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INVERSE PROBLEM FOR VISCOELASTIC SYSTEM IN A VERTICALLY LAYERED MEDIUM#

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Abstract. In this paper, we consider a three-dimensional system of first-order viscoelasticity equations written with respect to displacement and stress tensor. This system contains convolution integrals of relaxation kernels with the solution of the direct problem. The direct problem is an initial-boundary value problem for the given system of integro-differential equations. In the inverse problem, it is required to determine the relaxation kernels if some components of the Fourier transform with respect to the variables x_1 and x_2 of the solution of the direct problem on the lateral boundaries of the region under consideration are given. At the beginning, the method of reduction to integral equations and the subsequent application of the method of successive approximations are used to study the properties of the solution of the direct problem. To ensure a continuous solution, conditions for smoothness and consistency of initial and boundary data at the corner points of the domain are obtained. To solve the inverse problem by the method of characteristics, it is reduced to an equivalent closed system of integral equations of the Volterra type of the second kind with respect to the Fourier transform in the first two spatial variables x_1, x_2, for solution to direct problem and the unknowns of inverse problem. Further, to this system, written in the form of an operator equation, the method of contraction mappings in the space of continuous functions with a weighted exponential norm is applied. It is shown that with an appropriate choice of the parameter in the exponent, this operator is contractive in some ball, which is a subset of the class of continuous functions. Thus, we prove the global existence and uniqueness theorem for the solution of the stated problem.

Key words: viscoelasticity, resolvent, inverse problem, hyperbolic system, Fourier transform.

AMS Subject Classification: 35F61, 35L50, 42A38.

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Introduction

A perfectly elastic material does not exist in nature; in fact, inelasticity is always present. This inelasticity results in energy dissipation or damping. Therefore, for a wide class of materials, it is not enough to use an elastic model to study their mechanical behavior. Therefore, viscoelastic foundational models have often been used to model the behavior of polymeric materials with respect to time variable.

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Let be $\overline{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$. Let us denote by σ_{ij} the projection onto the x_i axis of the stress acting on the area with the normal parallel to the x_j axis, and $\overline{u}_i$ are the projection onto the x_i axis of the vector particle displacement. According to Hooke's law for viscoelastic media, stresses and deformations are related by the formulas [1, pp. 449–455], [2, ch. 3]:

$$\sigma_{ij}(\overline{x},t) = \mu\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) + \delta_{ij}\lambda \operatorname{div}\overline{u}$$
$$+\int\limits_0^t K_{ij}(t-\tau)\left[\mu\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) + \delta_{ij}\lambda \operatorname{div}\overline{u}\right](\overline{x},\tau)\, d\tau, \quad i,j = 1,2,3, \quad (1)$$

here $\mu = \mu(x_3)$, $\lambda = \lambda(x_3)$ are Lame coefficients, δ_{ij} is Kronecker symbol, $K_{ij}(t)$ are functions responsible for the viscosity of the medium and $K_{ij} = K_{ji}$, $i,j = 1,2,3$.

The equations of motion of a viscoelastic body particles in the absence of external forces have the form

$$\rho\frac{\partial^2 \overline{u}_i}{\partial t^2} = \sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j}, \quad i = 1,2,3, \quad (2)$$

where $\rho = \rho(x_3)$ is medium density, $\overline{u}(\overline{x},t) = (\overline{u}_1(\overline{x},t), \overline{u}_2(\overline{x},t), \overline{u}_3(\overline{x},t))$ is displacement vector. Throughout this work, μ, λ, ρ are considered to be given functions.

Note that (1) can be considered as integral Volterra equations of the second kind with respect to the expression $\mu\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) + \delta_{ij}\lambda \operatorname{div}\overline{u}$, $i,j = 1,2,3$. For each fixed pair (i,j) solving these equations, we get

$$\sigma_{ij}(\overline{x},t) = \mu\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) + \delta_{ij}\lambda \operatorname{div}\overline{u} + \int\limits_0^t r_{ij}(t-\tau)\sigma_{ij}(\overline{x},\tau)\, d\tau, \quad i,j = 1,2,3, \quad (3)$$

where r_{ij} are the resolvents of the kernels K_{ij} and they are related by the following integral relations [3, 4]:

$$r_{ij}(t) = -K_{ij}(t) - \int\limits_0^t K_{ij}(t-\tau) r_{ij}(\tau)\, d\tau, \quad i,j = 1,2,3. \quad (4)$$

From the condition $K_{ij} = K_{ji}$ implies the $r_{ij} = r_{ji}$.

Differentiating (3) with respect to t and introducing the notation $u_i = \frac{\partial}{\partial t}\overline{u}_i$, we get

$$\frac{\partial}{\partial t}\sigma_{ij}(\overline{x},t) = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \delta_{ij}\lambda \operatorname{div} u + r_{ij}(0)\sigma_{ij}(\overline{x},t) + \int\limits_0^t r'_{ij}(t-\tau)\sigma_{ij}(\overline{x},\tau)\, d\tau. \quad (5)$$

Then the system of equations (1) and (2) for the velocity u_i and strain σ_{ij} ($\sigma_{ij} = \sigma_{ji}$) in view of (3)–(5) can be written as a system of first-order integro-differential equations.

$$\left(A\frac{\partial}{\partial t} + B\frac{\partial}{\partial x_1} + C\frac{\partial}{\partial x_2} + D\frac{\partial}{\partial x_3} + F\right) U(\overline{x},t) = \int\limits_0^t R(t-\tau) U(\overline{x},\tau)\, d\tau, \quad (6)$$

where $U=(u_1,u_2,u_3,\sigma_{11},\sigma_{12},\sigma_{13},\sigma_{22},\sigma_{23},\sigma_{33})^*$, $*$ is the transposition sign,

$$A=\begin{pmatrix}(\rho I)_{3\times3} & (\mathbf{O})_{3\times3} & (\mathbf{O})_{3\times3}\\ (\mathbf{O})_{3\times3} & (I)_{3\times3} & (\mathbf{O})_{3\times3}\\ (\mathbf{O})_{3\times3} & (\mathbf{O})_{3\times3} & (I)_{3\times3}\end{pmatrix},\quad B=\begin{pmatrix}(\mathbf{O})_{3\times3} & (-I)_{3\times3} & (\mathbf{O})_{3\times3}\\ (B_1)_{3\times3} & (\mathbf{O})_{3\times3} & (\mathbf{O})_{3\times3}\\ (B_2)_{3\times3} & (\mathbf{O})_{3\times3} & (\mathbf{O})_{3\times3}\end{pmatrix},$$

$$B_1=\begin{pmatrix}-(\lambda+2\mu) & 0 & 0\\ 0 & -\mu & 0\\ 0 & 0 & -\mu\end{pmatrix},\quad B_2=\begin{pmatrix}-\lambda & 0 & 0\\ 0 & 0 & 0\\ -\lambda & 0 & 0\end{pmatrix},$$

$$C=\begin{pmatrix}(\mathbf{O})_{3\times3} & (C_1)_{3\times3} & (C_2)_{3\times3}\\ (C_3)_{3\times3} & (\mathbf{O})_{3\times3} & (\mathbf{O})_{3\times3}\\ (C_4)_{3\times3} & (\mathbf{O})_{3\times3} & (\mathbf{O})_{3\times3}\end{pmatrix},\quad C_1=\begin{pmatrix}0 & -1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$

$$C_2=\begin{pmatrix}0 & 0 & 0\\ -1 & 0 & 0\\ 0 & -1 & 0\end{pmatrix},\quad C_3=\begin{pmatrix}0 & -\lambda & 0\\ -\mu & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\quad C_4=\begin{pmatrix}0 & -(\lambda+2\mu) & 0\\ 0 & 0 & -\mu\\ 0 & -\lambda & 0\end{pmatrix},$$

$$D=\begin{pmatrix}(\mathbf{O})_{3\times3} & (D_1)_{3\times3} & (D_2)_{3\times3}\\ (D_3)_{3\times3} & (\mathbf{O})_{3\times3} & (\mathbf{O})_{3\times3}\\ (D_4)_{3\times3} & (\mathbf{O})_{3\times3} & (\mathbf{O})_{3\times3}\end{pmatrix},\quad D_1=\begin{pmatrix}0 & 0 & -1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$

$$D_2=\begin{pmatrix}0 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix},\quad D_3=\begin{pmatrix}0 & 0 & -\lambda\\ 0 & 0 & 0\\ -\mu & 0 & 0\end{pmatrix},\quad D_4=\begin{pmatrix}0 & 0 & -\lambda\\ 0 & -\mu & 0\\ 0 & 0 & -(\lambda+2\mu)\end{pmatrix},$$

$$F=\begin{pmatrix}(\mathbf{O})_{3\times3} & (\mathbf{O})_{3\times6}\\ (\mathbf{O})_{6\times3} & \mathrm{diag}\,(r_{11}(0),r_{22}(0),r_{33}(0),r_{12}(0),r_{13}(0),r_{23}(0))\end{pmatrix},$$

$$R(t)=\begin{pmatrix}(\mathbf{O})_{3\times3} & (\mathbf{O})_{3\times6}\\ (\mathbf{O})_{6\times3} & \mathrm{diag}\,(r'_{11},r'_{22},r'_{33},r'_{12},r'_{13},r'_{23})\end{pmatrix}.$$

The system (6) can be reduced to a symmetric hyperbolic system [5].

