O‘ZBEKISTON RESPUBLIKASI

OLIY TA’LIM, FAN VA INNOVATSIYALAR VAZIRLIGI

BUXORO DAVLAT UNIVERSITETI

Z.R.Bozorov

INTEGRAL ALMASHTIRISHLARDAN MASALALAR TO‘PLAMI

Uslubiy qo‘llanma

“Durdona” nashriyoti

Buxoro – 2024

Ushbu uslubiy qo‘llanma Oliy ta’lim muassasalarining 70540101 - Matematika(yo’nalishlar bo’yicha) mutaxassisligida tahsil olayotgan magistrantlar uchun mo‘ljallab yozilgan. Qo‘llanmada xosmas integrallar, maxsus funksiyalar va bir nechta turdagi integral almashtirishlarga doir misol masalalar keltirilgan.

Taqrizchilar:

Teshayev Muxsin Xudoyberdiyevich, Buxoro muhandislik- texnologiya instituti Oliy matematika kafedrasi professori, (DSc).

Nuriddinov Javlon Zafarovich, Buxoro davlat universiteti Differensial tenglamalar kafedrasi dotsenti, (PhD).

KIRISH

Integral almashtirishlar matematika va texnikada kuchli vosita bo'lib, differentsial tenglamalar, signallarni tahlil qilish, tasvirni qayta ishlash va boshqa muammolarni hal qilish uchun ishlatiladi. Integral almashtirishlar ko'p hollarda masalalarni tahlil qilish va yechishda juda qulaybo’lib, Ular funktsiyalarni bir fazodan boshqasiga almashtirishga imkon beradi,. Integral almashtirishlarning asosiy turlariga Furye, Laplas, Xartli va Hilbert kabi almashtirishlar kiradi.

Integral almashtirishlarning qo'llanilish sohalari juda keng.

Differensial tenglamalarni yechishda integral almashtirishlar murakkab differensial tenglamalarni algebraik tenglamalarga aylantirish orqali yechish imkonini beradi.

Signal tahlil qilishda Furye almashtirishi va uning diskret versiyasi signallarning chastota komponentlarini tahlil qilish uchun signalni qayta ishlashda keng qo'llaniladi.

Rasmlarga ishlov berishda integral almashtirishlar tasvirlarni filtrlash, siqish va tiklash uchun ishlatiladi.

Boshqarish nazariyasida Laplas integral almashtirishi dinamik tizimlarni tahlil qilish va boshqaruv tizimlarini sintez qilishga yordam beradi.

Optika va kvant mexanikasida integral almashtirishlar to'lqin funktsiyalari va to'lqin tarqalishini tahlil qilish uchun ishlatiladi.

Integral almashtirishlar matematika va texnikaning asosiy vositalari bo'lib, fan va texnikaning turli sohalaridagi murakkab muammolarni chuqurroq tushunish va samarali hal qilishga yordam beradi.

I. XOSMAS INTEGRALLAR

1. Xosmas integrallar

Funksiyaning aniq integrali (Riman integrali) tushunchasini kiritishda integrallash oralig‘ining chekli bo’lishi talab etilgan edi.

Endi cheksiz oraliqda ($[a,+\infty)$; $(-\infty,a]$; $(-\infty,+\infty)$ oraliqlarda) berilgan funksiyaning shu oraliq bo‘yicha integral tushunchasini keltiramiz.

Chegaralari cheksiz xosmas integral tushunchasi. $f(x)$ funksiya $[a,+\infty)$ oraliqda $(a\in R)$ berilgan bo‘lib, ixtiyoriy $[a,t]$ da $(a\le t<+\infty)$ integrallanuvchi bo‘lsin: $f(x)\in R([a,t])$.

Ushbu

$$F(t)=\int_a^t f(x)dx$$

belgilashni kiritamiz.

1-ta’rif. Agar $t\to+\infty$ da $F(t)$ funksiyaning limiti mavjud bo‘lsa, bu limiti $f(x)$ funksiyaning $[a,+\infty)$ cheksiz oraliq bo‘yicha xosmas integrali deyiladi va

$$\int_a^{+\infty} f(x)dx$$

kabi belgilanadi:

$$\int_a^{+\infty} f(x)dx=\lim_{t\to+\infty}F(t)=\lim_{t\to+\infty}\int_a^t f(x)dx. \qquad (1)$$

(1) integralni chegarasi cheksiz xosmas integral ham deb yuritiladi.

Qulaylik uchun, bundan keyin “chegarasi cheksiz xosmas integral” o‘rniga “integral” deymiz.

2-ta’rif. Agar $t\to+\infty$ da $F(t)$ funksiyaning limiti mavjud va chekli bo‘lsa, (1) integral yaqinlashuvchi deyiladi.

Agar $t\to+\infty$ da $F(t)$ funksiyaning limiti cheksiz yoki mavjud bo‘lmasa, (1) integral uzoqlashuvchi deyiladi.

1-misol. Ushbu

$$\int_0^{+\infty} e^{-x}dx$$

integralni qaraylik. Bu holda

$$F(t)=\int_0^t e^{-x}dx=-e^{-t}+1$$

bo‘lib,

$$\lim_{t\to+\infty}F(t)=1$$

bo‘ladi.

Demak, berilgan integral yaqinlashuvchi va

$$\int_0^{+\infty} e^{-x}dx=1.$$

2-misol. Ushbu $\int_a^{+\infty} \frac{dx}{x^\alpha}$ $(a > 0, \ \alpha > 0)$

integral uchun

$$F(t) = \int_a^t \frac{dx}{x^\alpha} = \begin{cases} \ln t - \ln a, & \text{агар} \quad \alpha = 1 \text{ бўлса} \\ \dfrac{t^{-\alpha+1}}{-\alpha+1} - \dfrac{a^{-\alpha+1}}{-\alpha+1}, & \text{агар} \ \alpha \neq 1 \text{ бўлса} \end{cases},$$

bo‘lib, $t \to +\infty$ da

$$F(t) \to \frac{a^{1-\alpha}}{\alpha - 1} \qquad (\alpha > 1),$$

$$F(t) \to +\infty \qquad (\alpha \le 1)$$

bo‘ladi. Demak,

$$\int_a^{+\infty} \frac{dx}{x^\alpha}$$

integral $\alpha > 1$ bo‘lganda yaqinlashuvchi, $\alpha \le 1$ bo‘lganda uzoqlashuvchi bo‘ladi.

3-misol. Ushbu

$$\int_0^{+\infty} \cos x dx$$

integral uzoqlashuvchi bo‘ladi, chunki $t \to +\infty$ da

$$F(t) = \int_0^t \cos x dx = \sin t$$

funksiyaning limiti mavjud emas.

Yuqoridagidek,

$$\int_{-\infty}^{a} f(x)dx, \quad \int_{-\infty}^{+\infty} f(x)dx$$

xosmas integrallar va ularning yaqinlashuvchiligi, uzoqlashuvchiligi ta’riflanadi:

$$\int_{-\infty}^{a} f(x)dx = \lim_{t \to -\infty} \int_t^a f(x)dx \ ,$$

$$\int_{-\infty}^{+\infty} f(x)dx = \lim_{\substack{u \to +\infty \\ v \to -\infty}} \int_v^u f(x)dx \ .$$

Yaqinlashuvchi xosmas integralning xossalari. Xosmas integralning turli xossalarini $f(x)$ funksiyaning $[a, +\infty)$ oraliq bo‘yicha olingan

$$\int_a^{+\infty} f(x)dx$$

integrali uchun keltiramiz. Bu xossalar

$$\int_{-\infty}^{a} f(x)dx, \quad \int_{-\infty}^{+\infty} f(x)dx$$

integrallar uchun ham o’xshash keltiriladi.

1-xossa. Agar $\int_a^{+\infty} f(x)dx$ integral yaqinlashuvchi bo‘lsa, u holda

$$\int_b^{+\infty} f(x)dx \quad (a < b)$$

integral ham yaqinlashuvchi bo‘ladi va aksincha . Bunda

$$\int_a^{+\infty} f(x)dx = \int_a^b f(x)dx + \int_b^{+\infty} f(x)dx \qquad (2)$$

tenglik bajariladi.

2-xossa. Agar $\int_a^{+\infty} f(x)dx$ integral yaqinlashuvchi bo‘lsa, u holda $\int_a^{+\infty} C \cdot f(x)dx$ ham $(C = const)$ yaqinlashuvchi bo‘lib,

$$\int_a^{+\infty} C \cdot f(x)dx = C \int_a^{+\infty} f(x)dx$$

bo‘ladi.

3-xossa. Agar $\int_a^{+\infty} f(x)dx$ integral yaqinlashuvchi bo‘lib, $\forall x \in [a,+\infty)$ da $f(x) \geq 0$ bo‘lsa, u holda

$$\int_a^{+\infty} f(x)dx \geq 0$$

bo‘ladi.

4-xossa. Agar $\int_a^{+\infty} f(x)dx$ va $\int_a^{+\infty} g(x)dx$ integrallar yaqinla-shuvchi bo‘lsa, u holda $\int_a^{+\infty} (f(x) \pm g(x))dx$ integral ham yaqinla-shuvchi bo‘lib,

$$\int_a^{+\infty} (f(x) \pm g(x))dx = \int_{\mathrm{a}}^{+\infty} f(x)dx \pm \int_a^{+\infty} g(x)dx$$

bo‘ladi.

5-xossa. Agar $\forall x \in [a,+\infty)$ da $f(x) \leq g(x)$ bo‘lib, $\int_a^{+\infty} f(x)dx$ va $\int_a^{+\infty} g(x)dx$ integrallar yaqinlashuvchi bo‘lsa, u holda

$$\int_a^{+\infty} f(x)dx \leq \int_a^{+\infty} g(x)dx$$

bo‘ladi.

Faraz qilaylik, $f(x)$ va $g(x)$ funksiyalar $[a,+\infty)$ da berilgan bo‘lib, $f(x)$ funksiya chegaralangan $(m \leq f(x) \leq M, \quad x \in [a,+\infty))$, $g(x)$ funksiya esa o‘z ishorasini o‘zgartirmasin $(\forall x \in [a,+\infty)$ da har doim $g(x) \geq 0$ yoki $g(x) \leq 0)$.

6-xossa. Agar $\int\limits_a^{+\infty} f(x)\cdot g(x)dx$ va $\int\limits_a^{+\infty} g(x)dx$ integrallar yaqin-lashuvchi bo‘lsa, u holda shunday o‘zgarmas $\mu(m \le \mu \le M)$ topiladiki,

$$\int\limits_a^{+\infty} f(x)\cdot g(x)dx = \mu \int\limits_a^{+\infty} g(x)dx \qquad (3)$$

bo‘ladi.

Odatda, bu xossa o‘rta qiymat haqidagi teorema deyiladi.

Xosmas integralning yaqinlashuvchiligi haqida quyidagi teorema o’rinli.

Teorema (Koshi teoremasi). $\int\limits_a^{+\infty} f(x)dx$ integralning yaqinla-shuvchi bo‘lishi uchun $\forall \varepsilon > 0$ son olinganda ham shunday $t_0 \in R$ $(t_0 > a)$ topilib, ixtiyoriy $t' > t_0$, $t'' > t_0$ bo‘lganda

$$\left| \int\limits_{t'}^{t''} f(x)dx \right| < \varepsilon$$

tengsizlikning bajarilishi zarur va yetarli.

Xosmas integralning absolyut yaqinlashuvchiligi. Aytaylik, $f(x)$ funksiya $[a,+\infty)$ oraliqda berilgan bo‘lsin. Bunda, $\forall x \in [a,+\infty)$ uchun $f(x) \ge 0$ bo‘lishi shart emas

Ta’rif. Agar

$$\int\limits_a^{+\infty} |f(x)|dx$$

integral yaqinlashuvchi bo‘lsa, $\int\limits_a^{+\infty} f(x)dx$ integral absolyut yaqinlashuvchi deyiladi.

Agar $\int\limits_a^{+\infty} f(x)dx$ yaqinlashuvchi bo‘lib, $\int\limits_a^{+\infty} |f(x)|dx$ uzoqlashuvchi bo‘lsa, u holda $\int\limits_a^{+\infty} f(x)dx$ shartli yaqinlashuvchi integral deyiladi.

Teorema. Agar integral absolyut yaqinlashuvchi bo‘lsa, u yaqinlashuvchi bo‘ladi.

Isbot. Aytaylik,

$$\int\limits_a^{+\infty} |f(x)|dx$$

integral yaqinlashuvchi bo‘lsin. Berilgan $f(x)$ va $|f(x)|$ funk-siyalar yordamida ushbu

$$\varphi(x) = \frac{1}{2}(f(x) + |f(x)|) \quad ,$$

$$\psi(x) = \frac{1}{2}(-f(x) + |f(x)|)$$

funksiyalarni tuzamiz.

Bu funksiyalar uchun, $\forall x \in [a,+\infty)$ da

1) $\varphi(x) \geq 0$, $\psi(x) \geq 0$

2) $\varphi(x) \leq |f(x)|$, $\psi(x) \leq |f(x)|$

3) $\varphi(x) - \psi(x) = f(x)$

bo‘ladi. Yuqorida keltirilgan 2-teoremadan foydalanib, quyidagi

$$\int_a^{+\infty} \varphi(x)dx, \quad \int_a^{+\infty} \psi(x)dx$$

integral yaqinlashuvchiligini topamiz.

Unda

$$\int_a^{+\infty} (\varphi(x) - \psi(x))dx$$

integral ham yaqinlashuvchi bo‘ladi. Demak,

$$\int_a^{+\infty} f(x)dx$$

yaqinlashuvchi bo‘ladi. Teorema isbotlandi.

Integralning yaqinlashuvchiligi alomatlari. Integralning bosh qiymati

1^0. Dirixle alomati. Faraz qilaylik, $f(x)$ va $g(x)$ funksiyalar $[a,+\infty)$ oraliqda berilgan bo‘lsin.

