating (40)–(43) with respect to λ:

$$\frac{\partial^2 v_\lambda}{\partial t^2} = \frac{\partial^2 v_\lambda}{\partial z^2} - 2\lambda v - \lambda^2 v_\lambda - \int_z^t h(t-\tau)v_\lambda(\tau,\lambda,z)d\tau, \quad (t,z)\in D_2, \tag{67}$$

$$v_\lambda|_{t=z+0} = -\lambda z, \tag{68}$$

$$\left.\frac{\partial v_\lambda}{\partial z}\right|_{z=0} = 0, \quad \left.\frac{\partial v_\lambda}{\partial z}\right|_{z=l} = 0, \tag{69}$$

$$\tilde{v}_\lambda(t,\lambda,0) = \tilde{g}_{0\lambda}(t,\lambda) + \int_0^t k_0(t-\tau)\tilde{g}_{0\lambda}(\tau,\lambda)\,d\tau. \tag{70}$$

Lemma 2. *In the domain D_2, the function $\tilde{v}_\lambda(t,\lambda,z)$ belongs to $C^3(D_2)$, and one has the integral equation*

$$v_\lambda(t,\lambda,z) = \frac{1}{2}\left[\tilde{g}_{0\lambda}(t+z,\lambda) + \tilde{g}_{0\lambda}(t-z,\lambda) + \int_0^{t+z} k_0(t+z-\tau)\tilde{g}_{0\lambda}(\tau,\lambda)\,d\tau + \int_0^{t_z} k_0(t-z-\tau)\tilde{g}_{0\lambda}(\tau,\lambda)\,d\tau\right]$$

$$-\frac{1}{2}\int_0^z\int_{t-z+\xi}^{t+z-\xi}\left[\lambda^2 v_\lambda(\tau,\lambda,\xi) + 2\lambda v(\tau,\lambda,\xi) + \int_\xi^\tau h(\tau-\alpha)v(\alpha,\lambda,\xi)\,d\alpha + \int_\xi^\tau k_1(\tau-\alpha)\,d\alpha\right]d\tau d\xi, \tag{71}$$

Proof. Let us use the equivalent description of the domain D_2 in the form $D_2 = \{(t,z)\,|\,0<t<2l,\ t<z<2l-t\}$. Using d'Alembert's formula, from Eq. (67) and the initial conditions (70), (69) (we mean the first of the two conditions in (69)) we obtain the linear integral equation (71) in the domain D_2. It follows from the theory of integral equations that Eq. (71) has a unique continuous solution in D_2. The smoothness of the solution can be established by differentiating Eq. (71) sufficiently many times. One can readily verify that the right-hand side of the differentiated equation will be continuous, and consequently, so will be the left-hand side [38, Ch. 2]. Thus, $v_\lambda \in C^3[D_2]$. □

4. PROBLEM OF DETERMINING THE FUNCTIONS k_1 AND $\tilde{u}_1$

We will assume that the functions k_0 and $\tilde{u}_0$ are given. Let us introduce a new function $w(t,\lambda,z)$, just as in the previous section, in the following manner:

$$w(t,\lambda,z) := \tilde{u}_1(t,\lambda,z) + \int_0^t k_0(t-\tau)\tilde{u}_1(\tau,\lambda,z)\,d\tau.$$

Then the function $\tilde{u}_1(t,\lambda,z)$ can be expressed via $w(t,\lambda,z)$ by the formula

$$\tilde{u}_1(t,\lambda,z) = w(t,\lambda,z) + \int\limits_0^t r(t-\tau)w(\tau,\lambda,z)\,d\tau,$$

where

$$r(t) = -k_0(t) - \int\limits_0^t k_0(t-\tau)r(\tau)\,d\tau.$$

For the new functions $w(t,\lambda,z)$ and $r(t)$, Eqs. (24)–(27), in view of (28), acquire the form