We reduce the system (6) to canonical form with respect to the variables t and x_3. To do this, multiply (6) on the left by A^{-1} and compose the equation

$$\left|A^{-1}D-\nu I\right|=0, \tag{7}$$

where I is the identity matrix of dimension 9. The last equation with respect to ν has following solutions:

$$\nu_1=-\nu_9=-\nu_p=-\sqrt{\frac{\lambda+2\mu}{\rho}},\quad \nu_{2,3}=-\nu_{7,8}=-\nu_s=-\sqrt{\frac{\mu}{\rho}},\quad \nu_{4,5,6}=0, \tag{8}$$

here ν_s and ν_p define velocities of the transverse and longitudinal seismic wave, respectively.

Now we choose a nondegenerate matrix $\Upsilon(x_3,t)$ so that the equality

$$\Upsilon^{-1}A^{-1}D\Upsilon=\Lambda \tag{9}$$

is hold, where Λ is a diagonal matrix, the diagonal of which contains the eigenvalues (for each fixed x_3) (8) of the matrix $A^{-1}D$ that is $\Lambda=\mathrm{diag}\,(-\nu_p,-\nu_s,-\nu_s,0,0,0,\nu_s,\nu_s,\nu_p)$.

From the formula (9) implies the equality

$$A^{-1}D\Upsilon=\Upsilon\Lambda,$$

which means that the column with the number i of the matrix Υ is an eigenvector of the matrix $A^{-1}D\Upsilon$, corresponding to the eigenvalue λ_i. Direct calculations show that the matrix Υ, satisfying the above conditions, can be chosen as (not uniquely)

$$\Upsilon(x_3)=\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\ \frac{\lambda}{\nu_p} & 0 & 0 & 1 & 0 & 1 & 0 & 0 & -\frac{\lambda}{\nu_p}\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & \rho\nu_s & 0 & 0 & 0 & 0 & -\rho\nu_s & 0\\ \frac{\lambda}{\nu_p} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -\frac{\lambda}{\nu_p}\\ 0 & \rho\nu_s & 0 & 0 & 0 & 0 & -\rho\nu_s & 0 & 0\\ \rho\nu_p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\rho\nu_p \end{pmatrix}.$$

We introduce the vector function ϑ by the equality

$$U=\Upsilon\vartheta.$$

Making this change in the equation (6) and then multiplying it on the left by $\Upsilon^{-1}A^{-1}$, then we get

$$\left(I\frac{\partial}{\partial t}+\Lambda\frac{\partial}{\partial x_3}+B_1\frac{\partial}{\partial x_1}+C_1\frac{\partial}{\partial x_2}+F_1\right)\vartheta(\overline{x},t)=\int\limits_0^t R_1(t-\tau,x_3)\vartheta(\overline{x},\tau)\,d\tau, \tag{10}$$

where

$$B_1(x_3)=\Upsilon^{-1}A^{-1}B\Upsilon=(b_{ij}),\quad C_1(x_3)=\Upsilon^{-1}A^{-1}C\Upsilon=(c_{ij}),$$
$$F_1(x_3)=\Upsilon^{-1}A^{-1}D\frac{\partial\Upsilon}{\partial x_3}+\Upsilon^{-1}A^{-1}F\Upsilon=(p_{ij}),$$
$$R_1(x_3,t)=\Upsilon^{-1}A^{-1}R\Upsilon=(\widetilde{r}_{ij})$$

$$=\begin{pmatrix} \frac{r'_{33}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{r'_{33}}{2}\\ 0 & \frac{r'_{23}}{2} & 0 & 0 & 0 & 0 & -\frac{r'_{23}}{2} & 0 & 0\\ 0 & 0 & \frac{r'_{13}}{2} & 0 & 0 & 0 & 0 & -\frac{r'_{13}}{2} & 0\\ \frac{\lambda(r'_{11}-r'_{22})}{\nu_p} & 0 & 0 & r'_{11} & 0 & r'_{11}-r'_{22} & 0 & 0 & \frac{\lambda(r'_{22}-r'_{11})}{\nu_p}\\ 0 & 0 & 0 & 0 & r'_{12} & 0 & 0 & 0 & 0\\ \frac{\lambda(r'_{22}-r'_{33})}{\nu_p} & 0 & 0 & 0 & 0 & r'_{22} & 0 & 0 & \frac{\lambda(r'_{33}-r'_{22})}{\nu_p}\\ 0 & -\frac{r'_{23}}{2} & 0 & 0 & 0 & 0 & \frac{r'_{23}}{2} & 0 & 0\\ 0 & 0 & -\frac{r'_{13}}{2} & 0 & 0 & 0 & 0 & \frac{r'_{13}}{2} & 0\\ -\frac{r'_{33}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{r'_{33}}{2} \end{pmatrix}. \tag{11}$$

The purpose of this article is to study the direct and inverse problems for the system (11). Moreover, the direct problem is an initial-boundary value problem for this system in domain $D=\{(x_1,x_2,x_3,t):\ (x_1,x_2)\in\mathbb{R}^2,\ x_3\in(0,H),\ t>0\}$, $H=\text{const}$, and in the inverse problem, the elements of the matrix R are assumed to be unknown, which are included in the definition of the matrix R_1 (12).

The is organized as follows. Section 1 presents the formulations of the direct and inverse problems and investigates the direct problem. In Section 2, the inverse problem is reduced to solving of an equivalent closed system of integral equations. In Section 3, we present the formulation and proof of the main result, which consists in the unique global solvability of the inverse problem. At the end there is a list of literatures used in the article.

1. Statement of the Direct and Inverse Problems

Consider the system of equations (10) in the domain D with a bounded $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$:

$$\Gamma_0 = \left\{(\overline{x}, t) :\ (x_1, x_2) \in \mathbb{R}^2,\ 0 \leqslant x_3 \leqslant H,\ t = 0\right\},$$

$$\Gamma_1 = \left\{(\overline{x}, t) :\ (x_1, x_2) \in \mathbb{R}^2,\ x_3 = 0,\ t > 0\right\},$$

$$\Gamma_2 = \left\{(\overline{x}, t) :\ (x_1, x_2) \in \mathbb{R}^2,\ x_3 = H,\ t > 0\right\}.$$

For this system **the direct problem** we pose as follows: determine the solution of the system of equations (10) at the following initial and boundary conditions:

$$\vartheta_i\big|_{t=0} = \varphi_i(\overline{x}), \quad i = 1, \ldots, 9, \tag{12}$$

$$\vartheta_i\big|_{x_3=H} = g_i(x_1, x_2, t),\ \ i = 1, 2, 3, \quad \vartheta_i\big|_{x_3=0} = g_i(x_1, x_2, t),\ \ i = 7, 8, 9. \tag{13}$$

Here $\varphi_i(\overline{x})$, $g_i(x_1, x_2, t)$ are given functions. It is known that [5, 6] the problem (10), (12), (13) is posed well.

The inverse problem is to determine the nonzero components of the matrix kernel R, that is $r_{ij}(t)$, $i, j = 1, 2, 3$ (R is included in R_1 according to the formula (12)) in (10) if the following conditions are known:

$$\vartheta_i\big|_{x_3=0} = h_i(x_1, x_2, t), \quad i = 1, \ldots, 6, \tag{14}$$

where $h_i(x_1, x_2, t)$, $i = 1, \ldots, 6$, are the given functions.

In the inverse problem, the numbers $r_{ij}(0)$, $i, j = 1, 2, 3$, are also considered to be given.

Currently, the problems of determining kernels from one hyperbolic integro-differential equations of the second order [7–22] have been widely studied. One- and multidimensional inverse problems are investigated and unique solvability theorems are obtained. Typically, second-order equations are derived from systems of first-order partial differential equations under some additional assumptions.

The inverse problem of determining the convolution kernels of integral terms from a system of first-order integro-differential equations of general form with two independent variables was studied in [23]. The theorem of local existence and global uniqueness is obtained. In the work of [24], the method for studying the work of [23] was applied to the investigating of the inverse problem of determining the diagonal relaxation matrix from the system of Maxwell's integro-differential equations.

It seems completely natural to study inverse problems on the determination of the kernels of integral terms of a system of integro-differential equations directly in terms of the system itself. This article is a natural continuation of this circle of problems and to a certain extent generalizes the results of [23] to the case of a three-dimensional system of viscoelasticity equations (1), (2).

Let functions $\varphi(\overline{x})$, $g_i(x_1, x_2, t)$ included in the right-hand side of (10) and the data (12), (13) are compact support in x_1, x_2 for each fixed x_3, t. From the existence for the system (10) of a compact support domain of dependence and compact support with respect to x_1, x_2 of the right-hand side (10) and data (12), (13) implies the compact support in x_1, x_2 solutions to the problem (10)–(13).

Let us study the property of solution to this problem. More precisely, we restrict ourselves to studying the Fourier transform in the variables x_1, x_2 of the solution. In what follows, for convenience, we put $x_3 = z$ and introduce the notation

$$\widehat{\vartheta}\,(\eta_1, \eta_2, z, t) = \int\limits_{\mathbb{R}^2} \vartheta(x_1, x_2, z, t) e^{i[\eta_1 x_1 + \eta_2 x_2]}\, dx_1\, dx_2,$$

where η_1, η_2 are transformation parameters. We fix η_1, η_2 and for convenience, we introduce the notation $\widehat{\vartheta}(\eta_1, \eta_2, z, t) = \widehat{\vartheta}(z, t)$.