1-teorema (**Dirixle alomati**). $f(x)$ va $g(x)$ funksiyalar quyidagi shartlarni qanoatlantirsin:

1) $f(x)$ funksiya $[a,+\infty)$ da uzluksiz va uning shu oraliqdagi boshlang‘ich $F(x)$ $(F'(x) = f(x))$ funksiyasi chegara-langan;

2) $g(x)$ funksiya $[a,+\infty)$ da uzluksiz $g'(x)$ hosilaga ega ;

3) $g(x)$ funksiya $[a,+\infty)$ da kamayuvchi;

4) $\lim\limits_{x\to+\infty} g(x) = 0$.

U holda

$$\int_a^{+\infty} f(x)g(x)dx$$

integral yaqinlashuvchi bo‘ladi.

Misol. Ushbu

$$J = \int_1^{+\infty} \frac{\sin x}{x^\alpha}dx \qquad (\alpha > 0)$$

integralni yaqinlashuvchilikka tekshirilsin.

Yechish: Berilgan integralni quyidagicha

$$J = \int_1^{+\infty} \sin x \frac{1}{x^\alpha}dx \qquad (\alpha > 0)$$

yozib, $f(x) = \sin x$, $g(x) = \dfrac{1}{x^\alpha}$ deymiz. Bu funksiyalar yuqorida keltirilgan teoremaning barcha shartlarini qanoatlanti-radi.

1) $f(x)=\sin x$ funksiya $[1,+\infty)$ oraliqda uzluksiz va uning boshlang‘ich funksiyasi $F(x)=-\cos x$ funksiya $[1,+\infty)$ da chegara-langan;

2) $g(x)=\dfrac{1}{x^{\alpha}}$ $(\alpha>0)$ funksiya $[1,+\infty)$ da

$$g'(x)=-\frac{\alpha}{x^{\alpha+1}}$$

hosilaga ega va u uzluksiz;

3) $g(x)=\dfrac{1}{x^{\alpha}}$ $(\alpha>0)$ funksiya $[1,+\infty)$ da kamayuvchi;

4) $\lim\limits_{x\to+\infty} g(x)=\lim\limits_{x\to+\infty}\dfrac{1}{x^{\alpha}}=0$. $(\alpha>0)$

Unda Dirixle alomatiga ko‘ra

$$\int_1^{+\infty}\frac{\sin x}{x^{\alpha}}dx \qquad (\alpha>0)$$

integral yaqinlashuvchi bo‘ladi.

2^0. Abel alomati. Faraz qilaylik, $f(x)$ va $g(x)$ funksiyalar $[a,+\infty)$ oraliqda berilgan bo‘lsin.

2-teorema (Abel alomati). $f(x)$ va $g(x)$ funksiyalar quyidagi shartlarni qanoatlantirsin:

1) $f(x)$ funksiya $[a,+\infty)$ da uzluksiz bo‘lib, $\int\limits_a^{+\infty} f(x)dx$ integral yaqinlashuvchi;

2) $g(x)$ funksiya $[a,+\infty)$ da uzluksiz $g'(x)$ hosilaga ega va bu hosila $[a,+\infty)$ da o‘z ishorasini saqlasin;

3) $g(x)$ funksiya $[a,+\infty)$ da chegaralangan.

U holda

$$\int_a^{+\infty} f(x)g(x)dx$$

integral yaqinlashuvchi bo‘ladi.

Misol: $\int_0^{+\infty}(2x+1)e^{-x}dx$ xosmas integralni hisoblang

Yechish: $f(x)=2x+1, g'(x)=e^{-x}$, deb olib bo’laklab integrallaymiz.

$$\int_0^{+\infty}(2x+1)e^{-x}dx=-(2x+1)e^{-x}\ \Big|_0^{+\infty}+\int_0^{\infty}2e^{-x}dx=3$$

Ba’zi xosmas integrallarning qiymatlari. Quyida ba’zi xosmas integ-rallarning qiymatlarini keltiramiz:

1. $\int_0^{+\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$,

2. $\int_0^{+\infty} \sin x^2 dx = \frac{\sqrt{\frac{\pi}{2}}}{2}$,

3. $\int_0^{+\infty} \cos x^2 dx = \frac{\sqrt{\frac{\pi}{2}}}{2}$,

4. $\int_0^{+\infty} \frac{\cos xy}{1+x^2}dx = \frac{\pi}{2}e^{-|y|}$,

5. $\int_0^{+\infty} \frac{x\sin xy}{1+x^2}dx = \frac{\pi}{2} sign\, ye^{-|y|}$,

6. $\int_0^{+\infty} \frac{\sin x^2}{x}dx = \frac{\pi}{4}$,

7. $\int_0^{+\infty} \frac{\sin^4 ax - \sin^4 bx}{x}dx = \frac{3}{8}\ln\frac{a}{b}$,

8. $\int_0^{+\infty} \frac{e^{-ax} - e^{-bx}}{x}\sin nxdx = arctg\frac{b}{n} - arctg\frac{a}{n} \quad (a > 0,\ b > 0,\ n \neq 0)$,

9. $\int_0^{+\infty} \frac{e^{-ax} - e^{-bx}}{x}\cos nxdx = \frac{1}{2}\ln\frac{b^2+n^2}{a^2+n^2}$.

MAVZUGA DOIR MISOLLAR

Xosmas integrallarni hisoblang:

1. $\int_{\infty}^{\infty} \frac{dx}{1+x^2}$

2. $\int_{2}^{+\infty} \frac{dx}{x^2-1}$

3. $\int_{-\infty}^{-1} \frac{dx}{x^2+1}$

4. $\int_{0}^{+\infty} e^{ax} sinbx, \quad a > 0, \qquad b \neq 0$

5.

$$\int_{0}^{+\infty} e^{-x} cosxdx$$

Xosmas integral yaqinlashuvchiligi ta'rifidan foydalanib xosmas integralni yaqinlashishga tekshiring

7.

$$\int_{0}^{+\infty} \frac{dx}{x^2+x+1}$$

8.

$$\int_{-\infty}^{+\infty} \frac{dx}{x^2-x+1}$$

9.

$$\int_{-\infty}^{0} \frac{(x+2)\,dx}{x^2+4}$$

10.

$$\int_{e}^{+\infty} \frac{1}{x^2 ln^2 x} dx$$

11.

$$\int_{0}^{\frac{\pi}{2}} \ln{(cosx)}dx$$

12.

$$\int_{0}^{1} \frac{x^3 arcsinx}{\sqrt{1-x^2}} dx$$

13.

$$\int_{1}^{2} \frac{cosx}{\sqrt{sinx}} dx$$

14.

$$\int_{1}^{\infty} (\sqrt{x+1}-\sqrt{x-1})\,dx$$

15.

$$\int_{1}^{\infty} (\sqrt{x^5+9x}-\sqrt{x^5-5x})\,dx$$

16.

p parametrga nisbatan xosmas integralni yaqilashuvchilikka tekshiring

17.

$$\int_{1}^{\infty} \frac{\ln{(1+sin\frac{1}{x^3})}}{x^p} dx$$

18.

$$\int_{e}^{\infty} \sin\left(\frac{1}{x^3}\right)\frac{1}{\ln^p x} dx$$

2. Xosmas integrallarni hisoblash

1. Nyuton-Leybnis formulasi. Ushbu

$$\int_a^{+\infty} f(x)dx$$

xosmas integral yaqinlashuvchi bo‘lib, uni hisoblash talab etilsin.

$f(x)$ funksiya $[a,+\infty)$ oraliqda boshlang‘ich $F(x)$ funksiyaga ega va $x \to +\infty$ da $F(x)$ funksiya chekli limiti mavjud bo‘lsin:

$$\lim_{x\to+\infty} F(x) = F(+\infty).$$

Unda

$$\int_a^{+\infty} f(x)dx = \lim_{x\to+\infty}\int_a^t f(x)dx =$$
$$= \lim_{t\to+\infty}(F(t) - F(a)) = F(+\infty) - F(a) = F(x)\Big|_a^{+\infty} \qquad (1)$$

bo‘ladi.

(1) formula Nyuton-Leybnis formulasi deyiladi.

1-misol. Ushbu,

$$\int_{\frac{2}{\pi}}^{+\infty} \frac{1}{x^2}\sin\frac{1}{x}dx$$

integral hisoblansin.

Ravshanki, $F(x) = \cos\frac{1}{x}$ funksiya $[\frac{2}{\pi},+\infty)$ oraliqda $f(x) = \frac{1}{x^2}\sin\frac{1}{x}$ funksiyaning boshlang‘ich funksiyasi bo‘ladi.

(1) formuladan foydalanib topamiz:

$$\int_{\frac{2}{\pi}}^{+\infty} \frac{1}{x^2}\sin\frac{1}{x}dx = \cos\frac{1}{x}\Big|_{\frac{2}{\pi}}^{+\infty} = 1.$$

2-misol. Ushbu,

$$\int_{-\infty}^{+\infty} sinx\, dx$$

Integral hisoblansin.

Yechish. $F(x) = -cosx$ funksiya $(-\infty; +\infty)$ oraliqda $sinx$ funksiyaning boshlang’chi, Nyuton -Leybnits formulasiga ko’ra

$$\int_{-\infty}^{+\infty} sinx\, dx = -cosx|_{-\infty}^{+\infty}$$

$x \to \pm\infty$ da $cosx$ funksiya limiti mavjud emas, demak integral uzoqlashuvchi.

2. Bo‘laklab integrallash. Faraz qilaylik, $f(x)$ va $g(x)$ funksiyalar $[a,+\infty)$ oraliqda uzluksiz va uzluksiz, $f'(x)$ va $g'(x)$ hosilalarga ega bo‘lsin.

Agar

1) $\int_a^{+\infty} f(x)\cdot g'(x)dx$ $(\int_a^{+\infty} f'(x)g(x)dx)$ integral yaqinlashuvchi;

2) ushbu $\lim_{x\to+\infty}(f(x)g(x))$ limit mavjud va chekli bo‘lsa, u holda

$$\int_a^{+\infty} f'(x)\cdot g(x)dx \qquad \left(\int_a^{+\infty} f(x)g'(x)dx\right)$$

integral yaqinlashuvchi bo‘lib,

$$\int_a^{+\infty} f'(x)\cdot g(x)dx = \lim_{x\to+\infty}(f(x)g(x)) - f(a)\cdot g(a) - \int_a^{+\infty} f(x)g'(x)dx \qquad (2)$$

$$\left(\int_a^{+\infty} f(x)g'(x)dx = \lim_{x\to+\infty}(f(x)g(x)) - f(a)\cdot g(a) - \int_a^{+\infty} f'(x)g(x)dx\right)$$

bo‘ladi.

2-misol . Ushbu

$$\int_0^{+\infty} xe^{-x}dx$$

integral hisoblansin.

Yechish: Agar $g(x)=x, \quad f'(x)=e^{-x}$ deb olsak, unda

$$g'(x)=1, \quad f(x)=-e^{-x}$$

bo‘lib, (2) formulaga ko‘ra $(a=0)$

$$\int_0^{+\infty} xe^{-x}dx = \lim_{t+\infty}(-te^{-t}) - 0 + \int_0^{+\infty} e^{-x}dx = 1$$

bo‘ladi.

MAVZUGA DOIR MISOLLAR

Xosmas integrallarni bo‘laklab integrallashdan foydalanib hisoblang

1. $\int_0^{+\infty} \frac{e^x\,dx}{1-e^{2x}}$
2. $\int_0^{+\infty} xe^{-x^2}dx$
3. $\int_0^{+\infty} x^3\, e^{-x^2}dx$
4. $\int_0^{+\infty} (3x+2)e^{-x}dx$
5. $\int_0^{+\infty} xe^{-2x}dx$

Xosmas integrallarni Nyuton-Leybnits formulasidan foydalanib hisoblang

6. $\int_1^2 \frac{2x+1}{\sqrt{x^2+x-2}}dx$
7. $\int_0^1 \frac{1}{(2-x)\sqrt{1-x}}dx$

8. $\int_0^{0.5} \frac{1}{x ln^2 x} dx$

9. $\int_0^1 \frac{lnx}{\sqrt{x}} dx$

II-BOB. MAXSUS FUNKSIYALAR.

Klassik maxsus funksiyalarni o'rganish matematik tahlilning bir qismi bo'lib, uning barchasi gipergeometrik funksiyaga qaratilgan bo'lib, ularning differensial tenglamalari mos ravishda uchta va ikkita yagona nuqtalarni, shuningdek, maxsus holatlarni taqdim etadi. Quyida ba’zi maxsua funksiyalar va ularning xossalarini eslatib o’tamiz.

Beta funksiya va uning xossalari.

1-ta’rif.

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx, \quad (a>0, b>0) \quad (1)$$

(1) integral Beta funksiya yoki birinchi tur Eyler integrali deyiladi va $B(a,b)$ kabi belgilanadi, $B(a,b)$ funksiya R^2 fazodagi

$$M = \{(a,b) \in R^2 : \ a \in (0; +\infty), b \in (0; +\infty)\}$$

to’plamda berilgandir. $B(a,b)$ funksiya $M = \{(a,b) \in R^2 : a \in (0,+\infty), b \in (0,+\infty)\}$ to‘plamda uzluksiz bo‘ladi.

Endi funksiyaning xossalarini ko’rib chiqamiz.

1^0. $B(a,b)$ integralni olamiz.

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx \quad (2)$$

Bu integral a va b ga nisbatan simmetrik funksiyalardan iborat, ya’ni

$$B(a,b) = B(b,a) \quad (3)$$

2^0. (1) integral

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$$

ixtiyoriy $M = \{(a,b)\epsilon R^2 :\ a \in (0;\ +\infty), b \in (0;\ +\infty)\}\ (a_0 > 0, b_0 > 0)$ to'plamda tekis yaqinlashuvchi bo'ladi.

3^0 . B (a, b) funksiya $\mathrm{M} = \{(\mathrm{a},\mathrm{b})\epsilon R^2 : a\epsilon(0;+\infty), b\epsilon[0;+\infty)\}$ to'plamda uzluksiz funksiyadir.

1-teorema. $f(x,y)$ funksiya M_0 to'plamda uzluksiz va

$$I_1(y) = \int_a^b f(x,y)dx$$

Integral $[c,d]$ da tekis yaqinlashuvchi bo'lsin. U holda $I_1(y)$ funksiya $[c,d]$ oraliqda uzluksiz bo'ladi.