$$\begin{aligned}\frac{\partial^2 w}{\partial t^2} &= \frac{\partial^2 w}{\partial z^2} - \lambda^2 w - \int\limits_0^t h(t-\tau)w(\tau,\lambda,z)\,d\tau \\ &\quad - i\int\limits_0^t k_1(t-\tau)\left[2\lambda\widetilde{u}_0(\tau,\lambda,z) + \lambda^2\tilde{u}_{0\lambda}(\tau,\lambda,z) - \frac{\partial^2\widetilde{u}_{0\lambda}}{\partial z^2}\right]d\tau, \quad (z,t)\in D_2, \quad \lambda\in\mathbb{R},\end{aligned} \tag{72}$$

$$w|_{t<0} \equiv 0, \tag{73}$$

$$\left[\frac{\partial w}{\partial z} - i\int\limits_0^t k_1(t-\tau)\frac{\partial\tilde{u}_{0\lambda}}{\partial z}\,d\tau\right]\Bigg|_{z=0} = 0, \tag{74}$$

$$\left[\frac{\partial w}{\partial z} - i\int\limits_0^t k_1(t-\tau)\frac{\partial\tilde{u}_{0\lambda}}{\partial z}\,d\tau\right]\Bigg|_{z=l} = 0, \tag{75}$$

$$w|_{z=0} = a_1(t,\lambda), \tag{76}$$

$$w|_{t=z} = 0, \tag{77}$$

where $h(t) := r''(t)$ and

$$a_1(t,\lambda) := \tilde{g}_1(t,\lambda) + \int\limits_0^t k_0(t-\tau)\tilde{g}_1(\tau,\lambda)\,d\tau. \tag{78}$$

Using formulas (65)–(69), we transform the last terms in (72), (74), and (75) as follows:

$$\int\limits_0^t k_1(t-\tau)\frac{\partial\tilde{u}_{0\lambda}}{\partial z}\,d\tau = \int\limits_z^t k_1(t-\tau)\left[\frac{\partial v_\lambda(\tau,\lambda,z)}{\partial z}\,d\tau - \int\limits_z^\eta r(\tau-\eta)\frac{\partial v_\lambda(\tau,\lambda,\eta)}{\partial z}\,d\eta\right]d\tau,$$

$$\int\limits_0^t k_1(t-\tau)\frac{\partial^2\tilde{u}_{0\lambda}}{\partial z^2}\,d\tau = \int\limits_z^t k_1(t-\tau)\left[\frac{\partial^2 v_\lambda}{\partial z^2}\,d\tau - \int\limits_z^\eta r(\tau-\eta)\frac{\partial^2 v_\lambda}{\partial z^2}\,d\eta\right]d\tau,$$

$$\int\limits_0^t \lambda^2 k_1(t-\tau)u_{0\lambda}(\tau,\lambda,z)\,d\tau = \int\limits_z^t \lambda^2 k_1(t-\tau)\left[v(\tau,\lambda,z) - \int\limits_z^\eta r(\tau-\eta)v(\tau,\lambda,\eta)\,d\eta\right]d\tau,$$

$$\int\limits_0^t 2\lambda k_1(t-\tau)\widetilde{u}_0(\tau,\lambda,z) = -2\lambda k_1(t-z) - 2\lambda\int\limits_z^t k_1(t-\tau)r(\tau-z)d\tau$$

$$+ 2\lambda \int\limits_z^t k_1(t-\tau)\left[v(\tau,\lambda,z) + \int\limits_z^\tau r(\tau-\eta)v(\eta,\lambda,z)\,d\eta\right]d\tau.$$

Thus, problem (72)–(77) can be rewritten as

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial z^2} - \lambda^2 w + \lambda_1 k_1(t-z) - \int\limits_z^t h(t-\tau)w(\tau,\lambda,z)\,d\tau - \int\limits_z^t k_1(t-\tau)s(\tau,\lambda,z)\,d\tau, \quad (z,t)\in D_2, \quad \lambda\in\mathbb{R}, \quad \lambda_1 := 2i\lambda, \tag{79}$$

$$w|_{t<0} \equiv 0, \tag{80}$$

$$\frac{\partial w}{\partial z}\,|_{z=0}= 0, \quad \frac{\partial w}{\partial z}\,|_{z=l}= 0, \tag{81}$$