In terms of the function $\widehat{\vartheta}$ we write the equations (10) as

$$\left(\frac{\partial}{\partial t} + \nu_j \frac{\partial}{\partial z}\right) \widehat{\vartheta}_j(z, t) = \sum_{k=1}^{9} \widehat{p}_{jk}(z) \widehat{\vartheta}_k(z, t) + \int\limits_0^t \sum_{k=1}^{9} \widetilde{r}_{jk}(z, \tau) \widehat{\vartheta}_k(z, t - \tau)\, d\tau, \qquad j = 1, \dots, 9, \tag{15}$$

where $\widehat{p}_{jk} = -i\eta_1 b_{jk} - i\eta_2 c_{jk} - p_{jk}$.

We will use a similar notations for the Fourier images of functions included in the initial, boundary and additional conditions (12)–(14):

$$\widehat{\vartheta}_i\big|_{t=0} = \widehat{\varphi}_i(z), \quad i = 1, \dots, 9, \tag{16}$$

$$\widehat{\vartheta}_i\big|_{z=H} = \widehat{g}_i(t), \ \ i = 1, 2, 3, \quad \widehat{\vartheta}_i\big|_{z=0} = \widehat{g}_i(t), \ \ i = 7, 8, 9, \tag{17}$$

$$\widehat{\vartheta}_i\big|_{z=0} = \widehat{h}_i(t), \quad i = 1, \dots, 6. \tag{18}$$

Where $\widehat{\varphi}_i(z)$, $i = 1, \dots, 9$, $\widehat{g}_i(t)$, $i = 1, 2, 3, 7, 8, 9$, $\widehat{h}_i(t)$, $i = 1, \dots, 6$, are the Fourier images of the corresponding functions from (12)–(14) for $\eta_1 = 0$, $\eta_2 = 0$. We also denote by D_H the projection of D onto the plane z, t. In what follows, we will consider the system of equations (15) in the domain $D_H \cup \widetilde{\Gamma}$ under the conditions (16) and (17). Where $\widetilde{\Gamma}_0 = \{(z, t) :\ 0 \leqslant z \leqslant H,\, t = 0\}$, $\widetilde{\Gamma}_1 = \{z, t) : z = 0,\, t > 0\}$, $\widetilde{\Gamma}_2 = \{(z, t) : z = H,\, t > 0\}$, $\widetilde{\Gamma} = \widetilde{\Gamma}_0 \cup \widetilde{\Gamma}_1 \cup \widetilde{\Gamma}_2$.

For the purpose of further research let us introduce the vector function $\omega(z, t) = \frac{\partial \widehat{\vartheta}}{\partial t}(z, t)$. To obtain a problem for a function $\omega(z, t)$ similar to (15)–(18) differentiate the equations (15) and the boundary conditions (17) with respect to the variable t, and the condition for $t = 0$ is found using the equations (15) and the initial conditions (16). In this case, we get

$$\left(\frac{\partial}{\partial t} + \nu_j \frac{\partial}{\partial z}\right) \omega_j(z, t) = \sum_{k=1}^{9} \widehat{p}_{jk}(z) \omega_k(z, t) + \sum_{k=1}^{9} \widetilde{r}_{jk}(z, t) \widehat{\varphi}_k(z) + \int\limits_0^t \sum_{k=1}^{9} \widetilde{r}_{jk}(z, \tau) \omega_k(z, t - \tau)\, d\tau, \quad j = 1, \dots, 9, \tag{19}$$

$$\omega_i\big|_{t=0} = -\nu_j \frac{\partial \widehat{\varphi}_i(z)}{\partial z} + \sum_{j=1}^{9} \widehat{p}_{ij}(z) \widehat{\varphi}_j(z) =: \Phi_i(z), \quad i = 1, \dots, 9, \tag{20}$$

$$\omega_i\big|_{z=H} = \frac{d}{dt} \widehat{g}_i(t), \ \ i = 1, 2, 3, \quad \omega_i\big|_{z=0} = \frac{d}{dt} \widehat{g}_i(t), \ \ i = 7, 8, 9. \tag{21}$$

For functions ω_i additional conditions (18) gets

$$\omega_i\big|_{z=0} = \frac{d}{dt}\,\widehat{h}_i(t), \quad i = 1, \dots, 6. \tag{22}$$

Let us pass from the equalities (19)–(22) to the integral relations for the components of the vector $\widehat{\vartheta}$ with integration flux along the corresponding characteristics of the equations of the system (19). Recall that the characteristics corresponding to ν_p and ν_s have a positive slope, and the characteristics corresponding to $-\nu_p$ and $-\nu_s$ have a negative slope. We denote

$$\mu_i(z) = \int\limits_0^z \frac{d\beta}{\nu_i(\beta)}, \quad i = 1,2,3,7,8,9, \quad \mu_i(z) = 0, \ \ i = 4,5,6.$$

Inverse functions to $\mu_i(z)$ will be denoted by $z = \mu_i^{-1}(\cdot)$. Using the introduced functions, the equations of characteristics passing through the points (z,t) on the plane of variables ξ, τ can be written in the form

$$\tau = t + \mu_i(\xi) - \mu_i(z), \quad i = 1, \dots, 9. \tag{23}$$

Consider an arbitrary point $(z,t) \in D_H$ on the plane of variables ξ, τ and draw through it the characteristic of the i th of the system (15) equation tell to intersection in the domain $\tau \leqslant t$ with boundary $\widetilde{\Gamma}$. The intersection point is denoted by (z_0^i, t_0^i). Integrating the equations of the system (15) along the corresponding characteristics from the point (z_0^i, t_0^i) to the point (z,t) we find

$$\begin{gathered}
\omega_i(z,t) = \omega_i(z_0^i, t_0^i) + \int\limits_{t_0^i}^{t} \sum_{k=1}^{9} \widehat{p}_{ik}\omega_k(\xi,\tau)\Big|_{\xi=\mu_i^{-1}[\tau - t + \mu_i(z)]} d\tau \\
+ \int\limits_{t_0^i}^{t} \left[\sum_{k=1}^{9} \widetilde{r}_{ik}(\xi,\tau)\widehat{\varphi}_i(\xi) + \int\limits_0^{\tau} \sum_{k=1}^{9} \widetilde{r}_{ik}(\xi,\tau-\alpha)\omega_k(\xi,\alpha)\,d\alpha \right]\Bigg|_{\xi=\mu_i^{-1}[\tau - t + \mu_i(z)]} d\tau, \\
i = 1, \dots, 9.
\end{gathered} \tag{24}$$

We define in (24) t_0^i. It depends on the coordinates of the point (z,t). It is not difficult to see that $t_0^i(z,t)$ has the form

$$t_0^i(z,t) = \begin{cases} t - \mu_i(z) + \mu_i(H), & t \geqslant \mu_i(z) - \mu_i(H), \\ 0, & 0 < t < \mu_i(z) - \mu_i(H), \end{cases} \qquad i = 1,2,3,$$

$$t_0^i(z,t) = 0, \ \ i = 4,5,6, \quad t_0^i(z,t) = \begin{cases} t - \mu_i(z), & t \geqslant \mu_i(z), \\ 0, & 0 < t < \mu_i(z), \end{cases} \qquad i = 7,8,9.$$

Then, from the condition that the pair (z_0^i, t_0^i) satisfies the equation (23) it follows

$$z_0^i(z,t) = \begin{cases} H, & t \geqslant \mu_i(z) - \mu_i(H), \\ \mu_i^{-1}\left(\mu_i(z) - t\right), & 0 < t < \mu_i(z) - \mu_i(H), \end{cases} \qquad i = 1,2,3,$$

$$z_0^i(z,t) = z, \ \ i = 4,5,6, \quad z_0^i(z,t) = \begin{cases} 0, & t \geqslant \mu_i(z), \\ \mu_i^{-1}\left(\mu_i(z) - t\right), & 0 < t < \mu_i(z), \end{cases} \qquad i = 7,8,9.$$

The free terms of the integral equations (24) are defined through the initial and boundary conditions (20) and (21) as follows:

$$\omega_i(z_0^i, t_0^i) = \begin{cases} \frac{\partial}{\partial t}\,\widehat{g}_i\left(t-\mu_i(z)+\mu_i(H)\right), & t \geqslant \mu_i(z)-\mu_i(H), \\ \Phi_i\left(\mu_i^{-1}\left(\mu_i(z)-t\right)\right), & 0<t<\mu_i(z)-\mu_i(H), \end{cases} \quad i=1,2,3,$$

$$\omega_i(z_0^i, t_0^i) = \Phi_i(z), \quad i=4,5,6,$$

$$\omega_i(z_0^i, t_0^i) = \begin{cases} \frac{\partial}{\partial t}\,\widehat{g}_i\left(t-\mu_i(z)\right), & t \geqslant \mu_i(z), \\ \Phi_i\left(\mu_i^{-1}\left(\mu_i(z)-t\right)\right), & 0<t<\mu_i(z), \end{cases} \quad i=7,8,9.$$

Let the following conditions hold

$$\widehat{\varphi}_i(H) = \widehat{g}_i(0) \quad \text{and} \quad \left.\frac{\partial \widehat{g}_i(t)}{\partial t}\right|_{t=0} = -\nu_j \left.\frac{\partial \widehat{\varphi}_i(z)}{\partial z}\right|_{z=H} + \sum_{j=1}^{9} \widehat{p}_{ij}(H)\widehat{\varphi}_j(H), \quad i=1,2,3, \quad (25)$$

$$\widehat{\varphi}_i(0) = \widehat{g}_i(0) \quad \text{and} \quad \left.\frac{\partial \widehat{g}_i(t)}{\partial t}\right|_{t=0} = -\nu_j \left.\frac{\partial \widehat{\varphi}_i(z)}{\partial z}\right|_{z=0} + \sum_{j=1}^{9} \widehat{p}_{ij}(0)\widehat{\varphi}_j(0), \quad i=7,8,9. \quad (26)$$

It is easy to see that the conditions for matching the initial and boundary data (16), (17) (20), (21) in corner points of the domain D_H coincide with the relations (25) and (26). Hence it is clear that at the fulfillment of the same equalities (25) and (26) equations (24) will have unique continuous solutions $\omega_i(z,t)$, or the same $\frac{\partial}{\partial t}\vartheta_i(z,t)$, $i=1,\dots,9$.