4^0 . $B(a,b)$ funksiya quyidagicha ham ifodalanadi

$$B(a,b) = \int_0^{+\infty} \frac{t^{a-1}}{1+t^{a+b}} dt \qquad (4)$$

Isbot. (1) integralda $x = \frac{t}{1+t}$ almashtirish bajarilsa, u holda

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx = \int_0^{+\infty} \left(\frac{t}{1+t}\right)^{a-1} (1 - \frac{t}{1+t})^{b-1} \frac{dt}{(1+t)^2}$$

$$= \int_0^{+\infty} \frac{t^{a-1}}{(1+t)^{a+b}} dt$$

bo'ladi.

5^0 . $b = 1 - a\ (0 < a < 1)$ bo'lganda

$$B(a,1-a) = \int_0^{+\infty} \frac{t^{a-1}}{1+t} dt = \frac{\pi}{sina\pi} \qquad (5)$$

(5) munosabatdan quyidagini topamiz.

$$B(\frac{1}{2},\frac{1}{2}) = B(\frac{1}{2}, 1 - \frac{1}{2}) = \frac{\pi}{sin\frac{\pi}{2}} = \pi$$

Beta funksiyaning xususiy holda uzluksiz hosilalarga ega, ya'ni

$$\mathrm{B}(\mathrm{a}, 1-\mathrm{a}) = \int_0^{+\infty} \frac{t^{a-1}}{1+t} dt = \frac{\pi}{sina\pi}$$

$$B'_a(a, 1-a) = \int_0^{+\infty} \frac{t^{a-1}}{1+t} lntdt = \left(\frac{\pi}{sina\pi}\right) = \frac{\pi^2 cosa\pi}{sin^2 a\pi}$$

$$B''_a(a, 1-a) = \int_0^{+\infty} \frac{t^{a-1}}{1+t} ln^2 tdt = \left(\frac{\pi}{sina\pi}\right)'' = \frac{\pi^3 sin^3 a\pi + 2\pi^3 sina\pi cos^2 a\pi}{sin^4 a\pi}$$

va hokazo,

$$B^{(n)}(a, 1-a) = \int_0^{+\infty} \frac{t^{a-1}}{1+t} ln^n tdt = \left(\frac{\pi}{sina\pi}\right)^{(n)} \quad (n = 1, 2, 3, \ldots$$

bo'ladi.

6^0 . $\forall(a, b) \in M'(M' = \{(a, b) \in R2 \colon a \in (0; +\infty), b \in (1; +\infty)\}$ uchun

$$B(a, b) = \frac{b-1}{a+b-1} B(a, b-1) \tag{6}$$

bo'ladi.

Isbot. (1) integralni bo'laklab integrallaymiz.

$$B(a, b) = \int_0^1 x^{a-1} (1-x)^{b-1} dx = \int_0^1 (1-x)^{b-1} d\left(\frac{x^a}{a}\right)$$

$$= \frac{1}{a} x^a (1-x)^{b-1} I_0^1 + \frac{b-1}{a} \int_0^1 x^a (1-x)^{b-2} dx \quad (a > 0, \ \ b > 1)$$

Agar

$$x^a (1-x)^{b-2} = x^{a-1}[1-(1-x)](1-x)^{b-2}$$
$$= x^{a-1}(1-x)^{b-2} - x^{a-1}(1-x)^{b-1}$$

ekanligigni e'tiborga olsak, uholda

$$\int_0^1 x^a (1-x)^{b-2} dx = \int_0^1 x^{a-1}(1-x)^{b-2} dx - \int_0^1 x^{a-1}(1-x)^{b-2} dx$$
$$= B(a, b-1) - B(a, b)$$

bo’lib, natijada

$$B(a,b)=\frac{b-1}{a}[B(a,b-1)-B(a,b)]$$

bo’ladi. Bu tenglikdan esa

$$B(a,b)=\frac{b-1}{a+b-1}B(a,b-1)\ (a>0,b>1)$$

bo’lishini topamiz.

7. $\forall(a,b)\in$ M” uchun $M''=\{(a,b)\ \in R^2: a\in(1,\infty)b\in$(0, +∞)}

$$B(a,b)=\frac{a-1}{a+b-1}B(a-1,b)$$

bo’ladi.

Isbot. (1) integralni bo’laklab integrallaymiz.

$$B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx=\int_0^1 x^{a-1}d\left(\frac{(1-x)^b}{b}\right)=$$

$$=-\frac{1}{b}x^{a-1}(1-x)^b I_0^1$$

$$+\frac{a-1}{b}\int_0^1 x^{a-2}(1-x)^b dx=\frac{a-1}{b}\int_0^1 x^{a-2}(1-x)^b dx,$$

$$(a>1),(b>0)$$

$$x^{a-2}(1-x)^b=x^{a-2}(1-x)^{b-1}(1-x)=$$

$$=x^{a-2}(1-x)^{b-1}-x^{a-1}(1-x)^{b-1}$$

$$\int_0^1 x^{a-2}(1-x)^b dx=$$

$$=\int_0^1 x^{a-2}(1-x)^{b-1}dx-\int_0^1 x^{a-1}(1-x)^{b-1}dx=B(a-1,b)$$

$$-B(a,b)$$

$$B(a,b)=\frac{a-1}{a+b-1}B(a-1,b)\ (\text{a}>1,b>0)$$

bo’lishini topamiz.

Xususan , $b=n\ (n\in N)$ bo’lganda

$$\mathrm{B(a,b)}=\mathrm{B(a,n)}=\frac{n-1}{a+n-1}B(a,n-1)$$

bo’lib, (6) formulani takror qo’llab, quyidagini topamiz.

$$B(a,n)=\frac{n-1}{a+n-1}*\frac{n-2}{a+n-2}*\dots*\frac{1}{n+1}B(a,1)$$

Ma’lumki, $B(a,1)=\int_0^1 x^{a-1}dx=\frac{1}{a}$, demak

$$B(a,n)=\frac{1*2*\dots*(a-1)}{a(a+1)(a+2)\dots(a+n-1)}. \tag{7}$$

Agarda (7) da $a=m\ (m\in N)$ bo’lsa, u holda

$$B(m,n)=\frac{1*2*\dots*(n-1)}{m(m+1)\dots(m+n-1)}=\frac{(n-1)!\,(m-1)!}{(m+n-1)!}$$

8. $B(a,b)$ funksiya uchun ushbu

$$B(a+1,b)=\frac{a}{a+b}B(a,b)\ \ (a>0,b>0)$$

formula o‘rinli bo‘ladi.

Isbot: Ravshanki,

$$B(a+1,b)=\int_0^1 x^a(1-x)^{b-1}dx.$$

Bu integralni bo‘laklab integrallaymiz:

$$B(a+1,b)=\int_0^1 x^a(1-x)^{b-1}dx=-\frac{1}{b}\int_0^1 x^a d((1-x)^b)=-\frac{1}{b}x^a(1-x)^b\Big|_0^1+\frac{a}{b}\int_0^1 x^a(1-x)^b dx=$$

$$=\frac{a}{b}\int_0^1 x^{a-1}(1-x)^b=\frac{a}{b}\left[\int_0^1 x^{a-1}(1-x)^{b-1}dx-\int_0^1 x^a(1-x)^{b-1}dx\right]=\frac{a}{b}B(a,b)-\frac{a}{b}B(a+1,b).$$

Natijada

$$B(a+1,b)=\frac{a}{b}B(a,b)-\frac{a}{b}B(a+1,b)$$

bo‘lib, undan

$$B(a+1,b)=\frac{a}{a+b}B(a,b)$$

bo‘lishi kelib chiqadi.

Gamma funksiya va uning xossalari.

Biz

$$\int_0^{+\infty} x^{a-1}e^{-x}dx, \qquad (1)$$

xosmas integralni qaraylik. Bu chegaralanmagan funksiyaning (a<1 da $x = 0$ maxsus nuqta) cheksiz oraliq bo‘yicha olingan xosmas integrali bo‘lishi bilan birga a (parametr) ga ham bog‘liqdir.

1-ta'rif: (1) integral gamma funksiya yoki II tur Eyler integrali deb ataladi va Γ(a) kabi belgilanadi. Demak,

$$\Gamma(a)=\int_0^{+\infty} x^{a-1}e^{-x}dx; \qquad (1)$$

Shunday qilib, $\Gamma(a)$ funksiya (0; +∞) da berilgandir. $\Gamma(a)$ funksiya $(0,+\infty)$ da uzluksiz bo‘ladi.

Γ(a) funksiyaning xossalarini qaraymiz.

1°. (1) integral

$$\Gamma(a)=\int_0^{+\infty} x^{a-1}e^{-x}dx$$

ixtiyoriy $[a_0,b_0](0<a_0<b_0<+\infty)$ oraliqda tekis yaqinlashuvchi bo’ladi.

2°. $\Gamma(a\ )$ funksiya $(0;+\infty)$ da uzluksiz hamda barcha tartibdagi uzluksiz hosilalarga ega va

$$\Gamma^{(n)}(a)=\int_0^{+\infty} x^{a-1}e^{-x}(lnx)^n dx \ (n=1,2,\dots.)$$

3°. $\Gamma(a)$ funksiya uchun ushbu $\Gamma(a+1) = a\Gamma(a)$ $(a > 0)$ formula o'rinli.

Beta va gamma funksiyalar orasidagi bog'lanishni quyidagi teorema ifodalaydi.

Teorema. $\forall (a,b) \in \{(a,b) \in R^2 : a \in (0,+\infty), b \in (0,+\infty)\}$ uchun

$$B(a,b) = \frac{\Gamma(a)\cdot\Gamma(b)}{\Gamma(a+b)}$$

formula o'rinli bo'ladi.

Isbot. Ushbu

$$\Gamma(s) = \int_0^{+\infty} x^{s-1}e^{-x}dx$$

integralda $x = (1+u)t$, $(t > 0)$ almashtirish bajarib, s ni $a+b$ ga almashtiramiz. Natijada

$$\Gamma(a+b) = \int_0^{+\infty} (1+u)^{a+b-1} t^{a+b-1} e^{-(1+u)t}(1+u)dt$$

bo'lib,

$$\frac{\Gamma(a+b)}{(1+u)^{a+b}} = \int_0^{+\infty} t^{a+b-1}e^{-(1+u)t}dt$$

bo'ladi.

Bu tenglikning har ikkila tomonini u^{a-1} ga ko'paytirib, $(0,+\infty)$ oraliq bo'yicha integrallaymiz:

$$\Gamma(a+b)\int_0^{+\infty} \frac{u^{a-1}}{(1+u)^{a+b}}du = \int_0^{+\infty}\left[\int_0^{+\infty} t^{a+b-1}e^{-(1+u)t}dt\right]u^{a-1}du$$

yoki

$$\Gamma(a+b)\cdot B(a,b) = \int_0^{+\infty}\left[\int_0^{+\infty} t^{a+b-1}e^{-(1+u)t}dt\right]u^{a-1}du.$$

Tenglikning o'ng tomonidagi integrallarning o'rinlarini almashtirsak

$$\Gamma(a+b)\cdot B(a,b) = \int_0^{+\infty}\left[\int_0^{+\infty} u^{a-1}e^{-(1+u)t}du\right]t^{a+b-1}dt$$

bo'ladi. Integralda $ut = y$ almashtirish bajaramiz:

$$\Gamma(a+b)\cdot B(a,b) = \int_0^{+\infty}\left[\int_0^{+\infty} y^{a-1}t^{b-1}e^{-t}e^{-y}dy\right]dt = \int_0^{+\infty} t^{b-1}e^{-t}dt\cdot\int_0^{+\infty} y^{a-1}e^{-y}dy = \Gamma(b)\cdot\Gamma(a).$$

Demak,

$$B(a,b) = \frac{\Gamma(a)\cdot\Gamma(b)}{\Gamma(a+b)}.$$

1-misol. Ushbu Puasson integrali hisoblansin.

$$\int_0^{+\infty} e^{-x^2}dx$$

Yechish: Integralda $x^2 = t$ almashtirish bajarsak

$$dx = \frac{1}{2\sqrt{t}}dt = \frac{1}{2}t^{-\frac{1}{2}}dt$$

bo‘lib,

$$\int_0^{+\infty} e^{-x^2}dx = \frac{1}{2}\int_0^{+\infty} t^{-\frac{1}{2}}e^{-t}dt = \frac{1}{2}\int_0^{+\infty} t^{\frac{1}{2}-1}e^{-t}dt = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

ga ega bo’lamiz.

2-misol. Quyidagi integral hisoblansin

$$\int_0^{+\infty} \frac{dx}{1+x^3}$$

Yechish: Bu integralda

$$1+x^3 = \frac{1}{y}$$

almashtirish bajaramiz. Unda

$$x = \left(\frac{1-y}{y}\right)^{\frac{1}{3}},\ dx = \frac{1}{3}\left(\frac{1-y}{y}\right)^{-\frac{2}{3}} \cdot \left(-\frac{dy}{y^2}\right),$$

$$\int_0^{+\infty} \frac{dx}{1+x^3} = -\frac{1}{3}\int_1^0 y\frac{1}{3}\left(\frac{1-y}{y}\right)^{-\frac{2}{3}} \cdot \frac{dy}{y^2} = \frac{1}{3}\int_0^1 y^{-\frac{1}{3}}(1-y)^{-\frac{2}{3}}dy =$$

$$= \frac{1}{3}B\left(\frac{2}{3},\frac{1}{3}\right) = \frac{1}{3}\cdot\frac{\Gamma\left(\frac{1}{3}\right)\cdot\Gamma\left(\frac{2}{3}\right)}{\Gamma(1)} = \frac{1}{3}\cdot\frac{\pi}{\sin\frac{1}{3}\pi} = \frac{\pi}{3\cdot\frac{\sqrt{3}}{2}} = \frac{2\pi}{3\sqrt{3}}$$

bo‘ladi.

Mustaqil bajarish uchun topshiriqlar.