$$w|_{z=0} = a_1(t,\lambda), \tag{82}$$

$$w|_{t=z} = 0, \tag{83}$$

$$s(t,\lambda,z) := i\left[\lambda^2\left(v(t,\lambda,z) - \int\limits_z^t r(t-\tau)v(\tau,\lambda,z)\,d\tau\right) + \frac{\partial^2 v_\lambda}{\partial z^2} - \int\limits_z^t r(t-\tau)\frac{\partial^2 v_\lambda}{\partial z^2}\,d\tau + 2\lambda\int\limits_z^t k_1(t-\tau)r(\tau-z)\,d\tau - 2\lambda\int\limits_z^t k_1(t-\tau)\left[v(\tau,\lambda,z) + \int\limits_z^\tau r(\tau-\eta)v(\eta,\lambda,z)\,d\eta\right]\right]. \tag{84}$$

Lemma 3. *In the domain D_2, one has the relation*

$$s(t,\lambda,z)|_{t=z+0} = -\frac{i\lambda^2}{2}z. \tag{85}$$

Proof. For $t = z + 0$, all terms in (84) except for the first one vanish, and in the first term, if for $(v(z+0,\lambda,z)$ we substitute the known value $\tilde{v}(t,\lambda,z)|_{t=z+0} = -\frac{\lambda^2}{2}z$ readily following from (39), then we obtain (85). □

Note that the unknown functions occur in Eq. (24) linearly. Let us replace system (79)–(83) by equivalent integral equations. Using d'Alembert's formula, from (79), (81), and (82) we obtain the equation

$$w(t,\lambda,z) = \frac{1}{2}\big[a_1(t-z,\lambda) + a_1(t+z,\lambda)\big] - \frac{1}{2}\int\limits_0^z\int\limits_{t-z+\xi}^{t+z-\xi}\left[\lambda^2 w(\tau,\lambda,\xi) - \lambda_1 k_1(\tau-\xi) + \int\limits_\xi^\tau h(\tau-\alpha)w(\alpha,\lambda,\xi)\,d\alpha + \int\limits_\xi^\tau k_1(\tau-\alpha)s(\alpha,\lambda,\xi)\,d\alpha\right]d\tau d\xi. \tag{86}$$

Passing to the limit as $t \to z+0$ in this equation and taking into account the conditions $w|_{t=z} = 0$ and $a_1(0,\lambda) = 0$, we find

$$a_1(2z,\lambda) = \int\limits_0^z\int\limits_\xi^{2z-\xi}\left[\lambda^2 w(\tau,\lambda,\xi) - \lambda_1 k_1(\tau-\xi) + \int\limits_0^{\tau-\xi} h(\alpha)w(\tau-\alpha,\lambda,\xi)\,d\alpha + \int\limits_0^{\tau-\xi} k_1(\alpha)s(\tau-\alpha,\lambda,\xi)\,d\alpha\right]d\tau d\xi.$$

To obtain an integral equation for $k_1(t)$, we replace $2z$ by t in the last equation and differentiate it two times with respect to t, thus obtaining

$$\begin{aligned} k_1(t) = \frac{2}{\lambda_1} a_1''(t,\lambda) - \frac{2}{\lambda_1} \Bigg\{ \int_0^{t/2} \Bigg[\lambda^2 w_t(t-\xi,\lambda,\xi) \\ + \int_0^{t-2\xi} h(\alpha) w_t(t-\xi-\alpha,\lambda,\xi)\,d\alpha + \int_0^{t-2\xi} k_1(\alpha) s_t(t-\xi-\alpha,\lambda,\xi)\,d\alpha \Bigg] d\xi \Bigg\}. \end{aligned} \tag{87}$$

Differentiating (86) with respect to t and taking into account (85), we obtain equations for w_t,

$$\begin{aligned} w_t(t,\lambda,z) = \frac{1}{2}\big[a_1'(t-z,\lambda) + a_1'(t+z,\lambda)\big] - \frac{1}{2}\lambda_1 k_1(t-z)z \\ + \frac{1}{2}\int_0^z \Bigg\{ \lambda^2 w(t+z-\xi,\lambda,\xi) - \lambda^2 w(t-z+\xi,\lambda,\xi) \\ - \lambda_1 k_1(t+z-2\xi) - \int_0^{t+z-2\xi} h(\alpha) w(t+z-\xi-\alpha,\lambda,\xi)\,d\alpha \\ + \int_0^{t-z} h(\alpha) w(t-z+\xi-\alpha,\lambda,\xi)\,d\alpha + \int_0^{t+z-2\xi} k_1(\alpha) s(t+z-\xi-\alpha,\lambda,\xi)\,d\alpha \\ - \int_0^{t-z} k_1(\alpha) s(t-z+\xi-\alpha,\lambda,\xi)\,d\alpha \Bigg\} d\xi. \end{aligned} \tag{88}$$