Suppose that all given functions included in (24) are continuous functions of their arguments in D_H. Then this system of equations is a closed system of integral equations of the Volterra type of the second kind with continuous kernels and free terms. As usual, such a system has a unique solution in the bounded subdomain $D_{HT} = \{(z,t) : 0<z<H,\ 0<t<T\}$, $T>0$ are some fixed number.

Theorem 1. *Assume functions $\varphi(x)$, $g(x_1,x_2,t)$ have compact support in x_1, x_2 for each fixed z, t. Let be $\rho(z),\mu(z),\lambda(z),\widehat{\varphi}(z) \in C^1[0,H]$, $\widehat{g}(t) \in C^1[0,T]$, $\rho(z)>0$, $\lambda(z)>0$, $\mu(z)>0$, $r'_{ij}(t) \in C[0,T]$, $i,j=1,2,3$ and conditions (25), (26) are satisfied. Then there is a unique solution to the problem (19)–(21) in the domain D_{HT}.*

The problem (15)–(17) in the domain D_{HT} is equivalent to a linear integral equation of the second kind of Volterra type with respect to $\widehat{\vartheta}$. As follows from the theory of linear integral equations, it has a unique solutions [3]. So we drop it.

2. Reduction of the Inverse Problem

In this section, the inverse problem is reduced to solving of an equivalent closed system of integral equations. Consider an arbitrary point $(z,0) \in \widetilde{\Gamma}_0$ and draw through it the characteristics (23) for $i=1,2,3$, up to the intersection with the boundary of the domain D_{HT}. Integrating the first six components of the equation (19), we obtain

$$\omega_i(z,0) = \frac{d}{dt}\widehat{h}_i(t_1^i) - \int_0^{t_1^i} \left[\sum_{j=1}^{9} \widehat{p}_{ij}(\xi)\omega_j(\xi,\tau) + \sum_{j=1}^{9} \widetilde{r}_{ij}(\xi,\tau)\widehat{\varphi}_j(\xi)\right]\Bigg|_{\xi=\mu_i^{-1}[\tau+\mu_i(z)]} d\tau$$
$$- \int_0^{t_1^i}\int_0^{\tau} \sum_{j=1}^{9} \widetilde{r}_{ij}(\xi,\alpha)\omega_j(\xi,\tau-\alpha)\,d\alpha\Bigg|_{\xi=\mu_i^{-1}[\tau+\mu_i(z)]} d\tau, \quad i=1,\dots,6, \quad (27)$$

where $t_1^i = -\mu_i(z)$, $i=1,2,3$, $t_1^i = t$, $i=4,5,6$.

For the purpose of further research we introduce the following notation for the unknowns:

$$\upsilon_i^1(z,t)=\omega_i(z,t),\quad i=1,\dots,9,\quad \upsilon_1^2(t)=r'_{11}(t),\quad \upsilon_2^2(t)=r'_{12}(t),\quad \upsilon_3^2(t)=r'_{13}(t),\tag{28}$$

$$\upsilon_4^2(t)=r'_{22}(t),\quad \upsilon_5^2(t)=r'_{23}(t),\quad \upsilon_6^2(t)=r'_{33}(t),\quad \upsilon_i^3(z,t)=\frac{\partial}{\partial y}\omega_i(z,t),\quad i=4,5,6,\tag{29}$$

$$\upsilon_i^3(z,t)=\frac{\partial}{\partial z}\omega_i(z,t)-\frac{r'_{33}(t_0^i)}{2}\left(\widetilde{\varphi}_1(z_0^i)-\widetilde{\varphi}_9(z_0^i)\right)\frac{\partial}{\partial z}t_0^i,\quad i=1,9,\tag{30}$$

$$\upsilon_i^3(z,t)=\frac{\partial}{\partial z}\omega_i(z,t)-\frac{r'_{23}(t_0^i)}{2}\left(\widetilde{\varphi}_2(z_0^i)-\widetilde{\varphi}_7(z_0^i)\right)\frac{\partial}{\partial z}t_0^i,\quad i=2,7,\tag{31}$$

$$\upsilon_i^3(z,t)=\frac{\partial}{\partial z}\omega_i(z,t)-\frac{r'_{13}(t_0^i)}{2}\left(\widetilde{\varphi}_3(z_0^i)-\widetilde{\varphi}_8(z_0^i)\right)\frac{\partial}{\partial z}t_0^i,\quad i=3,8.\tag{32}$$

Taking into account these notations and the explicit forms of the functions $\widetilde{r}_{ij}(z,t)$ in terms of $r'_{ij}(t)$ by the formula (11), we rewrite the equations (24) in the form

$$\upsilon_i^1(z,t)=\upsilon_i^{01}(z,t)+\int\limits_{t_0^i}^{t}\left[\sum_{j=1}^{9}\widehat{p}_{ij}(\xi)\upsilon_j^1(\xi,\tau)-\frac{\upsilon_6^2(\tau)}{2}\left(\widehat{\varphi}_1-\widehat{\varphi}_9\right)(\xi)\right]\Bigg|_{\xi=\mu_i^{-1}[\tau-t+\mu_i(z)]}d\tau$$
$$-\int\limits_{t_0^i}^{t}\int\limits_0^{\tau}\frac{\upsilon_6^2(\alpha)}{2}\left(\upsilon_1^1-\upsilon_9^1\right)(\xi,\tau-\alpha)\,d\alpha\Bigg|_{\xi=\mu_i^{-1}[\tau-t+\mu_i(z)]}d\tau,\quad i=1,9,\tag{33}$$

$$\upsilon_i^1(z,t)=\upsilon_i^{01}(z,t)+\int\limits_{t_0^i}^{t}\left[\sum_{j=1}^{9}\widehat{p}_{ij}(\xi)\upsilon_j^1(\xi,\tau)-\frac{\upsilon_5^2(\tau)}{2}\left(\widehat{\varphi}_2-\widehat{\varphi}_7\right)(\xi)\right]\Bigg|_{\xi=\mu_i^{-1}[\tau-t+\mu_i(z)]}d\tau$$
$$-\int\limits_{t_0^i}^{t}\int\limits_0^{\tau}\frac{\upsilon_5^2(\alpha)}{2}\left(\upsilon_2^1-\upsilon_7^1\right)(\xi,\tau-\alpha)\,d\alpha\Bigg|_{\xi=\mu_i^{-1}[\tau-t+\mu_i(z)]}d\tau,\quad i=2,7,\tag{34}$$

$$\upsilon_i^1(z,t)=\upsilon_i^{01}(z,t)+\int\limits_{t_0^i}^{t}\left[\sum_{j=1}^{9}\widehat{p}_{ij}(\xi)\upsilon_j^1(\xi,\tau)-\frac{\upsilon_3^2(\tau)}{2}\left(\widehat{\varphi}_3-\widehat{\varphi}_8\right)(\xi)\right]\Bigg|_{\xi=\mu_i^{-1}[\tau-t+\mu_i(z)]}d\tau$$
$$-\int\limits_{t_0^i}^{t}\int\limits_0^{\tau}\frac{\upsilon_3^2(\alpha)}{2}\left(\upsilon_2^1-\upsilon_2^1\right)(\xi,\tau-\alpha)\,d\alpha\Bigg|_{\xi=\mu_i^{-1}[\tau-t+\mu_i(z)]}d\tau,\quad i=3,8,\tag{35}$$

$$\upsilon_4^1(z,t)=\int\limits_0^t\sum_{j=1}^{9}\widehat{p}_{4j}(z)\upsilon_j^1(z,\tau)\,d\tau+\int\limits_0^t\int\limits_0^{\tau}\upsilon_1^2(\alpha)\upsilon_4^1(z,\tau-\alpha)\,d\alpha\,d\tau$$
$$+\int\limits_0^t\int\limits_0^{\tau}\left(\upsilon_1^2-\upsilon_4^2\right)(\alpha)\left(\frac{\lambda}{\nu_p}\left(\upsilon_1^1-\upsilon_9^1\right)(z,\tau-\alpha)+\upsilon_6^1(z,\tau-\alpha)\right)d\alpha\,d\tau$$
$$+\int\limits_0^t\left[\left(\upsilon_1^2-\upsilon_4^2\right)(\tau)\left(\frac{\lambda}{\nu_p}\left(\widehat{\varphi}_1-\widehat{\varphi}_9\right)(z)+\widehat{\varphi}_6(z)\right)+\upsilon_1^2(\tau)\widehat{\varphi}_4(z)\right]d\tau,\tag{36}$$