1. Quyidagi Lejandr formulasini isbotlang

$$\Gamma(x)\Gamma\left(x+\frac{1}{2}\right) = \sqrt{\pi}2^{1-2x}\Gamma(2x).$$

2. $a > 0$ va $a + b > 0$ bo'lsin. U holda ushbu tenglik bajarilishini isbotlang.

$$\int_0^{+\infty} \xi^{a-1}\Gamma(b,\xi)\, d\xi = \frac{\Gamma(a+b)}{a}.$$

Gamma va beta funksiyalar va ularning xossalarini qo'llab integrallarni hisoblang.

3. $$\int_0^{a} x^2\sqrt{a^2 - x^2}\, dx, a > 0.$$

4. $$\int_0^{+\infty} \frac{\sqrt[4]{x}}{(1+x)^2}\, dx,$$

5. $$\int_0^{\frac{\pi}{2}} \sin^m x cos^n x dx$$

6. $$\int_0^{+\infty} x^p\, e^{-ax} lnx dx\,,\ a > 0.$$

7. $$\int_0^{1} \sqrt{x - x^2} dx$$

8. $$\int_0^{+\infty} \frac{dx}{1 + x^3}$$

9. $$\int_0^{+\infty} \frac{x^2 dx}{1 + x^4}$$

10 $$\int_0^{\frac{\pi}{2}} \sin^4 x cos^4 x dx$$

11. $$\int_0^{1} \frac{dx}{\sqrt[n]{1 - x^n}}, n > 1$$

12. $$\int_0^{+\infty} x^{2n}\, e^{-x^2} dx\,, n \in Z, n > 0.$$

13. $$\int_0^{\frac{\pi}{2}} tg^n x dx$$

14. $$\int_0^{1} \left(ln\frac{1}{x}\right)^p$$

15. $$\int_0^{+\infty} \frac{x^{p-1}\ln^2 x\, dx}{1 + x}$$

16. $$\int_0^{+\infty} \frac{xlnxdx}{1 + x^3}$$

Bessel funksiyalari va ularning xossalari.

Birinchi tur Bessel funksiyasi: $J_\nu(z)$ birinchi tur Bessel funksiyasi quyidagicha:

$$J_\nu(z) \coloneqq \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\nu+1)} \left(\frac{z}{2}\right)^{2k+\nu} \tag{1}$$

Bu yerda, z-kompleks o'zgaruvchi, ν- parametr bo'lib, haqiqiy yoki kompleks qiymatlarni qabul qiladi. ν- butun son bo'lganda, Bessel funksiyasi analitik funksiya bo'ladi:

$$J_{-n}(z) = (-1)^n J_n(z)\,, \qquad n = 1,2,\dots \tag{2}$$

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+n)!} \left(\frac{z}{2}\right)^{2k+n}$$

$$J_{-n}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k-n)!} \left(\frac{z}{2}\right)^{2k-n} = \sum_{s=0}^{\infty} \frac{(-1)^{n+s}}{(s+n)!\,s!} \left(\frac{z}{2}\right)^{2s+n}$$

Agar $\nu \notin Z$ bo'lsa, Bessel funksiyasi $\left(\frac{z}{2}\right)^\nu$ ko'paytuvchi tufayli $z = 0$ nuqtada turli xil qiymatlarni qabul qiladi. Shuning uchun, kompleks sohaning manfiy yarim o'qini kesib, z $|arg\ (z)| < \pi$ shart bilan olinadi. Trikomi ta'rifiga ko'ra, $\left(\frac{z}{2}\right)^\nu$ singulyar ko'paytuvchini (1) dan ajratib olishimiz mumkin:

$$J_\nu^T(z) \coloneqq \left(\frac{z}{2}\right)^{-\nu} J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\nu+1)} \left(\frac{z}{2}\right)^{2k} \tag{3}$$

Bessel funksiyalari bu Fucks-Frobeniusning ikkinchi tartibli differensial tenglamasining ildizi sifatida ham tushuniladi:

$$\frac{d^2}{dz^2}u(z) + p(z)\frac{d}{dz}u(z) + q(z)u(z) = 0 \tag{4}$$

$p(z)$ va $q(z)$ lar ma'lum analitik funksiyalar. Agar $p(z)$ va $q(z)$ larni

$$p(z) = \frac{1}{z} \text{ va } q(z) = 1 - \frac{\nu^2}{z^2} \tag{5}$$

ko'rinishda olsak, (4) differensial tenglamadan, (3) ko'rinishidagi qatorga ega bo'lamiz. Natijada, 1-tur Bessel funksiyasi quyidagi differensial tenglamani qanoatlantiradi:

$$u''(z) + \frac{1}{z}u'(z) + \left(1 - \frac{\nu^2}{z^2}\right)u(z) = 0 \tag{6}$$

(6) tenglama, **Bessel differensial tenglamasi** deyiladi.

1-tur Bessel funksiyasining xossalari:

1. $\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x)$
2. $\frac{d}{dx}\left[x^{-p} J_p(x)\right] = -x^{-p} J_{p+1}(x)$
3. $x J_p'^{(x)} + pJ_p(x) = x J_{p-1}(x)$
4. $x J_p'^{(x)} - pJ_p(x) = -x J_{p+1}(x)$
5. $J_{p-1}(x) - J_{p+1}(x) = 2 J_p'(x)$
6. $J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x)$
7. $\int x^{p+1} J_p(x)dx = x^{p+1} J_{p+1}(x) + C$
8. $\int x^{-p+1} J_p(x)dx = -x^{-p+1} J_{p-1}(x) + C$

Ikkinchi tur Bessel funksiyasi: $Y_\nu(z)$.

$\nu = n$, $(n = 0, \pm 1, \pm 2, \dots)$ da J_ν dan chiziqli erkli bo'lgan (6) ikkinchi tartibli differensial tenglamaning yechimini olish uchun, 2-tur Bessel funksiyasini kiritamiz:

$$Y_\nu(z) := \frac{J_{-\nu}(z)\cos(\nu\pi) - J_{-\nu}(z)}{sin(\nu\pi)}. \tag{7}$$

$\nu \in Z$ ar uchun tenglikning o'ng tomoni no'malum bo'lib qoladi, shuning uchun $Y_n(z)$ ni limit ko'rinishida yozib olamiz:

$$Y_n(z) := \lim_{\nu \to n} Y_\nu(z) = \frac{1}{\pi}\left[\frac{\partial J_\nu(z)}{\partial \nu}\Big|_{\nu=n} - (-1)^n \frac{\partial J_{-\nu}(z)}{\partial \nu}\Big|_{\nu=n}\right] \tag{8}$$

(8) dan quyidagi kelib chiqadi:

$$Y_{-n}(z) = (-1)^n Y_n(z) \tag{9}$$

ixtiyoriy haqiqiy son bo'lganda, (6) ikkinchi tartibli differensial tenglamaning umumiy yechimi quyidagicha bo'ladi:

$$u(z) = \gamma_1 \mathrm{J}_\nu(\mathrm{z}) + \gamma_2 \mathrm{Y}_\nu(\mathrm{z}), \quad \gamma_1, \gamma_2 \in C \tag{10}$$

va bunga mos Wronskiy determinanti quyidagicha bo'ladi:

$$W\{\mathrm{J}_\nu(\mathrm{z}), \mathrm{Y}_\nu(\mathrm{z})\} = \frac{2}{\pi z} \tag{11}$$

Uchinchi tur Bessel funksiyalari: $H_\nu^{(1)}, H_\nu^{(2)}$.

1- va 2-tur Bessel funksiyalariga qo'shimcha ravishda, 3-tur Bessel funksiyalari yoki Hankel funksiyalari quyidagicha kiritiladi:

$$H_\nu^{(1)} := \mathrm{J}_\nu(\mathrm{z}) + i\mathrm{Y}_\nu(\mathrm{z}), \quad H_\nu^{(2)} := \mathrm{J}_\nu(\mathrm{z}) - i\mathrm{Y}_\nu(\mathrm{z}) \tag{12}$$

Bu funksiyalar Wronskiy determinanti orqali chiziqli erkli

$$W\{H_\nu^{(1)}, H_\nu^{(2)}\} = -\frac{4i}{\pi z} \tag{13}$$

(12) formulalarda $\mathrm{Y}_\nu(\mathrm{z})$ larning o'rniga (7) formuladan foydalansak, quyidagilarga ga bo'lamiz:

$$\begin{cases} H_\nu^{(1)}(z) := \dfrac{J_{-\nu}(z) - e^{-i\nu\pi} J_\nu(z)}{i\sin(\nu\pi)} \\ H_\nu^{(2)}(z) := \dfrac{e^{+i\nu\pi} J_\nu(z) - J_{-\nu}(z)}{i\sin(\nu\pi)} \end{cases} \tag{4}$$

(4) dan quyidagi muhim formulalarga ega bo'lamiz:

$$H_{-\nu}^{(1)}(z) = e^{+i\nu\pi} H_\nu^{(1)}(z), \quad H_{-\nu}^{(2)}(z) = e^{-i\nu\pi} H_\nu^{(2)}(z)$$

Mustaqil ishlash uchun topshiriqlar:

Mashqlar: Quyidagilarni isbot qiling:

1. $\left(\frac{\mathrm{d}}{\mathrm{xdx}}\right)^{\mathrm{m}} \left[\mathrm{x^p J_p(x)}\right] = \mathrm{x^{p-m} J_{p-m}(x)}$
2. $\left(\frac{\mathrm{d}}{\mathrm{xdx}}\right)^{\mathrm{m}} \left[\frac{\mathrm{J_p(x)}}{\mathrm{x^p}}\right] = (-1)^{\mathrm{m}} \frac{\mathrm{J_{p+m}(x)}}{\mathrm{x^{p+m}}}$

Quyida berilgan integrallarni soddalashtiring:

1. $\int x\, J_0(x)dx$
2. $\int x^4 J_3(x)dx$
3. $\int J_1(x)dx$
4. $\int x^{-2} J_3(x)dx$

Quyidagilarni isbotlang.

1. $J_{\frac{-1}{2}}(x) = \sqrt{\frac{2}{\pi x}} cosx$
2. $J_{-\frac{3}{2}}(x) = -\sqrt{\frac{2}{\pi x}}\left(\frac{cosx}{x} + sinx\right)$
3. $J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{3-x^2}{x^2} - \frac{3}{x} cosx\right)$
4. $\int_0^x sJ_0(s)ds = xJ_1(x).$
5. $J_{-n}(x) = (-1)^n J_n(x), n \in N$
6. $\frac{d}{dx}\left(\frac{J_p(x)}{x^p}\right) = -\frac{J_{p+1}}{x^p}$
7. $\int xJ_0(x)dx = xJ_1(x) + C$
8. $\int x^2 J_1(x)dx = -x^2 J_0(x) + 2xJ_1(x) + C$
9. $\int x^3 J_0(x)dx = 2x^2 J_0(x) + (x^3 - 4x)J_1(x) + C$
10. $\int xJ_p^2(\alpha x)dx = 0.5\left[x^2\left(J'_p(\alpha x)\right)^2 + \left(x^2 - \frac{p^2}{\alpha^2}\right)\left(J_p(\alpha x)\right)^2\right] + C$

Gipergeometrik funksiya va uning xossalari

Quyidagi

$$\boldsymbol{x(1-x)y'' + [c - (a+b+1)x]y' - aby = 0} \qquad (1)$$

gipergeometrik funksiya yoki *Gauss tenglamasi* deb ataluvchi tenglamani qaraymiz. Bu yerda a, b, c- ixtiyoriy parametrlar bo‘lib, haqiqiy yoki kompleks qiymatlar qabul qiladi. (a, b) tenglamada simmetrik ishtirok etadi.

(1) tenglamaning umumiy yechimini quyidagi ko‘rinishda yozish mumkin:

$$\boldsymbol{y = c_1 F(a, b, c; x) + c_2 x^{1-c} F(a - c + 1, b - c + 1, 2 - c; x)}$$

bu yerda c_1 va c_2–∀ o‘zgarmaslar.

Agar $\boldsymbol{a = -n_1, b = -n_2}$, bunda $\boldsymbol{n_1 > 0; n_2 > 0 -}$ butun sonlar bo‘lsa, u holda gipergeometrik qator ko‘phadga aylanib, uning darajasi $\boldsymbol{n_1, n_2}$ sonlarning kichigiga teng bo‘ladi:

$$\boldsymbol{F(a,b,c;x) = \frac{ab}{c} F(a+1, b+1, c+1; x)}$$

formulani hosil qilamiz.

Gipergeometrik funksiyalar uchun quyidagi formulalar o’rinli:

$$\begin{cases} \boldsymbol{\frac{d}{dx}[x^a F(a,b,c;x)] = a x^{a-1} F(a+1,b,c,x)} \\ \boldsymbol{\frac{d}{dx}[x^b F(a,b,c;x)] = b x^{b-1} F(a,b+1,c,x)} \\ \boldsymbol{\frac{d}{dx}[x^{c-1} F(a,b,c;x)] = (c-1) x^{c-2} F(a,b,c-1,x)} \end{cases}$$

Mustaqil yechish uchun misol va masalalar:

Quyida keltirilgan elementar funksiyalarni gipergeometrik funksiya orqali ifodalarini isbotlang.

1. $(1+x)^n = F(-n, 1; 1, -x)$
2. $ln\frac{1+x}{1-x} = 2xF(0.5, 1; 1.5;\ x^2)$
3. $\ln(1+x) = xF(1,1;2,-x)$
4. $e^z = \lim_{b\to\infty} F(1, b; 1,\ \frac{x}{b})$
5. $arcsinx = xF(0.5, 0.5; 1.5; x^2)$
6. $arctgx = xF(0.5, 1; 1.5; -x^2)$

Mittag-Leffler funksiyalari

Mittag-Lefler funksiyasi ko‘rsatkichli funksiyani kasrli umumlashtirishdir. 1903 yilda Mittag-Lefler tomonidan kiritilgan, bitta parametrni o'z ichiga olgan va bugungi kunda o’z nomini olgan funksiyani eksponensial funksiyani umumlashtirish deb hisoblash mumkin. Mittag-Leffler funksiyasini va ikki, uch parametrli Mittag-Leffler funksiyasilarini ta’riflab, xossalarini keltiramiz.