Equations (86)–(88) form a closed linear system of integral Volterra equations of the second kind in the domain D_2 for the functions $w(t,x,z)$, $w_t(t,x,z)$, and $k_1(t)$ with a given λ.

Since $s_t(t,\lambda,z)$ occurs in Eq. (87), for further reasoning we must show that $s(t,\lambda,z)$ belongs to the class $C^1[D_2]$. Consequently, we need to show that $v_\lambda(t,\lambda,z) \in C^3[D_2]$. The fact that $v_\lambda(t,\lambda,z)$ belongs to the class $C^3[D_2]$ has been proved in Lemma 2.

The main results of this section are Theorems 3 and 4 on the unique global solvability and stability of the inverse problem of determining $k_1(t)$.

Theorem 3. *Let* $\tilde{g}_{00}(t,\lambda) \in C^3[0,2l]$, $\tilde{g}_{00}(+0,\lambda) = 0$, $\tilde{g}_{00}'(+0,\lambda) = -\lambda^2/2$, $\tilde{g}_1(+0,\lambda) \in C^2[0,2l]$, *and* $\tilde{g}_1(+0,\lambda) = \tilde{g}_1'(+0,\lambda) = 0$. *Then there exists a unique solution* $k_1(t) \in C^2[0,2l]$ *of the inverse problem* (24)–(28) *for each* $l > 0$.

Proof. System (86)–(88) is a closed system of integral equations in D_2. We write this system in the form of the operator equation

$$\psi = F\psi, \tag{89}$$

where

$$\psi = \big[\psi_1(t,\lambda,z), \psi_2(t,\lambda,z), \psi_3(t)\big] = \Big[w(t,\lambda,z), w_t(t,\lambda,z) + \frac{1}{2}\lambda_1 k_1(t-z)z, k_1(t)\Big]$$

is a vector function with components ψ_i, $i = 1,\dots,3$, and the operator F is defined on the set of functions $\varphi \in C[D_2]$ and, in accordance with Eqs. (86)–(88), has the form $F = (F_1, F_2, F_3)$. Here

$$F_1\psi = \psi_{01} - \frac{1}{2}\int\limits_0^z \int\limits_{t-z+\xi}^{t+z-\xi} \Bigg[\lambda^2\psi_1(\tau,\lambda,\xi) - \lambda_1\varphi_3(\tau-\xi) + \int\limits_\xi^\tau h(\tau-\alpha)\psi_1(\alpha,\lambda,\xi)\,d\alpha + \int\limits_\xi^\tau \psi_3(\tau-\alpha)s(\alpha,\lambda,\xi)\,d\alpha\Bigg]\,d\tau d\xi, \tag{90}$$

$$\begin{aligned} F_2\psi = \psi_{02} - \frac{1}{2}\int\limits_0^z \Bigg\{ &\lambda^2\psi_1(t+z-\xi,\lambda,\xi) - \lambda^2\psi_1(t-z+\xi,\lambda,\xi) \\ &- \int\limits_0^{t+z-2\xi} h(\alpha)\psi_1(t+z-\xi-\alpha,\lambda,\xi)\,d\alpha + \int\limits_0^{t-z} h(\alpha)\psi_1(t-z+\xi-\alpha,\lambda,\xi)\,d\alpha \\ &+ \int\limits_0^{t+z-2\xi} \varphi_3(\alpha)s(t+z-\xi-\alpha,\lambda,\xi)\,d\alpha - \int\limits_0^{t-z} \varphi_3(\alpha)(\alpha)s(t-z+\xi-\alpha,\lambda,\xi)\,d\alpha \Bigg\}\,d\xi, \end{aligned} \tag{91}$$