$$\upsilon_5^1(z,t) = \int\limits_0^t \sum_{j=1}^9 \widehat{p}_{5j}(z)\upsilon_j^1(z,\tau)\,d\tau + \int\limits_0^t\int\limits_0^\tau \upsilon_2^2(\alpha)\upsilon_5^1(z,\tau-\alpha)\,d\alpha\,d\tau + \int\limits_0^t \upsilon_2^2(\tau)\widehat{\varphi}_1(z)\,d\tau, \quad (37)$$

$$\begin{aligned}\upsilon_6^1(z,t) = &\int\limits_0^t\int\limits_0^\tau \left[\frac{\lambda}{\nu_p}\left(\upsilon_4^2-\upsilon_6^2\right)(\alpha)\left(\upsilon_1^1-\upsilon_9^1\right)(z,\tau-\alpha)+\upsilon_1^2(\alpha)\upsilon_6^1(z,\tau-\alpha)\right]d\alpha\,d\tau\\ &+\int\limits_0^t \sum_{j=1}^9 \widehat{p}_{6j}(z)\upsilon_j^1(z,\tau)\,d\tau + \int\limits_0^t \left[\frac{\lambda}{\nu_p}\left(\upsilon_4^2-\upsilon_6^2\right)(\tau)\left(\widehat{\varphi}_1-\widehat{\varphi}_9\right)(z)+\upsilon_1^2(\tau)\widehat{\varphi}_6(z)\right]d\tau,\end{aligned} \quad (38)$$

where $\upsilon_i^{01}(z,t)=\omega_i(z_0^i,t_0^i)$, $i=1,2,3,7,8,9$.

Consider (27) the initial conditions (20), we differentiate (27) with respect to z for $i=1,2,3$ and for t for $i=4,5,6$. After simple calculations, taking into account (28)–(32), we pass to integral equations

$$\begin{aligned}\upsilon_1^2(t) = &\,\upsilon_1^{02}(t)-\frac{\lambda}{\nu_p}M_1\int\limits_0^t \upsilon_6^2(\tau)\left(\frac{d}{dt}\widehat{h}_1-\frac{d}{dt}\widehat{g}_9\right)(t-\tau)\,d\tau - M_1\int\limits_0^t \upsilon_1^2(\tau)\frac{d}{dt}\widehat{h}_4(t-\tau)\,d\tau\\ &- M_1\int\limits_0^t\left[\frac{\lambda}{\nu_p}\left(\upsilon_1^2-\upsilon_4^2\right)(\tau)\left(\frac{d}{dt}\widehat{h}_1-\frac{d}{dt}\widehat{g}_9\right)(t-\tau)-\left(\upsilon_1^2-\upsilon_4^2\right)(\tau)\frac{d}{dt}\widehat{h}_6(t-\tau)\right]d\tau\\ &+\lambda M_1\int\limits_0^t \upsilon_6^2(\tau)\frac{\partial}{\partial z}\left(\widehat{\varphi}_1-\widehat{\varphi}_9\right)(\xi)\Big|_{\xi=\mu_1^{-1}[t-\tau]}d\tau + 2\lambda M_1\int\limits_0^t \frac{\partial}{\partial z}\sum_{j=1}^9 \widehat{p}_{1j}(\xi)\omega_j(\xi,\tau)\Big|_{\xi=\mu_1^{-1}[t-\tau]}d\tau\\ &+\lambda M_1\int\limits_0^t\int\limits_0^\tau \upsilon_4^2(\alpha)\left[\left(\upsilon_1^3-\upsilon_9^3\right)(\xi,\tau-\alpha)-\upsilon_6^2(t)\left(\widehat{\varphi}_1-\widehat{\varphi}_9\right)(\xi)\right]d\alpha\Big|_{\xi=\mu_1^{-1}[t-\tau]}d\tau\\ &- M_1\int\limits_0^t\left[\frac{\lambda}{\nu_p}\left(\upsilon_4^2(\tau)-\upsilon_6^2(\tau)\right)\left(\frac{d}{dt}\widehat{h}_1-\frac{d}{dt}\widehat{g}_9\right)(t-\tau)+\upsilon_4^2(\tau)\frac{d}{dt}\widehat{h}_6(t-\tau)\right]d\tau,\end{aligned} \quad (39)$$

$$\upsilon_2^2(t)=\upsilon_2^{02}(t)-M_2\int\limits_0^t \upsilon_3^2(\tau)\frac{d}{dt}\widehat{h}_5(t-\tau)\,d\tau, \quad (40)$$

$$\begin{aligned}\upsilon_3^2(t) = &\,\upsilon_3^{02}(t)-M_4\int\limits_0^t \upsilon_3^2(\tau)\left(\frac{d}{dt}\widehat{h}_3-\frac{d}{dt}\widehat{g}_8\right)(t-\tau)\,d\tau\\ &+M_3\int\limits_0^t\int\limits_0^\tau \upsilon_3^2(\alpha)\left[\left(\upsilon_3^3-\upsilon_8^3\right)(\xi,\tau-\alpha)-\upsilon_3^2(t)\left(\widehat{\varphi}_3-\widehat{\varphi}_8\right)(\xi)\right)d\alpha\Big|_{\xi=\mu_2^{-1}[t-\tau]}d\tau\\ &-M_3\int\limits_0^t \upsilon_6^2(\tau)\frac{\partial}{\partial z}\left(\widehat{\varphi}_3-\widehat{\varphi}_8\right)(\xi)\Big|_{\xi=\mu_2^{-1}[t-\tau]}d\tau + 2M_3\int\limits_0^t\frac{\partial}{\partial z}\sum_{j=1}^9 \widehat{p}_{3j}(\xi)\upsilon_j^1(\xi,\tau)\Big|_{\xi=\mu_2^{-1}[t-\tau]}d\tau,\end{aligned} \quad (41)$$

$$\upsilon_4^2(t) = \upsilon_4^{02}(t) - \frac{\lambda}{\nu_p} M_5 \int\limits_0^t \upsilon_4^2(\tau) \left(\frac{d}{dt}\widehat{h}_1 - \frac{d}{dt}\widehat{g}_9 \right)(t-\tau)\, d\tau +$$

$$+ \lambda M_5 \int\limits_0^t \int\limits_0^\tau \upsilon_6^2(\alpha) \left[\left(\upsilon_1^3 - \upsilon_9^3 \right)(\xi, \tau-\alpha) - \upsilon_6^2(t) \left(\widehat{\varphi}_1 - \widehat{\varphi}_9 \right)(\xi) \right] d\alpha \Big|_{\xi=\mu_1^{-1}[t-\tau]} d\tau \tag{42}$$

$$+ \lambda M_5 \int\limits_0^t \upsilon_6^2(\tau) \frac{\partial}{\partial z} \left(\widehat{\varphi}_1 - \widehat{\varphi}_9 \right)(\xi) \Big|_{\xi=\mu_1^{-1}[t-\tau]} d\tau + 2\lambda M_5 \int\limits_0^t \frac{\partial}{\partial z} \sum_{j=1}^{9} \widehat{p}_{1j}(\xi) \upsilon_j^1(\xi,\tau) \Big|_{\xi=\mu_1^{-1}[t-\tau]} d\tau$$

$$- M_5 \int\limits_0^t \left[\frac{\lambda}{\nu_p} \left(\upsilon_4^2 - \upsilon_6^2 \right)(\tau) \left(\frac{d}{dt}\widehat{h}_1(t-\tau) - \frac{d}{dt}\widehat{g}_9(t-\tau) \right) + \upsilon_4^2(\tau) \frac{d}{dt}\widehat{h}_6(t-\tau) \right] d\tau,$$

$$\upsilon_5^2(t) = \upsilon_5^{02}(t) + M_6 \int\limits_0^t \upsilon_5^2(\tau) \frac{\partial}{\partial z} \left(\widehat{\varphi}_2 - \widehat{\varphi}_7 \right)(\xi) \Big|_{\xi=\mu_2^{-1}[t-\tau]} d\tau$$

$$+ M_6 \int\limits_0^t \int\limits_0^\tau \upsilon_5^2(\alpha) \left[\left(\upsilon_2^3 - \upsilon_7^3 \right)(\xi, \tau-\alpha) - \upsilon_5^2(t) \left(\widehat{\varphi}_2 - \widehat{\varphi}_7 \right)(\xi) \right] d\alpha \Big|_{\xi=\mu_2^{-1}[t-\tau]} d\tau \tag{43}$$

$$- M_7 \int\limits_0^t \upsilon_5^2(\tau) \left(\frac{d}{dt}\widehat{h}_2 - \frac{d}{dt}\widehat{g}_7 \right)(t-\tau)\, d\tau + 2M_6 \int\limits_0^t \frac{\partial}{\partial z} \sum_{j=1}^{9} \widehat{p}_{2j}(\xi) \upsilon_j^1(\xi,\tau) \Big|_{\xi=\mu_2^{-1}[t-\tau]} d\tau,$$