Ta'rif 1. α parametrga ($Re\{\alpha\} > 0$) bog'liq bo'lgan quyidagi darajali qator bir parametrli Mittag-Leffler funksiyasi deyiladi va $E_\alpha(x)$ ko'rinishda belgilanadi:

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)} \quad \alpha \in \mathbb{C}, Re(\alpha) > 0$$

yoki quyidagi ko'rinishda ham aniqlanadi:

$$E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\Gamma(\alpha k + 1)} \quad \alpha \in \mathbb{C}, Re(\alpha) > 0.$$

Bir parametrli Mittag-Leffler funksiyasining xususiy hollarini keltirib o'tamiz.

1)α=0 da

$$E_0(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(1)} = \sum_{k=0}^{\infty} x^k = \begin{cases} \frac{1}{1-x}, & |x| < 1 \\ \infty, & |x| \geq 1 \end{cases}.$$

2)α=1 da

$$E_1(x) = \sum_{n=0}^{\infty} \frac{x^k}{\Gamma(k+1)} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = e^x.$$

3)α=2 da

$$E_2(x) = \sum_{n=0}^{\infty} \frac{x^k}{\Gamma(2k+1)} = 1 + \frac{x}{2!} + \frac{x^2}{4!} + \frac{x^3}{6!} + \cdots = ch\sqrt{x} = \frac{e^{\sqrt{x}} + e^{-\sqrt{x}}}{2}.$$

Mittag-Leffler funksiyasining bir nechta turlari mavjud bo'lin, ular bir-biridan parametrlar va o'zgaruvchilar soni bilan farq qiladi. Ikki parametrli Mittag Leffler funksiyasini 1905-yilda Wiman tomonidan o'rganilgan.

Ta'rif 2. Faraz qilaylik $x \in \mathbb{C}$ va ikkita parametrlar $\alpha \in \mathbb{C}, \beta \in \mathbb{C}, \mathbb{R}e(\alpha) > 0, Re(\beta) > 0$ bo'lsin. U holda ikki parametrli Mittag Leffler funksiyasi ikki parametrlar bilan quyidagidarajali qator ko'rinishida aniqlanadi:

$$E_{\alpha,\beta}(x)=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\alpha k+\beta)},\qquad \alpha,\beta\in\mathbb{C},\quad Re(\alpha)>0, Re(\beta)>0.$$

Bu ikki parametrli Mittag-Leffler funksiyasi bir parametrli Mittag-Leffler funksiyaning umumlashmasidir.

Ikki parametrli Mittag-Leffler funksiyasiga doir misollar

1) α=1,β=2;

$$E_{1,2}(x)=\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(n+2)}=1+\frac{x}{2!}+\frac{x^2}{3!}+\frac{x^3}{4!}+\cdots=\frac{1}{x}(1+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots-1)=$$

$$=\frac{1}{x}(\sum_{n=0}^{\infty}\frac{x^n}{n!}-1)=\frac{1}{x}(e^x-1)$$

2) α=2,β=2

$$E_{2,2}(x)=\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(2n+2)}=1+\frac{x}{3!}+\frac{x^2}{5!}+\frac{x^3}{7!}=\frac{1}{\sqrt{x}}(1+\frac{\sqrt{x}^3}{3!}+\frac{\sqrt{x}^5}{5!}+\cdots)=$$

$$=\frac{1}{2\sqrt{x}}\left(1+\sqrt{x}+\frac{\sqrt{x}^2}{2!}+\frac{\sqrt{x}^3}{3!}-\frac{\sqrt{x}^4}{4!}+\cdots+(-1)+\sqrt{x}-\frac{\sqrt{x}^2}{2!}+\frac{\sqrt{x}^3}{3!}+\frac{\sqrt{x}^4}{4!}\dots\right)$$

$$=\frac{1}{2\sqrt{x}}\left(e^{\sqrt{x}}-e^{-\sqrt{x}}\right)=\frac{sh\sqrt{x}}{\sqrt{x}}$$

3) α=1,β=3

$$E_{1,3}(x)=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(k+3)}=\frac{1}{2!}+\frac{x}{3!}+\frac{x^2}{4!}+\frac{x^3}{5!}+\cdots=\frac{1}{x^2}(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+$$

$$+(-1-x))=\frac{1}{x^2}(\sum_{k=0}^{\infty}\frac{x^k}{k!}-(1+x))=\frac{1}{x^2}(e^x-x-1)$$

Mittag –Leffler funksiyasini Mellin-Barnes tipidagi integral orqali ham ifodalash mumkin va u quyidagicha ko’rinishda bo’ladi:

Bir parametrli Mittag –Leffler funksiyasi:

$$E_\alpha(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{t^{\alpha-1}e^t}{t^\alpha - z}dt\,;$$

Ikki parametrli Mittag –Leffler funksiyasi:

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{t^{\alpha-\beta}e^t}{t^\alpha - z}dt\,;$$

Bir parametrli Mittag –Leffler funksiyasining n-tartibli hosilasi quyidagicha aniqlanadi:

$$\left(\frac{d}{dx}\right)^k E_k(x^k) = E_k(x^k), k \in N.$$

Ikki parametrli Mittag –Leffler funksiyasining $\mu > 0$ va $1 \le n \le k$ uchun n -tartibli hosilasi quyidagicha aniqlanadi:

$$\left(\frac{d}{dx}\right)^k \left[x^{n-1}E_{k,n}(\mu x^k)\right] = \mu x^{n-1}E_{k,n}(\mu x^k).$$

Umumlashgan Mittag-Leffler funksiyasi

Umumlashgan Mittag-Leffler (3 parametrli Mittag-Leffler)funksiyasi quyidagi ko’rinishda aniqlanadi:

$$E^{\rho}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\rho)_k}{\Gamma(\alpha k + \beta)}\frac{z^k}{k!} \quad z \in \mathbb{C}\,; \alpha, \beta, \rho \in \mathbb{C}.$$

Bu yerda $(\rho)_k$- Pochhammer simvoli va u quyidagicha aniqlanadi

$$(\rho)_k = \rho(\rho+1)(\rho+2)\ldots(\rho+k-1);$$

Bu funksiya ikki parametrli Mittag-Leffler funksiyasining umumlashlasidir, ya'ni ρ=1 bo'lsa, $E^{1}_{\alpha,\beta}(z) = E_{\alpha,\beta}(z)$ bo'ladi.

Eslatma. Faraz qilaylik $x \in \mathbb{C}$, $\alpha,\beta \in \mathbb{C}, Re\{\alpha\} > 0, Re\{\beta\} > 0$ bo'lsin va $k = 0,1,2,\ldots$. U holda ikki va uch parametrli Mittag-Leffler funksiyalari orasida quyidagi munosabat o'rinlidir:

$$\frac{d^k}{dx^k} E_{\alpha,\beta}(x) = k!\, E^{k+1}_{\alpha,\beta+\alpha k}(x).$$

Mustaqil yechish uchun topshiriqlar.

Quyidagi tengliklarni isbotlang.

1. $E_{1,2}(x)1 + E_{1,3}(x)$

2.Faraz qilaylik $\alpha > 0$ va $x \in R$ bo'lsin. U holda Mittag-Leffler funksiyasi uchun ushbu tenglik bajarilishini isbotlang.

$$\frac{1}{2}\left[E_\alpha(\sqrt{x}) + E_\alpha(-\sqrt{x})\right] = E_{2\alpha}(x)$$

3. Hisoblang.

$$xE_{2,2}(-x^2) = sinx$$

4. Hisoblang.

$$E_{-\alpha,\beta}(x) = \frac{1}{\Gamma(\beta)} - E_{\alpha,\beta}\left(\frac{1}{x}\right)$$

5. Hisoblang.

$$E_{-2,1}\left(-\frac{1}{x^2}\right) = 1 - \cos x$$

6. Hisoblang.

$$E_{\alpha,\beta}(-x) = \frac{1}{\alpha\Gamma(\beta-\alpha)} \int_0^1 \left(1 - \varepsilon^{\frac{1}{\alpha}}\right)^{\beta-\alpha-1} E_{\alpha,\alpha}(-x\varepsilon)d\epsilon$$

7. Hisoblang.

$$E^{\rho}_{\alpha,\beta-\alpha}(x) - E^{\rho-1}_{\alpha,\beta-\alpha}(x) = xE^{\rho}_{\alpha,\beta}(x)$$

8. isbotlang

$$\int_0^x \varepsilon^{\beta-1} E^{\rho}_{\alpha,\beta-\alpha}(\mu\varepsilon^{\alpha})d\varepsilon = x^{\beta} E^{\rho}_{\alpha,\beta+1}(\mu x^{\alpha})$$

9. $\beta > 0$ uchun ushbu tenglik bajarilishini isbotlang.

$$\frac{d}{dz}E_{\beta}(z) = \frac{E_{\beta,\beta}(z)}{\beta}$$

11. $\alpha > 0$ son uchun quyidagini isbotlang:

$$E_{\alpha}(-x) = E_{2\alpha}(x^2) - xE_{2\alpha,\alpha+1}(x^2)$$

12. $x > 0\ va\ \alpha, \beta, \gamma > 0$ sonlar uchun quyidagini isbotlang:

$$\frac{1}{\Gamma(\gamma)}\int_0^x (x-\xi)^{\gamma-1}\,\xi^{\beta-1}E_{\alpha,\beta}(\xi^{\alpha})d\xi = x^{\beta+\gamma-1}E_{\alpha,\beta+\gamma}(x^{\alpha})$$

13. Isbotlang:

$$E_{\alpha,\beta}(-x) = \frac{1}{\alpha\Gamma(\beta-\alpha)}\int_0^1 \left(1-\xi^{\frac{1}{\alpha}}\right)^{\beta-\alpha-1} E_{\alpha,\alpha}(-x\xi)d\xi$$

14. Isbotlang:

$$\left(x\frac{d}{dx}+p\right)E^{p}_{\alpha,\beta}(x) = pE^{p+1}_{\alpha,\beta}(x)$$

15. Isbotlang:

$$E_{1,2}(x) = 1 + xE_{1,3}(x)$$

16. Isbotlang:

$$xE_{2,2}(-x)^2 = sinx$$

17. $\rho = 2, \alpha = 1\ va\ \beta = 2$ uchun $E^{\rho}_{\alpha,\beta}$ ni hisoblang.

18. $\frac{d^n}{dx^n}E_n(x^n) = E_n(x^n)$ ni n=2 uchun isbotlang.

Foks funksiyasi va uning xossalari.

Foksning H-funksiyasi kasr tartibli hisobning maxsus funksiyalaridan bo'lib, xususiy hollarda Mittag-Leffler funksiyasini ifodalaydi. H funksiya Fox tomonidan kiritilgan bo'lib Meyer funksiyasining umumlashmasini ifodalaydi. Bundan tashqari H-funksiya Mellin-Braus tipidagi integral bilan quyidagicha aniqlangan.

$$H_{p,q}^{m,n}\left[z\left|\begin{smallmatrix}(a_1,A_1),\dots,(a_p,A_p)\\(b_1,B_1),\dots,(b_q,B_q)\end{smallmatrix}\right.\right]=H_{p,q}^{m,n}\left[z\left|\begin{smallmatrix}(a_p,A_p)\\(b_q,B_q)\end{smallmatrix}\right.\right]=\frac{1}{2\pi i}\int_{\Omega}\theta\, z^{-s}ds$$

bunda:

$$\theta(s)=\frac{\prod_{j=1}^{m}\Gamma(b_j+B_js)\prod_{k=1}^{n}\Gamma(1-a_k-A_ks)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-B_js)\prod_{k=n+1}^{p}\Gamma(a_k+A_ks)}$$

bu yerda: $i=(-1)^{\frac{1}{2}}, z\neq 0, va\ z^{-s}=\exp\{-s[\ln|z|+iargz]\}, 0\leq n\leq p, 0\leq m\leq q, A_e, B_j\in R_+, a_e, B_j\in\mathbb{C}(R), e=1,2,\dots,p, j=1,2,\dots,q.$

Integral yaqinlashuvchi bo'lishi uchun quyidagi shartlar bajarilishi kerak:

1.) $C>0, |argz|<\frac{1}{2}\pi c\ va\ z\neq 0;$

2.) $C=0, pD+Re(E)<-1, argz=0\ va\ z\neq 0$

bunda

$$C:=\sum_{j=1}^{n}A_j-\sum_{j=n+1}^{p}A_j+\sum_{j=1}^{m}B_j-\sum_{j=m+1}^{q}B_j$$

$$D:=\sum_{j=1}^{q}B_j-\sum_{j=1}^{p}A_j\ E:=\sum_{j=1}^{q}b_j-\sum_{j=1}^{p}a_j+\frac{p-q}{2}$$

H-funksiya uchun quyidagi rekurent formulalar o'rinli :

$$H_{p,q}^{m,n}\left[z\left|\begin{smallmatrix}(a_1,A_1),\dots,(a_p,A_p)\\(b_1,B_1),\dots,(b_{q-1},B_{q-1}),(a_1,A_1)\end{smallmatrix}\right.\right]=H_{p-1,q-1}^{m,n-1}\left[z\left|\begin{smallmatrix}(a_2,A_2),\dots,(a_p,A_p)\\(b_1,B_1),\dots,(b_{q-1},B_{q-1})\end{smallmatrix}\right.\right]$$

$$H_{p,q}^{m,n}\left[z\left|\begin{smallmatrix}(a_p,A_p)\\(b_q,B_q)\end{smallmatrix}\right.\right]=H_{p,q}^{n,m}\left[\frac{1}{z}\left|\begin{smallmatrix}(1-b_q,B_q)\\(1-a_p,A_p)\end{smallmatrix}\right.\right]$$

Xossalari:

1-Xossa.H-funksiya

(a_1,α_1),...,(a_n,α_n);(a_{n+1},α_{n+1}),...,(a_p,α_p)((b_1,β_1),...,(b_m,β_m) va

(b_{m+1},β_{m+1}),...,(b_q,β_q) juftliklar to'plamida simetrikdir.