$$\begin{aligned} F_3\psi = \psi_{03} - \frac{2}{\lambda_1}\Bigg\{\int\limits_0^{t/2}\Bigg[&\lambda^2\Big(\psi_2(t-\xi,\lambda,\xi) - \frac{1}{2}\lambda_1\psi_3(t-2\xi)\xi\Big) \\ &+ \int\limits_0^{t-2\xi} h(\alpha)\Big(\psi_2(t-\xi-\alpha,\lambda,\xi) - \frac{1}{2}\lambda_1\psi_3(t-2\xi-\alpha)\xi\Big)\,d\alpha \\ &+ \int\limits_0^{t-2\xi} \psi_3(\alpha)s_t(t-\xi-\alpha,\lambda,\xi)\,d\alpha\Bigg]\,d\xi\Bigg\}, \end{aligned} \tag{92}$$

where

$$\psi_0(x,t) = (\psi_{01},\psi_{02},\psi_{03}) := \Big[a_1(t-z,\lambda) + a_1(t+z,\lambda), \frac{1}{2}\big[a_1'(t-z,\lambda) + a_1'(t+z,\lambda)\big], \frac{2}{\lambda_1}a_1''(t,\lambda)\Big].$$

Let us show that, for some $n \in \mathbb{N}$, the nth power of the linear mapping $F\psi$ is a contraction. Set

$$\|\psi\| = \max\Big\{\max_{(t,z)\in D_2}\big|\psi_j(t,\lambda,z)\big|,\ j=1,2,\ \max_{t\in[0,2l]}\big|\psi_3(t)\big|\Big\}.$$

Let ψ^1 and ψ^2 be two continuous vector functions in D_2 satisfying the linear system of integral equations (90)–(92). Set

$$\begin{aligned} \Delta(t,z) &= \big\{(\xi,\tau)\mid 0\le\xi\le z,\ t-z+\xi\le\tau\le t+z-\xi\big\}, \\ \Pi(t,\xi,z) &= \big\{\tau \mid (\xi,\tau)\in\Delta(t,z)\big\}. \end{aligned}$$

Then for $(t,z)\in D_2$ we have (in the estimates we use the fact that $t = 2z$ in Eq. (87))

$$\big|F_1\psi^{(1)} - F_1\psi^{(2)}\big|(t,\lambda,z) \le \gamma_1\int\limits_0^z \max\Big\{\max_{(t,\lambda,z)\in\Pi}\big|\psi_1^{(1)} - \psi_1^{(2)}\big|(\tau,\lambda,\xi), \big|\psi_3^{(1)} - \psi_3^{(2)}\big|(2\xi)\Big\}\,d\xi$$

$$\leq \gamma_1 \int_0^z \max\left\{ \max_{(\xi,\tau)\in D_2} \left|\psi_1^{(1)} - \psi_1^{(2)}\right|(\tau,\lambda,\xi),\ \max_{(\xi)\in[0,l]} \left|\psi_3^{(1)} - \psi_3^{(2)}\right|(2\xi) \right\} d\xi \leq \gamma_1 z \left\|\psi^{(1)} - \psi^{(2)}\right\|,$$

$$\left|F_2\psi^{(1)} - F_2\psi^{(2)}\right|(t,\lambda,z)$$
$$\leq \gamma_2 \int_0^z \max\left\{ \max_{(\xi,\tau)\in D_2} \left|\psi_1^{(1)} - \psi_1^{(2)}\right|(\tau,\lambda,\xi),\ \max_{(\xi)\in[0,l]} \left|\psi_3^{(1)} - \psi_3^{(2)}\right|(2\xi) \right\} d\xi \leq \gamma_2 z \left\|\psi^{(1)} - \psi^{(2)}\right\|,$$

$$\left|F_3\psi^{(1)} - F_3\psi^{(2)}\right|(2z)$$
$$\leq \gamma_3 \int_0^z \max\left\{ \max_{(\xi\in[0,l]} \left|\psi_2^{(1)} - \psi_2^{(2)}\right|(2l-\xi,\lambda,\xi),\ \max_{(\xi)\in[0,l]} \left|\psi_3^{(1)} - \psi_3^{(2)}\right|(2\xi) \right\} d\xi \leq \gamma_3 z \left\|\psi^{(1)} - \psi^{(2)}\right\|,$$