$$\upsilon_6^2(t) = \upsilon_6^{02}(t) + M_8 \int\limits_0^t \upsilon_6^2(\tau) \frac{\partial}{\partial z} \left(\widehat{\varphi}_1 - \widehat{\varphi}_9 \right)(\xi) \Big|_{\xi=\mu_1^{-1}[t-\tau]} d\tau$$

$$+ M_8 \int\limits_0^t \int\limits_0^\tau \upsilon_6^2(\alpha) \left[\left(\upsilon_1^3 - \upsilon_9^3 \right)(\xi, \tau-\alpha) - \upsilon_6^2(t) \left(\widehat{\varphi}_1 - \widehat{\varphi}_9 \right)(\xi) \right] d\alpha \Big|_{\xi=\mu_1^{-1}[t-\tau]} d\tau \tag{44}$$

$$- M_9 \int\limits_0^t \upsilon_6^2(\tau) \left(\frac{d}{dt}\widehat{h}_1 - \frac{d}{dt}\widehat{g}_9 \right)(t-\tau)\, d\tau + 2M_8 \int\limits_0^t \frac{\partial}{\partial z} \sum_{j=1}^{9} \widehat{p}_{1j}(\xi) \upsilon_j^1(\xi,\tau) \Big|_{\xi=\mu_1^{-1}[t-\tau]} d\tau,$$

where

$$\upsilon_1^{02}(t) = M_1 Q_1^1(t), \quad \upsilon_2^{02}(t) = M_2 Q_2^1(t), \quad \upsilon_3^{02}(t) = -2M_3 P_3(t),$$

$$\upsilon_4^{02}(t) = M_5 Q_4^1(t), \quad \upsilon_5^{02}(t) = -2M_6 P_2(t), \quad \upsilon_3^{02}(t) = -2M_8 P_1(t),$$

$$P_i(t) = \frac{d^2}{dt^2}\, \widehat{h}_i(t_1^i) - \frac{\partial}{\partial z} \Phi_i(0) - \frac{1}{\nu_i(z)} \sum_{j=1}^{9} \widehat{p}_{ij}(0) \omega_j(0,t),$$

and

$$Q_1^1 = \frac{d^2}{dt^2}\widehat{h}_4(t) - \sum_{j=1}^{9} \widehat{p}_{4j}(0)\omega_j(0,t) - M_5 \left(\frac{d^2}{dt^2}\widehat{h}_6(t) - \sum_{j=1}^{9} \widehat{p}_{6j}(0)\omega_j(0,t) - 2M_8 P_1(z) \right),$$

$$Q_2^1 = \frac{d^2}{dt^2}\widehat{h}_5(t) - \sum_{j=1}^{9} \widehat{p}_{5j}(0)\omega_j(0,t), \quad Q_4^1 = \frac{d^2}{dt^2}\widehat{h}_6(t) - \sum_{j=1}^{9} \widehat{p}_{6j}(0)\omega_j(0,t) - 2M_8 P_1(z),$$

$$M_2 = \frac{1}{\widehat{\varphi}_5(0)}, \quad M_1 = \frac{\nu_p}{\lambda\left(\widehat{\varphi}_1(0) - \widehat{\varphi}_9(0)\right) + \nu_p \widehat{\varphi}_6(0) + \nu_p \widehat{\varphi}_4(0)}, \quad M_3 = \frac{\nu_2}{\widehat{\varphi}_3(0) - \widehat{\varphi}_8(0)},$$

$$M_4 = \frac{1}{\widehat{\varphi}_3(0) - \widehat{\varphi}_8(0)}, \quad M_6 = \frac{\nu_2}{\widehat{\varphi}_2(0) - \widehat{\varphi}_7(0)}, \quad M_5 = \frac{\nu_p}{\lambda\left(\widehat{\varphi}_1(0) - \widehat{\varphi}_9(0)\right) + \nu_p \widehat{\varphi}_6(0)},$$

$$M_7 = \frac{1}{\widehat{\varphi}_2(0) - \widehat{\varphi}_7(0)}, \quad M_8 = \frac{\nu_1}{\widehat{\varphi}_1(0) - \widehat{\varphi}_9(0)}, \quad M_9 = \frac{1}{\widehat{\varphi}_1(0) - \widehat{\varphi}_9(0)}.$$

The equation (39)–(44) contains unknown functions $\frac{\partial \omega_j}{\partial z}$, $j = 1, \ldots, 9$. For them we will receive integral equations from (24) by differentiating them with respect to the variable z. Using the notation (28)–(32), we obtain the integral equations for them

$$\upsilon_i^3(z,t) = \upsilon_i^{03}(z,t) + \int\limits_{t_0^i}^{t} \frac{\partial}{\partial z}\left[\sum_{j=1}^{9} \widehat{p}_{ij}(\xi)\upsilon_j^1(\xi,\tau) - \frac{\upsilon_6^2(\tau)}{2}\left(\widehat{\varphi}_1 - \widehat{\varphi}_9\right)(\xi)\right]\Bigg|_{\xi=\mu_i^{-1}[\tau - t + \mu_i(z)]} d\tau$$
$$+ \frac{\partial}{\partial z} t_0^i \int\limits_0^{t_0^i} \left(\upsilon_1^1 - \upsilon_9^1\right)\left(z_0^i, t_0^i - \tau\right) d\tau \tag{45}$$
$$- \int\limits_{t_0^i}^{t}\int\limits_0^{\tau} \frac{\upsilon_6^2(\alpha)}{2}\frac{\partial}{\partial z}\left(\upsilon_1^1 - \upsilon_9^1\right)(\xi, \tau-\alpha)\, d\alpha\Bigg|_{\xi=\mu_i^{-1}[\tau - t + \mu_i(z)]} d\tau, \quad i = 1, 9,$$

$$\upsilon_i^3(z,t) = \upsilon_i^{03}(z,t) + \int\limits_{t_0^i}^{t} \frac{\partial}{\partial z}\left[\sum_{j=1}^{9} \widehat{p}_{ij}(\xi)\upsilon_j^1(\xi,\tau) - \frac{\upsilon_5^2(\tau)}{2}\left(\widehat{\varphi}_2 - \widehat{\varphi}_7\right)(\xi)\right]\Bigg|_{\xi=\mu_i^{-1}[\tau - t + \mu_i(z)]} d\tau$$
$$+ \frac{\partial}{\partial z} t_0^i \int\limits_0^{t_0^i} \left(\upsilon_2^1 - \upsilon_7^1\right)\left(z_0^i, t_0^i - \tau\right) d\tau \tag{46}$$
$$- \int\limits_{t_0^i}^{t}\int\limits_0^{\tau} \frac{\upsilon_5^2(\tau)}{2}\frac{\partial}{\partial z}\left(\upsilon_2^1 - \upsilon_7^1\right)(\xi, \tau-\alpha)\, d\alpha\Bigg|_{\xi=\mu_i^{-1}[\tau - t + \mu_i(z)]} d\tau, \quad i = 2, 7,$$

$$\upsilon_i^3(z,t) = \upsilon_i^{03}(z,t) + \int\limits_{t_0^i}^{t} \frac{\partial}{\partial z}\left[\sum_{j=1}^{9} \widehat{p}_{ij}(\xi)\upsilon_j^1(\xi,\tau) - \beta_i\frac{\upsilon_3^2(\tau)}{2}\left(\widehat{\varphi}_3 - \widehat{\varphi}_8\right)(\xi)\right]\Bigg|_{\xi=\mu_i^{-1}[\tau - t + \mu_i(z)]} d\tau$$
$$+ \frac{\partial}{\partial z} t_0^i \int\limits_0^{t_0^i} \left(\upsilon_3^1 - \upsilon_8^1\right)\left(z_0^i, t_0^i - \tau\right) d\tau \tag{47}$$
$$- \int\limits_{t_0^i}^{t}\int\limits_0^{\tau} \frac{\upsilon_3^3(\alpha)}{2}\frac{\partial}{\partial z}\left(\upsilon_3^1 - \upsilon_8^1\right)(\xi, \tau-\alpha)\, d\alpha\Bigg|_{\xi=\mu_i^{-1}[\tau - t + \mu_i(z)]} d\tau, \quad i = 3, 8,$$

$$\upsilon_4^3(z,t)=\int_0^t\sum_{j=1}^9\frac{\partial}{\partial z}\left[\widehat{p}_{4j}(z)\upsilon_j^1(z,\tau)\right]d\tau+\int_0^t\int_0^\tau\upsilon_1^2(\alpha)\frac{\partial}{\partial z}\upsilon_4^1(z,\tau-\alpha)\,d\alpha\,d\tau$$
$$+\int_0^t\int_0^\tau\left(\upsilon_1^2(\alpha)-\upsilon_4^2(\alpha)\right)\frac{\partial}{\partial z}\left[\frac{\lambda}{\nu_p}\left(\upsilon_1^1-\upsilon_9^1\right)(z,\tau-\alpha)+\upsilon_6^1(z,\tau-\alpha)\right]d\alpha\,d\tau$$
$$+\int_0^t\left[\left(\upsilon_1^2(\tau)-\upsilon_4^2(\tau)\right)\frac{\partial}{\partial z}\left(\frac{\lambda}{\nu_p}\left(\widehat{\varphi}_1-\widehat{\varphi}_9\right)(z)+\widehat{\varphi}_6(z)\right)+\upsilon_1^2(\tau)\frac{\partial}{\partial z}\widehat{\varphi}_4(z)\right]d\tau,\quad (48)$$