2-Xossa. Agar (a_i,α_i) dan biri (i=$\overline{1,n)}$ dan biri ((b_j,β_j) (j=m+1,q) yoki ((a_i,α_i) dan biri (i=n+1,p) (b_j,β_j) (j=1,m) dan biriga teng bo'lsa)lardan biriga teng bo'lsa u holda H-funksiya biror quyi tartibga kamaydi, ya'ni p,q va n(yoki m) birlik bilan kamayadi. Bunday qisqartirish formulalariga ikkita miso1 keltiramiz.

$$H_{p,q}^{m,n}\left[z\Big|_{(b_j,\beta_j)_{1,q-1},(a_1,\alpha_1)}^{(a_i,\alpha_i)_{1,p}}\right]=H_{p-1,q-1}^{m,n-1}\left[z\Big|_{(b_j,\beta_j)_{1,q-1}}^{(a_i,\alpha_i)_{2,p}}\right] \quad (1)$$

$n\geq 1$ va $q>m$ bo'lsin va

$$H_{p,q}^{m,n}\left[z\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p-1},(b_1,\beta_1)}\right]=H_{p-1,q-1}^{m-1,n}\left[z\Big|_{(b_j,\beta_j)_{2,q}}^{(a_i,\alpha_i)_{1,p-1}}\right] \quad (2)$$

$m\geq 1\; va\;\; p>n$ bo’ladi

3- xossa**.** Quyidagi tenglik o’rinli

$$H_{p,q}^{m,n}\left[\frac{1}{z}\Big|_{(b_j,\beta_j)_{1,q},}^{(a_i,\alpha_i)_{1,p}}\right]=H_{p,q}^{m,n}\left[z\Big|_{(1-a_i,\alpha_i)_{1,p}}^{(1-b_j,\beta_j)_{1,p}}\right] \quad (3)$$

4-xossa. $k>0$ uchun

$$H_{p,q}^{m,n}\left[z\Big|_{(b_j,\beta_j)_{1,q},}^{(a_i,\alpha_i)_{1,p}}\right]=kH_{p-1,q-1}^{m,n-1}\left[z^k\Big|_{(b_j,k\beta_j)_{1,q}}^{(a_i,k\alpha_i)_{1,p}}\right] \quad (4)$$

5- Xossa. $\sigma\in C$ uchun

$$z^{\sigma}H_{p,q}^{m,n}\left[z\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right]=H_{p,q}^{m,n}\left[z\Big|_{(b_j+\sigma\beta_j,\beta_j)_{1,q-1}}^{(a_i+\alpha_i,\alpha_i)_{2,p}}\right] \quad (5)$$

6- Xossa. $c\in\mathbb{C}, \alpha>0$ va $k=0,\pm1,\pm2,...,$ lar uchun

$$H_{p+1,q+1}^{m,n+1}\left[z\Big|_{(b_j,\beta_j)_{1,q},(c+k,\alpha)}^{(c,\alpha),(a_i,\alpha_i)_{1,p}}\right]=(-1)^k H_{p+1,q+1}^{m+1,n}\left[z\Big|_{(c+k,\alpha),(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p},(c,\alpha)}\right], \quad (7)$$

$$H_{p+1,q+1}^{m+1,n}\left[z\Big|_{(c+k,\alpha),(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p},(c,\alpha)}\right]=(-1)^k H_{p+1,q+1}^{m+1,n}\left[z\Big|_{(b_j,\beta_j)_{1,q},(c+k,\alpha)}^{(c,\alpha),(a_i,\alpha_i)_{1,p}}\right], \quad (8)$$

munosabat o’rinli.

7-Xossa. $a,b\in\mathbb{C}$ sonlar uchun quyidagi munosabatlar mavjud:

$$H_{p,q}^{m,n}\left[z\Big|_{(b_j,\beta_j)_{1,q}}^{(a,0),(a_i,\alpha_i)_{2,p}}\right]=\Gamma(1-\alpha)H_{p-1,q}^{m,n-1}\left[z\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{2,p}}\right], \quad (9)$$

bunda $Re(1-\alpha)>0$ va $n\geq 1$;

$$H_{p,q}^{m,n}\left[z\Big|_{(b,0),(b_j,\beta_j)_{2,q}}^{(a_i,\alpha_i)_{1,p}}\right]=\Gamma(b)H_{p,q-1}^{m-1,n}\left[z\Big|_{(b_j,\beta_j)_{2,q}}^{(a_i,\alpha_i)_{1,p}}\right], \quad (10)$$

bunda $Re(b)>0$ va $m\geq 1$;

$$H_{p,q}^{m,n}\left[z\Big|_{(b_j,\beta_j)_{1,q-1},(b,0)}^{(a_i,\alpha_i)_{1,p}}\right]=\frac{1}{\Gamma(1-b)}H_{p,q-1}^{m,n}\left[z\Big|_{(b_j,\beta_j)_{1,q-1}}^{(a_i,\alpha_i)_{1,p}}\right], \quad (11)$$

bunda $Re(1-b)>0$ va$p>n$.

Differensiallash formulalari

8- Xossa. $w,c\in C$ va $\sigma>0$ uchun

$$(\frac{d}{dz})^k\{z^wH_{p,q}^{m,n}\left[cz^\sigma|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right]=$$

$$=z^{w-k}H_{p+1,q+1}^{m,n+1}\left[cz^\sigma|_{(b_j,\beta_j)_{1,q},(k-w,\sigma)}^{(-w,\sigma),(a_i,\alpha_i)_{2,p}}\right] \quad (12)$$

$$=(-1)z^{w-k}H_{p+1,q+1}^{m+1,n}\left[cz^\sigma|_{(k-w),(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p},(-w,\sigma)}\right]. \quad (13)$$

$w,a,c_j\in C\ (j=1,\dots k),\sigma>0$ uchun

$$\prod_{j=1}^{k}\left(z\frac{d}{dz}-c_j\right)\{z^w\,H_{p,q}^{m,n}\left[az^\sigma|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right]\}=$$

$$=z^wH_{p+k,q+k}^{m,n+k}\left[az^\sigma|_{(b_j,\beta_j)_{1,q},\,(c_j+1-w,\sigma)_{1,k}}^{(c_j-w,\sigma)_{1,k},(a_i,\alpha_i)_{1,p}}\right], \quad (14)$$

$$=(-1)^kz^wH_{p+k,q+k}^{m+k,n}\left[az^\sigma|_{(c_j+1-w,\sigma)_{1,k},(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p},(\ c_j\ -w,\sigma)_{1,k}}\right] \quad (15)$$

$c,d\in C,\sigma>0$ lar uchun

$$\left(\frac{d}{dz}\right)^k H_{p,q}^{m,n}\left[(cz+d)^\sigma|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right]=$$

$$\frac{c^k}{(cz+d)^k}H_{p+1,q+1}^{m,n+1}\left[(cz+d)^\sigma|_{(b_j,\beta_j)_{1,q},(k,\sigma)}^{(0,\sigma),(a_i,\alpha_i)_{1,p}}\right], \quad (16)$$

$$\left(\frac{d}{dz}\right)^k H_{p,q}^{m,n}\left[\frac{1}{(cz+d)^\sigma}|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right]=\frac{c^k}{(cz+d)^k}H_{p+1,q+1}^{m,n}\left[\begin{matrix}(a_i,\alpha_i)_{1,p},(1-k,\sigma)\\(1,\sigma),(b_j,\beta_j)_{1,q}\end{matrix}\right], \quad (17)$$

tengliklar o'rinli.

9-Xossa. $m\geq 1$ va $\sigma=\beta$, bo'lganda $k>1$ uchun

$$\left(\frac{d}{dz}\right)^k(z^{-\sigma bq/q}H_{p,q}^{m,n}\left[z^\sigma|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right]=$$

$$(\frac{\sigma}{\beta_q})^kz^{-k-\sigma bq/\beta_q}\ H_{p,q}^{m,n}\left[z^\sigma|_{(b_j,\beta_j)_{1,q-1},(b_q+k,\beta_q)}^{(a_i,\alpha_i)_{1,p}}\right], \quad (18)$$

$n\geq 1$ va $\sigma=\alpha$, bo'lganda $k>1$ uchun

$$\left(\frac{d}{dz}\right)^k (z^{-\sigma(1-a_1}/_{\alpha_1} H_{p,q}^{m,n}\left[z^{-\sigma}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right] =$$

$$(-\frac{\sigma}{\beta_q})^k z^{-k-\sigma(1-\alpha_1)}/_{\alpha_1}\ H_{p,q}^{m,n}\left[z^{-\sigma}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_1-k,\alpha_1),(a_i,\alpha_i)_{2,p}}\right], \qquad (19)$$

$p > n$ va $\sigma = \alpha_p$ bo'lganda $k > 1$ uchun

$$\left(\frac{d}{dz}\right)^k (z^{-\sigma(1-a_p}/_{\alpha_p} H_{p,q}^{m,n}\left[z^{-\sigma}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right] =$$

$$(-\frac{\sigma}{\alpha_q})^k z^{-k-\sigma(1-\alpha_p)}/_{\alpha_p}\ H_{p,q}^{m,n}\left[z^{-\sigma}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p-1},(a_p-k,\alpha_p)}\right], \qquad (20)$$

bo'ladi.

(17)-(20) munosabatlar agar 2-xossani inobatga olsak,(21) va (22) ifodadan kelib chiqadi.

10- Xossa. $n \geq 1$ uchun

$$z\frac{d}{dz}\{H_{p,q}^{m,n}\left[z^{\sigma}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right]\} = \frac{\sigma(a_1-1)}{\alpha_1}H_{p,q}^{m,n}\left[z^{\sigma}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right] +$$

$$+\frac{\sigma}{\alpha_1}H_{p,q}^{m,n}\left[z^{p}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_1-1,\alpha_1)(a_i,\alpha_i)_{2,p}}\right]; \qquad (21)$$

$n \leq p - 1$ uchun

$$z\frac{d}{dz}\{H_{p,q}^{m,n}\left[z^{\sigma}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right]\} = \frac{\sigma(a_p-1)}{\alpha_p}H_{p,q}^{m,n}\left[z^{\sigma}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right]$$

$$-\frac{\sigma}{\alpha_p}H_{p,q}^{m,n}\left[z^{\sigma}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{i,p-1},(a_{p-1},\alpha_p)}\right]; \qquad (22)$$

$m \geq 1$ uchun

$$z\frac{d}{dz}\{H_{p,q}^{m,n}\left[z^{\sigma}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right]\} = \frac{\sigma b_1}{\beta_1}H_{p,q}^{m,n}\left[z^{\sigma}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right]$$

$$-\frac{\sigma}{\alpha_p}H_{p,q}^{m,n}\left[z^{\sigma}\Big|_{(b_1+1,\beta_1),(b_j,\beta_j)_{2,q}}^{(a_i,\alpha_i)_{1,\beta}}\right]; \qquad (23)$$

va $m \leq q - 1$ lar uchun

$$z\frac{d}{dz}\{H_{p,q}^{m,n}\left[z^{\sigma}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,q}}\right]\} = \frac{\sigma bq}{\beta_q}H_{p,q}^{m,n}\left[z^{\sigma}\Big|_{(b_j,\beta_j)_{1,q}}^{(a_i,\alpha_i)_{1,p}}\right]$$

$$+\frac{\sigma}{\beta_q}H_{p,q}^{m,n}\left[z^{\sigma}\Big|_{(b_j,\beta_j)_{1,q-1},(b_q+1,\beta_q)}^{(a_i,\alpha_i)_{1,p}}\right]; \qquad (24)$$

bo'ladi.

(11)-(14) formulalar quyidagi munosabatlar asosida o'rnatiladi:

$$-\alpha_1 s\Gamma(1-a_1-\alpha_1 s) = (a_1-1)\Gamma(1-a_1-\alpha_1 s) + \Gamma(2-a_1-\alpha_1 s);$$

$$-\frac{\alpha_p s}{\Gamma(a_p+\alpha_p s)} = \frac{a_p-1}{\Gamma(a_p+\alpha_p s)} - \frac{1}{\Gamma(a_p-1+\alpha_p s)};$$

$$-\beta_1 s\Gamma(b_1+\beta_1 s) = b_1\Gamma(b_1+\beta_1 s) - \Gamma(b_1+1+\beta_1 s);$$

$$-\frac{\beta_q s}{\Gamma(1-b_q-\beta_q s)} = \frac{b_q}{\Gamma(1-b_q-\beta_q s)} + \frac{1}{\Gamma(-b_q-\beta_q s)}$$

mos ravishda

$$z\Gamma(z) = \Gamma(z+1) \qquad (25)$$

munosabatlar kelib chiqadi.

Ba'zi maxsus funksiyalarni Foks funksiyasi yordamida olish.

Misollardan na'munalar.

H-funksiya o'z ichiga olgan soda misollardan eksponensial, Mittag-Leffler vaumumlashgan Mittag-Leffler funksiyalardir.

1-misol. Quyidagi

$$f(z) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(s)z^{-s}ds, \ |argz| < \frac{\pi}{2}, z \neq 0 \qquad (1)$$

integrallarni hisoblang, bu yerda integrallash yo'li $\Gamma(s)$ ning $s = -v, v = 0,1,2,\dots$ qutb nuqtalaridan o'ngda joylashgan $Re(s) = \gamma, \gamma > 0$ to'g'ri chiziq bo'ylab olinadi, natijani H-funksiya orqali ifodalang

Yechish.Qoldiqlar nazaryasi bo'yicha hisoblaymiz

$$f(z) = \sum_{v=0}^{\infty}\lim_{s\to -v}(s+v)\Gamma(s)z^{-s} = \sum_{v=0}^{\infty}\lim_{s\to -v}\frac{(s+v)(s+v-1)\dots s}{(s+v-1)\dots s}\Gamma(s)z^{-s} =$$

$$\sum_{v=0}^{\infty}\lim_{s\to -v}\frac{\Gamma(s+v+1)}{(s+v-1)\dots s}z^{-s} = \sum_{v=0}^{\infty}\frac{(-1)^v}{v!} = e^{-z}. \qquad (2)$$

integral H-funksiya orqali quyidagicha

$$e^{-z} = H_{0,1}^{1,0}\left[z|_{(0,1)}\right], \qquad (3)$$

ifodalanadi.