where the γ_j are constants depending on the quantities occurring in C (Theorem 2). Setting $M = \max\{\gamma_1, \gamma_2, \gamma_3\}$, we obtain

$$\max_{1\leq j\leq 3} \left|F_j\psi^{(1)} - F_j\psi^{(2)}\right|(t,\lambda,z) \leq Mz\|\psi^{(1)} - \psi^{(2)}\|.$$

Further,

$$\left|F_1^2\psi^{(1)} - F_1^2\psi^{(2)}\right|(t,\lambda,z)$$
$$\leq \gamma_1 \int_0^z \max\left\{ \max_{(t,\lambda,z)\in\Pi} \left|F_1\psi_1^{(1)} - F_1\psi_1^{(2)}\right|(\tau,\lambda,\xi), \left|F_1\psi_3^{(1)} - F_1\psi_3^{(2)}\right|(2\xi) \right\} d\xi$$
$$\leq \gamma_1 M \int_0^z \xi\|\psi^{(1)} - \psi^{(2)}\|\, d\xi \leq \gamma_1 M \frac{z^2}{2!}\|\psi^{(1)} - \psi^{(2)}\|,$$

$$\left|F_2^2\psi^{(1)} - F_2^2\psi^{(2)}\right|(t,\lambda,z)$$
$$\leq \gamma_2 \int_0^z \max\left\{ \max_{(t,\lambda,z)\in\Pi} \left|F_2\psi_1^{(1)} - F_2\psi_1^{(2)}\right|(\tau,\lambda,\xi), \left|F_2\psi_3^{(1)} - F_2\psi_3^{(2)}\right|(2\xi) \right\} d\xi$$
$$\leq \gamma_2 M \int_0^z \xi\|\psi^{(1)} - \psi^{(2)}\|\, d\xi \leq \gamma_2 M \frac{z^2}{2!}\|\psi^{(1)} - \psi^{(2)}\|,$$

$$\left|F_3^2\psi^{(1)} - F_3^2\psi^{(2)}\right|(2z)$$
$$\leq \gamma_3 \int_0^z \max\left\{ \max_{(\tau\in[\xi,2z-\xi]} \left|F_3\psi_2^{(1)} - F_3\psi_2^{(2)}\right|(2l-\xi,\lambda,\xi), \left|F_3\psi_3^{(1)} - F_3\psi_3^{(2)}\right|(2\xi) \right\} d\xi$$
$$\leq \gamma_3 M \frac{z^2}{2!}\|\psi^{(1)} - \psi^{(2)}\|.$$

Hence

$$\max_{1\leq j\leq 3} \left|F_j^2\psi^{(1)} - F_j^2\psi^{(2)}\right|(t,\lambda,\xi) \leq M^2 \frac{z^2}{2!}\|\psi^{(1)} - \psi^{(2)}\|, \quad (t,z)\in D_2,$$

and consequently,

$$\max_{1\le j\le 3}\left|F_j^n\psi^{(1)}-F_j^n\psi^{(2)}\right|(t,\lambda,\xi)\le M^n\frac{z^n}{n!}\|\psi^{(1)}-\psi^{(2)}\|,\quad (t,z)\in D_2,$$
$$\left|F^n\psi^{(1)}-F^n\psi^{(2)}\right|(t,\lambda,\xi)\le M^n\frac{l^n}{n!}\|\psi^{(1)}-\psi^{(2)}\|.$$

For any given l, the number n can be selected large enough that $M^n\dfrac{l^n}{n!}<1$. Then the mapping F^n is a contraction. According to a generalization of the contraction mapping principle, Eq. (89) has a unique solution in $C(D_2)$. This solution can be found by the successive approximation method. The proof of Theorem 3 is complete. □

Let $K(m_1)$ be the set of functions $k_1(t)\in C[0,2l]$ satisfying the condition $\|k_1\|_{C[0,2l]}\leqslant m_1$ with a constant $m_1>0$ for some $l>0$.