$$\upsilon_5^3=\int_0^t\sum_{j=1}^9\frac{\partial}{\partial z}\left(\widehat{p}_{5j}\upsilon_j^1(z,\tau)\right)d\tau+\int_0^t\int_0^\tau\upsilon_2^1(\alpha)\frac{\partial}{\partial z}\upsilon_5^1(z,\tau-\alpha)\,d\alpha\,d\tau+\int_0^t\upsilon_2^2(\tau)\frac{\partial}{\partial z}\widehat{\varphi}_1(z)\,d\tau,\quad (49)$$

$$\upsilon_6^3(z,t)=\int_0^t\sum_{j=1}^9\frac{\partial}{\partial z}\left[\widehat{p}_{6j}(z)\upsilon_j^1(z,\tau)\right]d\tau+\int_0^t\int_0^\tau\upsilon_1^2(\alpha)\frac{\partial}{\partial z}\upsilon_6^1(z,\tau-\alpha)\,d\alpha\,d\tau$$
$$+\int_0^t\int_0^\tau\left(\upsilon_4^2(\alpha)-\upsilon_6^2(\alpha)\right)\frac{\partial}{\partial z}\left[\frac{\lambda}{\nu_p}\left(\upsilon_1^1-\upsilon_9^1\right)(z,\tau-\alpha)\right]d\alpha\,d\tau$$
$$+\int_0^t\left[\left(\upsilon_4^2-\upsilon_6^2\right)(\tau)\frac{\partial}{\partial z}\left[\frac{\lambda}{\nu_p}\left(\widehat{\varphi}_1(z)-\widehat{\varphi}_9(z)\right)\right]+\upsilon_1^2(\tau)\frac{\partial}{\partial z}\widehat{\varphi}_6(z)\right]d\tau,\quad (50)$$

where

$$\upsilon_i^{03}(z,t)=\frac{\partial}{\partial z}\omega_i(z_0^i,t_0^i)-\frac{\partial}{\partial z}t_0^1\left[\sum_{j=1}^9\widehat{p}_{ij}(z_0^9)\,\omega_j(z_0^i,t_0^i)\right],\quad i=1,2,3,7,8,9.$$

$$\lambda\left[\widehat{\varphi}_1(0)-\widehat{\varphi}_9(0)\right]+\nu_p\widehat{\varphi}_6(0)+\nu_p\widehat{\varphi}_4(0)\neq0,\quad \widehat{\varphi}_5(0)\neq0,\quad \widehat{\varphi}_3(0)-\widehat{\varphi}_8(0)\neq0,\quad (51)$$
$$\lambda\left[\widehat{\varphi}_1(0)-\widehat{\varphi}_9(0)\right]+\nu_p\widehat{\varphi}_6(0)\neq0,\quad \widehat{\varphi}_2(0)-\widehat{\varphi}_7(0)\neq0,\quad \widehat{\varphi}_1(0)-\widehat{\varphi}_9(0)\neq0.\quad (52)$$

We require the fulfillment of the matching conditions

$$-\nu_j\frac{\partial\widehat{\varphi}_i(z)}{\partial z}\bigg|_{z=0}+\sum_{j=1}^9\widehat{p}_{ij}(0)\widehat{\varphi}_j(0)=\frac{d}{dt}\widehat{h}_i\bigg|_{t=0},\quad i=1,\ldots,6.\quad (53)$$

3. Main Result

The main result of this work is the following theorem:

Theorem 2. *Let the conditions of Theorem 1 are satisfied, besides function $h(x_1;x_2;t)$ have compact support in x_1, x_2 for each fixed t, $\widehat{\varphi}_i(z)\in C^2[0,H]$, $i=1,\ldots,9$, $\widehat{g}_i(t)\in C^2[0,\ H]$, $i=1,2,3,7,8,9$, $\widehat{h}_i(t)\in C^2[0,H]$, $i=1,\ldots,6$, equalities (51), (52) and matching conditions (25), (26), (53) hold. Then for any $H>0$ on the segment $\left[0,H\right]$ there is a unique solution to the inverse problems (15)–(18) from class $r'_{ij}(t)\in C[0,H]$, $i,j=1,2,3$.*

◁ Equations (33)–(50) supplemented by the initial and boundary conditions from the equalities (19) forms a closed system of equations for the unknowns $\omega_i(z,t)$, $i=1,\dots,9$, $r'_{ij}(t)$, $i,j=1,2,3$, $\frac{\partial}{\partial z}\omega_i(z,t)$, $i=1,\dots,9$. Consider now a square $D_0 := \{(z,t):\ 0\leqslant z\leqslant H,\, 0\leqslant t\leqslant H\}$.

Then, these equations show that the values of the functions $\omega_i(z,t)$, $i=1,\dots,9$, $r'_{ij}(t)$, $i,j=1,2,3$, $\frac{\partial}{\partial z}\omega_i(z,t)$, $i=1,\dots,9$ at $(z,t)\in D_0$ are expressed through integrals of some combinations of the same functions over segments lying in D_0.

The system of equations (33)–(50) we rewrite in the operator form

$$\upsilon = A\upsilon, \tag{54}$$

where the operator $A=(A_i^1,A_j^2,A_i^3)$, $i=1,\dots,9$, $j=1,\dots,6$, in accordance with the right-hand sides follow equations (33)–(50).

Let $C_s(D_0)$, $(s\geqslant 0)$ be the Banach space of continuous functions induced by the family at weighted norms $\|\cdot\|_s$,

$$\|\upsilon\|_s=\max\left\{\max_{1\leqslant i\leqslant 9}\sup_{(z,t)\in D_0}\left|\upsilon_i^1(z,t)e^{-st}\right|,\ \max_{1\leqslant i\leqslant 6}\sup_{t\in[0,H]}\left|\upsilon_i^2(t)e^{-st}\right|,\ \max_{1\leqslant i\leqslant 9}\sup_{(z,t)\in D_0}\left|\upsilon_i^3(z,t)e^{-st}\right|\right\}.$$

Obviously, C_s with $s=0$ is the usual space of continuous functions with the ordinary norm, denoted by $\|\cdot\|$ in what follows. Because $e^{-sH}\|\upsilon\|\leqslant\|\upsilon\|_s\leqslant\|\upsilon\|$, the norms $\|\upsilon\|_s$ and $\|\upsilon\|$ are equivalent for any $H\in(0,\infty)$. We choose that number s later.

Next, consider the set of functions $S(\upsilon^0,r)\subset C_s(D_0)$, satisfying the inequality

$$\left\|\upsilon-\upsilon^0\right\|_s\leqslant r, \tag{55}$$

where r is a known number, the vector function

$$\upsilon^0(z,t)=\left(\upsilon_i^{01}(z,t),\ i=1,\dots,9,\ \ \upsilon_i^{02}(t),\ i=1,\dots,6,\ \ \upsilon_i^{03}(z,t),\ i=1,\dots,9\right),$$

defined by the free terms of the operator equation (54). It is easy to see that for $\upsilon\in S(\upsilon^0,r)$ the estimate $\|\upsilon\|_s\leqslant\|\upsilon^0\|_s+r\leqslant\|\upsilon^0\|+r:=r_0$. Thus, r_0 is a known number.

Let us introduce the following notation:

$$\varphi_0:=\max_{i=1,\dots,9}\left\{\|\widehat{\varphi}_i\|_{C^2[0,H]}\right\},\quad g_0:=\max_{i=1,2,3,7,8,9}\left\{\|\widehat{g}_i\|_{C^2[0,H]}\right\},\quad h_0:=\max_{i=1,\dots,6}\left\{\left\|\widehat{h}_i\right\|_{C^2[0,H]}\right\},$$

$$M_1^0=\max_{1,\dots,9}\left\{\left\|M_i(z)\right\|_{C[0,H]}\right\},\quad M_2^0=\max_{i,j=1,\dots,9}\left\{\left\|\widehat{p}_{ij}(z)\right\|_{C[0,H]}\right\},$$

$$M_3^0=\max_{i=1,2,3,7,8,9}\left\{\left\|\frac{\lambda(z)}{\nu_p(z)}\right\|_{C[0,H]},\left\|\lambda(z)\right\|_{C^1[0,H]},\ \left\|\frac{\partial t_1^i}{\partial z}\right\|_{C[0,H]}\right\},\ M^0=\max\left\{M_1^0,M_2^0,M_3^0\right\}.$$