Mustaqil yechish uchun misollar.

1. Quyidagini isbotlang

$$(1-z)^{-a}=\frac{1}{2\pi i\Gamma(a)}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(-s)\Gamma(s+a)(-z)^{s}ds,\quad |\arg(-z)|<\pi,$$

Bu yerda $0<Re(\gamma)<Re(a)$ va kontur $\Gamma(-s)$ ning qutb nuqtalaridan $\Gamma(s+a)$ qutb nuqtalarini ajratib, $Re(s)=\gamma$ to'g'ri chiziq bo'ylab harakatlanadi.

2. Mellin-Barnes integralini hisoblang

$$f(z)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(s)\Gamma(1-s)}{\Gamma(1-\alpha s)}(-z)^{-s}ds,\quad |argz|<\pi$$

bu yerda $\alpha\in\mathbb{R}_{+}$ va $f(z)$-Mittag-Leffler funksiyasi ekanligini ko'rsating.

3. Quyidagi Mellin-Barnes integrali umumlashgan $E_{\alpha,\beta}(z)$-Mittag-Leffler funksiyasini ifodalanishini ko'rsating:

$$E_{\alpha,\beta}(z)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma(s)\Gamma(1-s)}{\Gamma(\beta-\alpha s)}(-z)^{-s}ds,\quad |argz|<\pi,$$

bu yerda $\alpha\in\mathbb{R}_{+},\beta\in\mathbb{C},Re(\beta)>0.$

4. Hisoblang: $H_{0,1}^{1,0}\left[z|_{(0,1)}\right]=$

5. Hisoblang: $H_{1,2}^{1,1}\left[z\Big|_{(0,1);(0,\alpha)}^{(0,1)}\right]=$

6. Hisoblang: $H_{1,2}^{1,1}\left[z\Big|_{(0,1);(1-\beta,\alpha)}^{(0,1)}\right]=$

7. Hisoblang: $H_{1,2}^{1,1}\left[-x\Big|_{(0,1);(1-\beta,\alpha)}^{(1-\rho,1)}\right]=$

III-BOB. INTEGRAL ALMASHTIRISHLAR

Furyening integral almashtirishi.

Ta'rif. *Haqiqiy o'zgaruvchili* $f(t) \epsilon L_1(-\infty, +\infty)$ *funksiyaning Furye almashtirishi deb*

$$F(f) = \hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-i\lambda t} f(t) dt, \quad \lambda \in R \qquad (1)$$

integralga aytiladi. $\hat{f} = F_+[f]$ *kabi ham belgilanadi*

Ta'rif. $g \in L_1(-\infty, +\infty)$ *funksiyaning teskari Furye almashtirishi deb*

$$\breve{g}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\lambda t} g(t) dt, \; \lambda \in R, \qquad (2)$$

integralga aytiladi, $\breve{g} = F_-[g]$ *kabi ham belgilanadi*

Furye almashtirishining xossalari

1. *Chiziqlilik xossasi.* Furye almashtirishi chiziqlidir.

$$F_{\pm}[af_1 + bf_2] = aF_+[\mathrm{f}_1] \pm bF_+[f_2], \qquad \mathrm{a, b} \in \mathrm{R}$$

2. *Ko'paytmaning Furye almashtirishi.*

$$F_+[f \cdot g\,] = \frac{1}{2\pi} F(f)(x) * F(g)(x)$$

3. *Furye o'ramasi.* Furye o'ramasi $*$ kabi belgilanadi. $f(t)$ *va* $g(t)$ $\mathrm{t} \in \mathbb{R}$ da

Direxle shartini qanoatlantirsin.

$$f(t) * g(t) := \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau = \int_{-\infty}^{\infty} f(t-\tau) g(\tau) d\tau$$

$$f * g(t) = g * f(t)$$

O'ramaning Furye almashtirishi.

$$F_+(f * g)(x) = F_+(f)(x) F_+(g)(x)$$

4. (*Hosilaning Furye almashtirishi*)

n ∈ N, n- tartibli hosilaning Furye va teskari Furye almashtirishi

$$F\left(f^{(n)}\right)(x) = (ix)^n F(f)(x);$$

$$F^{-1}\left(f^{(n)}\right)(x) = (ix)^n f(x)$$

5. $F^{-1}Ff = f \quad va \; F^{-1}Fg = g$

6. $F_+\left[e^{iat}f(t)\right](\lambda) = F_+[f(x)](\lambda - a)$

7. ℝ sonlar o'qida absolyut integrallanuvchi bo'lgan har qanday funksiya uchun quyidagi o'rinli

$$\lim_{\lambda\to\infty} \int_{-\infty}^{\infty} f(x)\, cos\, \lambda x dx = 0 \quad va \quad \lim_{\lambda\to\infty} \int_{-\infty}^{\infty} f(x)\, sin\, \lambda x\, dx = 0$$

8. $[a; b] \subset \mathbb{R}$ kesmada Riman bo'yicha integrallanuvchi bo'lgan har qanday f funksiya uchun

$$\lim_{\lambda\to\infty} \int_{a}^{b} f(x)\, cos\, \lambda x\, dx = \lim_{\lambda\to\infty} \int_{a}^{b} f(x)\, sin\, \lambda x\, dx = 0$$

tenglik o'rinli bo'ladi.

9. $F_+[f(x - a)](\lambda) = e^{-ia\lambda}F_+[f(x)](\lambda)$

10. $F_+\left[\frac{df}{dx}\right](\lambda) = (i\lambda)F_+[f(x)](\lambda)$

Ba'zi funksiyalarning Furye almashtirishlarini keltiramiz:

1-Misol: $f(x) = e^{-a|x|}$

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} e^{-a|x|}e^{-ix\lambda}dx = \int_{-\infty}^{0} e^{x(a-i\lambda)}dx + \int_{0}^{\infty} e^{-x(a+i\lambda)}dx =$$

$$= \frac{e^{x(a-i\lambda)}}{a - i\lambda}\Bigg|_{-\infty}^{0} + \frac{e^{-x(a+i\lambda)}}{e^{-(a+i\lambda)}}\Bigg|_{0}^{\infty} = \frac{1}{a - i\lambda} + \frac{1}{a + i\lambda} = \frac{2a}{a^2 + \lambda^2}$$

2-Misol : $\varphi(t) = \delta(t), t \in R,$ Furye almashtirishini toping.

Yechim: Bu yerda $\delta(\cdot) - Dirakning$ delta funksiya $\delta(t) = \begin{cases} 0, & t \neq 0. \\ \infty, & t = 0. \end{cases}$

Delta funksiya quyidagi xossaga ega:

$$\int\limits_R \delta(t - t_0)\varphi(t)dt = \varphi(t_0)$$

$\mathrm{F}[\delta(t)](\lambda) = \int_R e^{-i\lambda t}\,\delta(t)dt = e^{-i\lambda t}\,\Big|_{t=0}^{t=1}$ ya'ni $\mathrm{F}[\delta]\,(\lambda) = 1, \lambda \in R$

munosabatdan

$$1 = F[\delta(t)]\,(\lambda) \quad F[1] = \delta(-t) \qquad \text{ya'ni } F[1] = 2\pi\delta(\lambda).$$

3-misol. Berilgan funksiya Furye almashtirishini toping.

$$f(x) = \begin{cases} 1, & agar\ |x| \leq a, \\ 0, & agar\ |x| > a. \end{cases}$$

Yechish:

$$\boldsymbol{F}_+[\boldsymbol{f}](\boldsymbol{\lambda}) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} f(x)e^{-ix\lambda}dx = \frac{1}{\sqrt{2\pi}} \int\limits_{-a}^{+a} 1 \cdot e^{-ix\lambda}dx = \sqrt{\frac{2}{\pi}\frac{sina\lambda}{\lambda}}$$

4-misol. $f(x) = \begin{cases} 0, -\infty < x < 1 \\ 1, 1 < \ x < 0 \\ 0, \ 2 < x \end{cases}$ funksiyaning Furye almshtirishini toping.

Yechish. R da berilgan funksiya absolyut integrallanuvchi ekanligidan

$$\int^{+\infty} |f(x)|dx = \int_1^2 dx = 1$$

$$\int_R f(x)e^{-i\xi x}dx = \int_1^2 e^{-i\xi x}dx = \frac{i(e^{-2i\xi} - e^{-i\xi}}{\xi}$$

$$F_+[f](\xi) = \frac{i(e^{-2i\xi} - e^{-i\xi}}{2\pi\xi}$$

Ba'zi funksiyalr Furye almashtirishi uchun jadval keltiramiz:

$f(x)$	$F[f(x)](\xi)$
$f(x) \pm g(x)$	$F[f(x)](\xi) + F[g(x)](\xi)$
$\alpha f(x)$	$\alpha F[f(x)](\xi)$
$f(bx)$	$\frac{1}{b}F[f(x)](\frac{\xi}{b})$

$f(x+c)$	$e^{ic\xi}F[f(x)](\xi)$
$e^{ic\xi}f(x)$	$F[f(x)](\xi-c)$
$x^n f(x)$	$\frac{1}{(-i)^n}\cdot\frac{d^n}{d\xi^n}F[f(x)](\xi)$
$\frac{d^n}{dx^n}f(x)$	$(i\xi)^n F[f(x)](\xi)$

Mustaqil yechish uchun topshiriqlar.

Berilgan funksiyalar Furye almashtirishini toping.

1. $f(t)=e^{-|t|}$
2. $f(t)=e^{-t^2}$
3. $f(t)=\sin t$
4. $a\in R_+$ bo'lsin. $F(\omega)$- $f(t)$ funksiyaning Fure almashtirishi, $F_+[f(t-a)](\omega)=e^{i\omega a}F_+(\omega)$, ni isbotlang.
5. $a\in R$ bo'lsin.

$$F_+\left[e^{iat}f(t)\right](\omega)=F(\omega-a),$$

bu yerda $F(\omega)$- $f(t)$ funksiyaning Fure almashtirishi.

6. Ushbu tenglik bajarilishini isbotlang $f(-\omega)=\mathcal{F}[F(t)](\omega)$, bu yerda $F(\omega)$- $f(t)$ funksiyaning Fure almashtirishi.
7. $F(\omega)=\mathcal{F}[f(t)](\omega)$ va $G(\omega)=\mathcal{F}[g(t)](\omega)$ Furye almashtirishlari $f(t)$ va $g(t)$ funksiyalarning. Ushbu tenglik bajarilishini isbotlang: $\int_{-\infty}^{\infty}F(\omega)g(\omega)e^{i\omega t}d\omega=\int_{-\infty}^{\infty}f(\xi)G(\xi-t)d\xi.$
8. $(\omega)=\mathcal{F}[f(t)](\omega)$ va $G(\omega)=\mathcal{F}[g(t)](\omega)$ Furye almashtirishlari $f(t)$ va $g(t)$ funksiyalarning. Ushbu tenglik bajarilishini isbotlang: $\int_{-\infty}^{\infty}F(\omega)G(\omega)d\omega=\int_{-\infty}^{\infty}f(-\xi)g(\xi)d\xi.$
9. Tenglik bajarilishini isbotlang

$$F[f(t-a)](\omega)=e^{i\omega a}F(\omega).$$

10. Ushbu tenglik bajarilishini isbotlang

$$F[f(at)](\omega) = \frac{1}{|a|} F\left(\frac{\omega}{a}\right).$$

11.$f(x) = xe^{-x^2}$, funksiya Furye almashtirishini toping.

12.$f(x) = Ce^{-ax^2}, C \in R, a > 0$ funksiya Furye almashtirishini toping.

13.$f(x) = e^{-0.5x^2}\cos ax, a \in R$ funksiya Furye almashtirishini toping.

14. $f_a(x) = \frac{\theta(x)}{x^{1-\alpha}\, e^{\beta x}\Gamma(a)}$ bo'lsa $f_a * f_b = f_{a+b}$ ni isbotlang. Bu yerda $\theta(t)$ −Xevisayde funksiyasi.

15.$f(x) = \begin{cases} 0, & -\infty < x < 1 \\ 1, & 1 < x < 2 \\ 0, & 2 < x \end{cases}$ funksiyaning sinus va kosinus Furye almashtirishini toping.

16. $f(x) = \frac{sinax}{x}$, Furye almashtirishini toping bunda $f(0) = a \in R$ deb oling.

17. $f(x) = \begin{cases} cosx, & x \in (0,\pi) \\ 0, & x \notin (0,\pi) \end{cases}$ funksiyaning sinus Furye almashtirishini toping

18.$f(x) = \begin{cases} 0, & x \leq 0 \\ e^{-x}, & x > 0 \end{cases}$ funksiyaning kosinus Furye almashtirishini toping

19.$f(x) = \begin{cases} 0, & x \leq 0 \\ xe^{-x}, & x > 0 \end{cases}$ funksiyaning sinus Furye almashtirishini toping

20.$f(x) = \begin{cases} 0, & x \leq 0 \\ e^{-x}, & x > 0 \end{cases}$ funksiyaning Furye almashtirishini toping

21.$f(x) = \begin{cases} sinx, & 0 \leq x \leq \pi \\ 0, & x > \pi \end{cases}$ funksiyaning Furye almashtirishini toping

22.$f(x) = \begin{cases} 0, & x \leq 0 \\ e^{-x}cosx, & x > 0 \end{cases}$ funksiyaning Furye almashtirishini toping

23.$f(x) = \begin{cases} 0, & x \leq 0 \\ e^{-x}sinx, & x > 0 \end{cases}$ funksiyaning Furye almashtirishini toping

Laplas integral almashtirishi.

Haqiqiy o'zgaruvchili $\varphi(t), t \in (0 + \infty)$ funksiyaning Laplas almashtirishi

$$L^{-1}\{\varphi\}(s) = \tilde{\varphi}\,(s) \coloneqq \int_0^{+\infty} e^{-st}\, \varphi(t)dt, \qquad (s \in \mathbb{C}) \qquad (1)$$

integral yordamida aniqlanadi.