Theorem 4. *Let $k_1^1(t),k_1^2(t)\in K(m_1)$ be two solutions of the inverse problem (24)–(28) with data $\left(\tilde g_1^1,k_0^1,u_0^1\right)$ and $\left(\tilde g_1^2,k_0^2,u_0^2\right)$, respectively. Then there exists a positive number $C_1=C_1(k_1,l)$ such that the following inequality holds*:

$$\left\|k_1^1(t)-k_1^2(t)\right\|_{C[0,2l]}\leqslant C_1\Big[\left\|\tilde g_1^1-\tilde g_1^2\right\|_{C_1^3[0,2l]}+\left\|k_0^1(t)-k_0^2(t)\right\|_{C[0,2l]}\Big].$$

The proof of Theorem 4 is similar to that of Theorem 2.

REFERENCES

1. A. Lorenzi, "An identification problem related to a nonlinear hyperbolic integro-differential equation," Nonlinear Anal. Theory Meth. & Appl. **22** (1), 21–44 (1994).
2. J. Janno and L. von Wolfersdorf, "Inverse problems for identification of memory kernels in viscoelasticity," Math. Meth. Appl. Sci. **20** (4), 291–314 (1997).
3. D. K. Durdiev and Zh. D. Totieva, "The problem of determining the one-dimensional kernels of the viscoelasticity equation," Sib. Zh. Ind. Mat. **16** (2), 72–82 (2013).
4. D. K. Durdiev and Zh. Sh. Safarov, "Inverse problem of determining the one-dimensional kernel of the viscoelasticity equation in a bounded domain," Math. Notes **97** (6), 867–877 (2015).
5. D. K. Durdiev and A. A. Rahmonov, "Inverse problem for a system of integro-differential equations for SH waves in a visco-elastic porous medium: global solvability," Theor. Math. Phys. **195** (3), 923–937 (2018).
6. D. K. Durdiev and Z. D. Totieva, "The problem of determining the one-dimensional matrix kernel of the system of viscoelasticity equations," Math. Meth. Appl. Sci. **41** (17), 8019–8032 (2018).
7. Z. D. Totieva and D. K. Durdiev, "The problem of finding the one-dimensional kernel of the thermoviscoelasticity equation," Math. Notes. **103** (1–2), 118–132 (2018).
8. Z. S. Safarov and D. K. Durdiev, "Inverse problem for an integro-differential equation of acoustics," Differ. Equations **54** (1), P. 134–142 (2018).
9. D. K. Durdiev and Z. D. Totieva, "The problem of determining the one-dimensional kernel of viscoelasticity equation with a source of explosive type," J. Inverse Ill-Posed Probl. **28** (1), 43–52 (2020).
10. J. Sh. Safarov, "Global solvability of the one-dimensional inverse problem for the integro-differential equation of acoustics," J. Sib. Fed. Univ. Math. Phys. **11** (6), 753–763 (2018). https://doi.org/10.17516/1997-1397-2018-11-6-753-763
11. J. Sh. Safarov, "Inverse problem for an integro-differential equation with distributed initial data," Uzb. Math. J., No. 1, 117–124 (2019).
12. U. D. Durdiev, "An inverse problem for the system of viscoelasticity equations in homogeneous anisotropic media," J. Appl. Ind. Math. **13** (4), 623–628 (2019).

13. V. G. Romanov, "Inverse problems for equation with a memory," Eurasian J. Math. Comput. Appl. **2** (4), 51–80 (2014).

14. V. G. Romanov, "Problem of determining the permittivity in the stationary system of Maxwell equations," Dokl. Math. **95** (3), 230–234 (2017).

15. V. G. Romanov, "Stability estimates for the solution to the problem of determining the kernel of a viscoelastic equation," J. Appl. Ind. Math. **6** (3), 360–370 (2012).

16. V. G. Romanov, "On the determination of the coefficients in the viscoelasticity equations," Sib. Math. J. **55** (3), 503–510 (2014).

17. V. G. Romanov, "Problem of kernel recovering for the viscoelasticity equation," Dokl. Math. **86**, 608–610 (2012).

18. D. K. Durdiev and Z. D. Totieva, "The problem of determining the multidimensional kernel of the viscoelasticity equation," Vladikavkaz. Mat. Zh. **17** (4), 18–43 (2015).