Note that the operator A maps the space $C_s(D_0)$ into itself. Let us show that for a suitable choice of s (recall that $H>0$ — is an arbitrary fixed number) it is on the set $S(\upsilon^0,r)$ a contraction operator. First, let us make sure that the operator A takes the set $S(\upsilon^0,r)$ into itself, that is, it follows from the condition $\upsilon(z,t)\in S(\upsilon^0,r)$ that $A\upsilon\in S(\upsilon^0,r)$, if s satisfies some constraints. In fact, for any $(z,t)\in D_0$ and $\upsilon\in S(\upsilon^0,r)$ the following inequalities hold:

$$\left\|A_1^1\upsilon-\upsilon_1^{01}\right\|_s=\sup_{(z,t)\in D_0}\left|\left(A_1^1\upsilon-\upsilon_1^{01}\right)e^{-st}\right|$$

$$\leqslant\left|\int\limits_{t_0^i}^{t}\left[\sum_{j=1}^{9}\widehat{p}_{1j}(\xi)\upsilon_j^1(\xi,\tau)e^{-s\tau}+\frac{\upsilon_6^2(\tau)e^{-s\tau}}{2}\left(\widehat{\varphi}_1-\widehat{\varphi}_9\right)(\xi)\right]e^{-s(t-\tau)}\Big|_{\xi=\mu_i^{-1}[t-\tau+\mu_i(z)]}d\tau\right.$$

$$\left.+\int\limits_{t_0^i}^{t}e^{-s(t-\tau)}\int\limits_0^{\tau}\frac{\upsilon_6^2(\alpha)e^{-s\alpha}}{2}\left(\upsilon_1^1-\upsilon_9^1\right)(\xi,\tau-\alpha)e^{-s(\tau-\alpha)}\,d\alpha\Big|_{\xi=\mu_i^{-1}[t-\tau+\mu_i(z)]}d\tau\right|$$

$$\leqslant\left(9M^0\|\upsilon\|_s+\varphi_0\|\upsilon\|_s+\|\upsilon\|_s\|\upsilon\|_s\right)\int\limits_{t_0^1}^{t}e^{-s(t-\tau)}d\tau\leqslant\frac{r_0}{s}\left(9M^0+\varphi_0+r_0\right):=\frac{\gamma_{11}}{s}r_0.$$

Using similar calculations for the remaining equations. Finally, we get

$$\left\|A\upsilon-\upsilon^0\right\|_s\leqslant\frac{1}{s}\max\left\{\max_{j=1,\dots,9}\{\gamma_{1j}\},\ \max_{j=1,\dots,6}\{\gamma_{2j}\},\ \max_{j=1,\dots,9}\{\gamma_{3j}\}\right\}:=\frac{r_0}{s}\gamma^0,$$

where

$$\gamma_{1j}:=9M^0+\varphi_0+r_0,\ \ j=2,3,7,8,9,\quad \gamma_{14}:=13M^0+4M^0r_0+3r_0\varphi_0+3\varphi_0,$$

$$\gamma_{15}:=9M^0+\varphi_0+r_0,\quad \gamma_{16}:=9M^0+9M^0r_0+4M^0\varphi_0+\varphi_0,$$

$$\gamma_{21}:=2r_0+2\varphi_0r_0+5\left(M^0\right)^2(g_0+h_0)+4h_0+9\left(M^0\right)^3,\quad \gamma_{22}:=M^0h_0,$$

$$\gamma_{24}:=\left(M^0\right)^2(3g_0+3h_0+2\varphi_0+18),\quad \gamma_{2j}:=2M^0\left(r_0(1+\varphi_0)+\varphi_0+h_0+9\right),\ \ j=3,5,6,$$

$$\gamma_{3j}:=11M^0+\varphi_0+r_0,\ \ j=1,2,3,7,8,9,\quad \gamma_{34}:=13M^0+4M^0r_0+3\varphi_0(r_0+1),$$

$$\gamma_{35}:=9M^0+r_0+\varphi_0,\quad \gamma_{36}:=9M^0+9M^0r_0+\varphi_0(4M^0+1).$$

Choosing $s>(1/r)\gamma^0$, we get that the operator A maps the set $S\left(\upsilon^0,r\right)$ into itself.

Now, let υ and $\widetilde{\upsilon}$ be two arbitrary elements in $S(\upsilon^0,r)$. Using the obvious inequality

$$\left|\upsilon_i^k\upsilon_i^l-\widetilde{\upsilon}_i^k\widetilde{\upsilon}_i^l\right|e^{-st}\leqslant\left|\upsilon_i^k-\widetilde{\upsilon}_i^k\right|\left|\upsilon_i^l\right|e^{-st}+\left|\widetilde{\upsilon}_i^k\right|\left|\upsilon_i^l-\widetilde{\upsilon}_i^l\right|e^{-st}\leqslant 2r_0\|\upsilon-\widetilde{\upsilon}\|_s,\quad (z,t)\in D_0,$$

after some easy estimations, we find that for $(z,t)\in D_0$,

$$\left\|A_1^1\upsilon-A_1^1\widetilde{\upsilon}\right\|_s=\sup_{(z,t)\in D_0}\left|\left(A_1^1\upsilon-A_1^1\widetilde{\upsilon}\right)e^{-st}\right|\leqslant\frac{\|\upsilon-\widetilde{\upsilon}\|_s}{s}\left[9M^0+\varphi_0+4r_0\right]:=\frac{1}{s}\gamma_{41}\left\|\upsilon-\widetilde{\upsilon}\right\|_s$$

and hence we have

$$\|A\upsilon-A\widetilde{\upsilon}\|_s=\frac{\|\upsilon-\widetilde{\upsilon}\|_s}{s}\max\left\{\max_{j=1,\dots,9}\{\gamma_{4j}\},\max_{j=1,\dots,6}\{\gamma_{5j}\},\max_{j=1,\dots,9}\{\gamma_{6j}\}\right\}:=\frac{1}{s}\gamma^1\left\|\upsilon-\widetilde{\upsilon}\right\|_s,$$

where

$$\gamma_{4j}:=9M^0+\varphi_0+4r_0,\ \ j=2,3,7,8,9,\quad \gamma_{44}:=13M^0+4M^0r_0+12r_0\varphi_0+3\varphi_0,$$

$$\gamma_{45}:=9M^0+\varphi_0+4r_0,\quad \gamma_{46}:=9M^0+9M^0r_0+4M^0\varphi_0+4\varphi_0,$$

$$\gamma_{51} := 2r_0 + 4\varphi_0 r_0 + 10\left(M^0\right)^2(g_0+h_0) + 10h_0 + 9\left(M^0\right)^3, \quad \gamma_{52} := M^0 h_0,$$

$$\gamma_{54} := \left(M^0\right)^2(12g_0 + 3h_0 + 10\varphi_0 + 18), \quad \gamma_{5j} := M^0\left(r_0(1+\varphi_0) + \varphi_0 + 4h_0 + 9\right), \;\; j = 3,5,6,$$

$$\gamma_{6j} := 11M^0 + \varphi_0 + r_0, \;\; j = 1,2,3,7,8,9, \quad \gamma_{64} := 13M^0 + 4M^0 r_0 + 3\varphi_0(r_0+1),$$

$$\gamma_{65} := 9M^0 + r_0 + \varphi_0, \quad \gamma_{66} := 9M^0 + 9M^0 r_0 + \varphi_0(4M^0+1).$$

Choosing now $s > \gamma^1$, we get, that the operator A compresses the distance between the elements $\upsilon, \widetilde{\upsilon}$ to $S\left(\upsilon^0, r\right)$.

As follows from the performed estimates, if the number s is chosen from conditions $s > s^* := \max\{\gamma^0, \gamma^1\}$, then the operator A is contracting on $S\left(\upsilon^0, r\right)$. In this case, according to the principle Banach the equation [25, pp. 87–97] (54) has the only solution in $S\left(\upsilon^0, r\right)$ for any fixed $H > 0$. Theorem 2 is proved. $\rhd$

By the found functions $r'_{ij}(t)$, $i,j = 1,2,3$, the functions $r_{ij}(t)$, $i,j = 1,2,3$, are found by the formulas

$$r_{ij}(t) = r_{ij}(0) + \int_0^t r'_{ij}(\tau)\, d\tau, \quad i,j = 1,2,3.$$

Note that by the functions $r_{ij}(t)$, $i,j = 1,2,3$, the functions $K_{ij}(t)$, $i,j = 1,2,3$, are defined as solutions of integral equations (4).

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ОБРАТНАЯ ЗАДАЧА ДЛЯ ВЯЗКОУПРУГОЙ СИСТЕМЫ В ВЕРТИКАЛЬНО-СЛОИСТОЙ СРЕДЕ

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Аннотация. В данной работе рассматривается трехмерная система уравнений вязкоупругости первого порядка, написанная относительно перемещение и тензора напряжения. Эта система содержит свёрточные интегралы ядер релаксации с решением прямой задачи. Прямая задача есть начально-краевая задача для данной системы интегродифференциальных уравнений. В обратной задаче требуется определить ядра релаксации по заданным для некоторых компонент Фурье преобразования по переменным x_1 и x_2 решения прямой задачи на боковых границах рассматриваемой области. В начале методом сведения к интегральным уравнениям и последующим применением метода последовательных приближений изучаются свойства решения прямой задачи. Для обеспечения непрерывного решения получены условия гладкости и согласования начальных и граничных данных в угловых точках области. Чтобы решить обратную задачу методом характеристик она сводится к эквивалентной замкнутой системе интегральных уравнений вольтерровского типа второго рода относительно преобразования Фурье по первым двум пространственным переменным x_1, x_2, для решения прямой задачи и неизвестных обратной задачи. Далее к этой системе, написанной в виде операторного уравнения применяется метод сжимающих отображений в пространстве непрерывных функций с весовой экспоненциальной нормой. Показывается, что при подходящем выборе параметра в показателе экспоненты, этот оператор являются сжимающим в некотором шаре, который является подмножеством класса непрерывных функций. Таким образом, доказывается глобальная теорема существования и единственности решения поставленной задачи.

Ключевые слова: вязкоупругость, резольвента, обратная задача, гиперболическая система, преобразование Фурье.

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