19. D. K. Durdiev, "Some multidimensional inverse problems of memory determination in hyperbolic equations," Zh. Mat. Fiz. Anal. Geom. **3** (4), 411–423 (2007).

20. D. K. Durdiev and A. A. Rakhmonov, "The problem of determining the 2D kernel in a system of integro-differential equations of a viscoelastic porous medium," J. Appl. Ind. Math. **14** (2), 281–295 (2020).

21. D. K. Durdiev and A. A. Rahmonov, "A 2D kernel determination problem in a visco-elastic porous medium with a weakly horizontally inhomogeneity," Math. Meth. Appl. Sci. **43** (15), 8776–8796 (2020).

22. M. K. Teshaev, I. I. Safarov, N. U. Kuldashov, M. R. Ishmamatov, and T. R. Ruziev, "On the distribution of free waves on the surface of a viscoelastic cylindrical cavity," J. Vib. Eng. Technol. **8** (4), 579–585 (2020).

23. I. Safarov, M. Teshaev, E. Toshmatov, Z. Boltaev, and F. Homidov, "Torsional vibrations of a cylindrical shell in a linear viscoelastic medium," IOP Conf. Ser.: Mater. Sci. Eng. **883** (1), 012190 (2020).

24. M. K. Teshaev, I. I. Safarov, and M. Mirsaidov, "Oscillations of multilayer viscoelastic composite toroidal pipes," J. Serb. Soc. Comput. Mech. **13** (2), 104–115 (2019).

25. V. A. Dedok, A. L. Karchevsky, and V. G. Romanov, "A numerical method of determining permittivity from the modulus of the electric intensity vector of an electromagnetic field," J. Appl. Ind. Math. **13** (3), 436–446 (2019).

26. A. L. Karchevsky, "A frequency-domain analytical solution of Maxwell's equations for layered anisotropic media," Russ. Geol. Geophys. **48** (80), 689–695 (2007).

27. V. G. Romanov and A. L. Karchevsky, "Determination of permittivity and conductivity of medium in a vicinity of a well having complex profile," Eurasian J. Math. Comput. Appl. **6** (4), 62–72 (2018).

28. A. L. Karchevsky, "A numerical solution to a system of elasticity equations for layered anisotropic media," Russ. Geol. Geophys. **46** (3), 339–351 (2005).

29. A. L. Karchevsky, "The direct dynamical problem of seismics for horizontally stratified media," Sib. Electron. Mat. Izv., No. 2, 23–61 (2005).

30. A. L. Karchevsky, "Numerical solution to the one-dimensional inverse problem for an elastic system," Dokl. Earth Sci. **375** (8), 1325–1328 (2000).

31. A. L. Karchevsky, "Simultaneous reconstruction of permittivity and conductivity," Inverse Ill-Posed Probl. **17** (4), 387–404 (2009).

32. E. Kurpinar and A. L. Karchevsky, "Numerical solution of the inverse problem for the elasticity system for horizontally stratified media," Inverse Probl. **20** (3), 953–976 (2004).

33. A. L. Karchevsky, "Numerical reconstruction of medium parameters of member of thin anisotropic layers," Inverse III-Posed Probl. **12** (50), 519–534 (2004).

34. U. D. Durdiev, "Numerical method for determining the dependence of the dielectric permittivity on the frequency in the equation of electrodynamics with memory," Sib. Electron. Mat. Izv. **17**, 179–189 (2020). https://doi.org/10.33048/semi.2020.17.013

35. Z. R. Bozorov, "Numerical determining a memory function of a horizontally-stratified elastic medium with aftereffect," Eurasian J. Math. Comput. Appl. **8** (2), 4–16 (2020).

36. A. S. Blagoveshchenskii and D. A. Fedorenko, "The inverse problem for an acoustic equation in a weakly horizontally inhomogeneous medium," J. Math. Sci. **155** (3), 379–389 (2008).

37. V. G. Romanov, *Inverse Problem of Mathematical Physics* (Nauka, Moscow, 1984) [in Russian].