$F(s)=\int_0^{+\infty} e^{-st} f(t)dt,$ kabi ham yoziladi. Bunda $f(t)$ −funksiya original, $F(s)$ −originalning tasviri yoki tasvir funksiya ham deb yuritiladi.

Agar (1) integral $s_0 \in \mathbb{C}$ nuqtada yaqinlashuvchi bo'lsa, u holda $Res > Res_0$ shartni qanoatlantiruvchi $s \in \mathbb{C}$ larda absolyut yaqinlashuvchi bo'ladi. (1) Laplas integrali yaniqlashuvchi bo'ladigan s larning infinumi $\sigma_\varphi -$ yaqinlashish absissasi deyiladi. Shuning uchun (1) integral $Re\, s > \sigma_\varphi$ da yaqinlashuvchi $Res < \sigma_\varphi$ da esa uzoqlashuvchi bo'ladi.

Teskari Laplas almashtirish esa

$$L^{-1}\{g(s)\}(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\ g(s)ds, \quad (\gamma = Re(s) > \sigma_\varphi\ , \qquad (2)$$

integral yordamida beriladi.

Yetarlicha silliq $\varphi\ va\ g$ funksiyalar uchun to'g'ri va teskari Laplas almashtirishlari o'rtasida quyidagi tengliklar o'rinli.

$$L^{-1}L\{\varphi\} = \varphi \quad \text{va} \quad LL^{-1}\{g\} = g \qquad (22)$$

Laplas almashtirishining xossalari

1. $\mathcal{L}(af_1 \pm bf_2) = a\mathcal{L}(f_1) \pm b\mathcal{L}(f_2)$ a,b∈$\mathbb{R}$
2. a∈$\mathbb{R}$. $\mathcal{L}[f]$(s)=$F(s)$ bo'lsa,

$$\mathcal{L}[e^{-at}f(t)] = F(s+a)$$

3. $\forall\alpha > 0$

$$\mathcal{L}[f(\alpha t)] = \frac{1}{\alpha}\ F\left(\frac{s}{\alpha}\right)$$

4. a>0 $H(t-a)$ Heaviside funksiyasi

$$\mathcal{L}[f(t-a)H(t-a)] = e^{-as}F(s)$$

yoki

$$\mathcal{L}[f(t)H(t-a)] = e^{-as}\mathcal{L}[f(t)]$$

5. $t > 0$ larda $f(t)$ uzluksiz bo'lsin.

$$\mathcal{L}[f'(t)] = sF(s) - \mathrm{f}(0)$$

6. Funksiya differensialining Laplas almashtirishi $f\colon \mathbb{R} \to \mathbb{R}$ va (n-1)-tartibli hosilagacha yopiq $[0; c] \subset \mathbb{R}$ intervalda uzluksiz bo'lsin.U holda ∃M>0 soni mavjudki $t_0 > 0$ uchun

$$|f(t)| \leq Me^{at}\ ,\ \left|\frac{d}{dt}f(t)\right| \leq Me^{at}, \dots ,\ \left|\frac{d^{n-1}}{dt^{n-1}}f(t)\right| \leq\ Me^{bt}$$

$Re(s) > b$ baholar o'rinli bo'lsin.

$$\mathcal{L}\left[\frac{d^n}{dt^n}f\right](s) = s^n\mathcal{L}[f](s) - \sum_{k=0}^{n-1}\left[\left(\frac{d^{n-1-k}}{dt^{n-1-k}}f_{t=0}\right)\right]$$

7. O'ramaning Laplas almashtirishi.

$$\mathcal{L}[f(t) * g(t)](s) = \mathcal{L}[f](s)\, \mathcal{L}[g](s) = F(s)G(s)$$

$$f(t) * g(t) \coloneqq \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t g(t-\tau)f(\tau)d\tau$$

8. Laplas almashtirishining hosilasi

$$\mathcal{L}[t^n f(t)](s) = (-1)^n \frac{d^n}{ds^n}F(s); \quad n = 0,1,2,3, \dots$$

9. Laplas almashtirishining integrali

$$\mathcal{L}\left[\frac{f(t)}{t}\right](s) = \int_s^\infty F(\xi)d\xi$$

10. Integralning Laplas almashtirishi

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\}(s) = \frac{F(s)}{s}$$

Teskari Laplas almashtirishi

$$\mathcal{L}^{-1}[F](t)=\frac{1}{2\pi i}\lim_{\tau\to\infty}\int_{\sigma-i\tau}^{\sigma+i\tau} e^{st}F(s)ds. \quad Re(s)=\sigma>0$$

Fure almashtirishiga o'xshash Laplas almashtirishini t∈ R^n uchun umumlashtirish mumkin. Laplas almashtirishga oid bir nechta misollar ko'rib o'tamiz.

Misol. $f(t)=\mathrm{e}^{2\mathrm{t}}\cos\mathrm{t}$ funksiyaning Laplas almashtirishini hisoblang.

$$\mathcal{L}[\mathrm{e}^{2\mathrm{t}}\cos\mathrm{t}]=\int_0^\infty e^{-st}e^{2t}\cos tdt=\int_0^\infty e^{-t(s-2)}\cos tdt$$

$$\mathcal{L}[\mathrm{e}^{2\mathrm{t}}\cos\mathrm{t}]=Re\{e^{-(s-2-i)t}dt\}=Re\left\{\frac{1}{s-2-i}\right\}=Re\left\{\frac{s-2+i}{(s-2)^2+i}\right\}$$

$$\mathcal{L}[\mathrm{e}^{2\mathrm{t}}\cos\mathrm{t}]=\frac{s-2}{(s-2)^2+1}$$

Misol. Quyidagi Volterra tipidagi integral tenglamani qaraymiz

$$\varphi(x)=f(x)+\int_0^x k(x-t)\varphi(t)dt, \quad x>0. \qquad (1)$$

$f(x)$ и $k(x)$ funksiyalar $R_+=(0,\infty)$, da uzluksiz va $x\to+\infty$ bo'lganda nolga intiladi. Quyidagi baholar o'rinli

$$|f(x)|\le Ae^{-ax}, \qquad |k(x)|\le Be^{-bx}, \qquad (2)$$

Bu yerda $A>0,\ B>0,\ a\ge 0,\ b\ge 0$ lar o'zgarmaslar. m va M sonlari $x\ge 0$ bo'lganda $|f(x)|$ и $|k(x)|$ larning yuqori chegaralari:

$$m=\sup_{x\ge0}|f(x)|, \qquad M=\sup_{x\ge0}|k(x)|. \qquad (3)$$

(1) tenglamaga ketma-ket yaqinlashishlar usulini qo'llab $x\ge 0$ bo'lganda $\varphi(x)$ uchun baho olamiz:

$$|\varphi(x)|\le\sum_{n=0}^\infty|\varphi_n(x)|\le\sum_{n=0}^\infty m\frac{(Mx)^n}{n!}=me^{-Mx}.$$

Bundan esa $\varphi(x)$, $f(x)$,$k(x)$ funksiyalarga $\sigma>max\{a,b,M\}$ larda Laplas almashtirishini qo'llash mumkinlagi kelib chiqadi. (1) tenglikning ikkala tomoniga Laplas almashtirishini, o'rama formulasini qo'llaymiz va belgilashlar kiritamiz:

$$\Phi(z) = (L\varphi)(z), \qquad F(z) = (Lf)(z), \qquad K(z) = (Lk)(z), \qquad (4)$$

$$\Phi(z) = F(z) + K(z)\Phi(z),$$

$$\Phi(z) = \frac{F(z)}{1 - K(z)} \qquad (5)$$

$\Phi(z)$ funksiya $Re(z) > M$ da analitik bo'lishi kerak. (5) tenglikdagi kasrning maxraji yuqorida aytilgan yarim tekislikda ildizga ega emasligi kelib chiqadi. $\varphi(x)$ uchun quyidagiga ega bo'lamiz:

$$\varphi(x) = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} \Phi(z) e^{zx}\, dx = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} \frac{F(z)}{1 - K(z)} e^{zx}\, dx \quad (6\ )$$

Demak, (1) tenglama yechimi $\varphi(x)$ funksiya (6) tenglik bilan aniqlanar ekan. Bu yechim uchun boshqa formula keltiramiz. Buning uchun (1) tenglama uchun barcha takroriy yadrolar$(x - t)$ ayirmaga bog'liqligini ko'rsatamiz.

$$K_1(x,t) = K(x - t), \quad K_2(x,t) = \int_t^x K_1(x,\tau) K_1(\tau, t)\, d\tau =$$

$$= \int_t^x K(x - \tau) K(\tau - t) d\tau = \int_0^{x - t} K(x - t - s) K(s) ds = K_2(x - t)$$

$$Kn(x,t) = Kn(x - t) \ \ (n = 1,2,3,4, \dots .)$$

$\lambda = 1$ da (1) tenglama rezolventasi $R(x,t;\lambda)$ faqat $x - t$ ayirmadan bog'liq bo'ladi. Belgilash kiritamiz:

$$R(x,t;1) = r(x - t) \qquad (7)$$

(1) tenglama yechimini quyidagicha yozish mumkin

$$\varphi(x) = f(x) + \int_0^x r(x - t) f(t)\, dt \qquad (9)$$

Tenglikning ikkala tomoniga Laplas almashtirishini qo'llab (4) hisobga olsak

$$R(z) = (Lr)(x) \qquad (10)$$

Shu bilan birga o'ramaning Laplas almashtirishi formulasini qo'llab

$$\Phi(z) = F(z) + R(z) F(z) .$$

Tenglikka ega bo'lamiz. (5) dan foydalanib $R(z)$ ni quyidagicha yozamiz:$R(z) = \frac{\Phi(z)-F(z)}{f(z)} = \frac{1}{F(z)}\left[\frac{F(z)}{1-K(z)} - F(z)\right] = \frac{K(z)}{1-K(z)}$

$$R(z) = \frac{K(z)}{1-K(z)} \qquad (11)$$

$r(x)$ rezolventa:

$$r(x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{K(z)}{1-K(z)} e^{zx}\, dx \qquad (12)$$

Misol. Quyidagi tenglamani qaraymiz.

$$\varphi(x) = f(x) + \lambda\int_0^x (x-t)\varphi(t)dt \qquad (x>0,\ \lambda>0). \qquad (13)$$

Bu (1) ko'rinishdagi tenglama: $k(x) = \lambda x$. $K(z) = \lambda\int_0^\infty e^{-zx}xdx = \frac{\lambda}{z^2}$

ReZ>0. (10 formulaga ko'ra

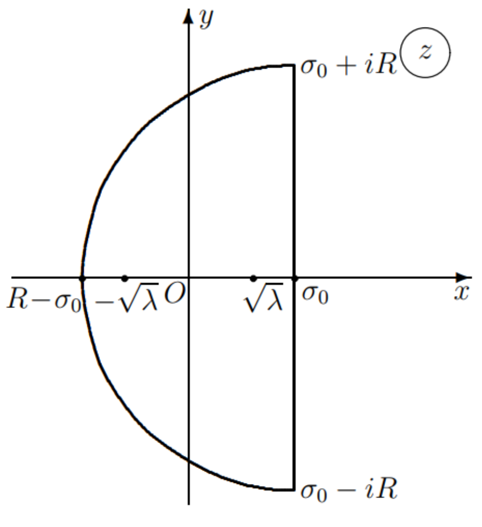
$$R(z) = \frac{\lambda}{z^2-\lambda},$$

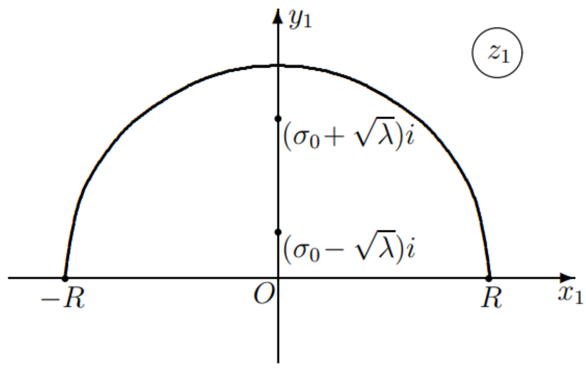
Va (11) ni hisobga olib $r(x)$ rezolventa quyidagicha:

$$r(x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\lambda e^{xz}}{z^2-\lambda}dz \qquad (x>0), \qquad (14)$$

Bu yerda σ -ixtiyoriy yetarlicha katta musbat son.

$$r(x) = \frac{\sqrt{\lambda}}{2}\left(e^{\sqrt{\lambda}x} - e^{-\sqrt{\lambda}x}\right).$$





(13) tenglama yechimi (8) ni inobatga olsak

$$\varphi(x) = f(x) + \frac{\sqrt{\lambda}}{2}\int_0^x \left[e^{\sqrt{\lambda}(x-t)} - e^{-\sqrt{\lambda}(t-x)}\right] f(t)dt.$$

Formula bilan aniqlaniladi.

Ba'zi funksiyalar Laplas almashtirishlari uchun Jadval keltiramiz

$f(t)$	$\mathcal{L}[f(t)] = F(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{\Gamma(1+n)}{s^{1+n}}$
e^{bt}	$\frac{1}{s-b}$
$\sin(bt)$	$\frac{b}{s^2+b^2}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$
$sh(bt)$	$\frac{b}{s^2-b^2}$
$ch(bt)$	$\frac{s}{s^2-b^2}$
$e^{bt}f(t)$	$F(s-b)$
$tf(t)$	$-F'(s)$

Mustaqil yechish uchun topshiriqlar.

1. $f(t) = e^{4t}\cos 3t$ funksiyaning Laplas almashtirishini toping.
2. $f(t) = \sin^2 t$ funksiya Laplas almashtirishini toping.
3. $f(t) = ch3tsin3\,t$ funksiya Laplas almashtirishini toping.
4. $f(t) = \frac{e^{-3t}-e^{-2t}}{t}$ funksiya Laplas almashtirishini toping.

5. Quyidagi funksiyaning Laplas almashtirishini hisoblang. $f(t) = e^{3t}(1-4t)$.
6. Quyidagi funksiyaning Laplas almashtirishini hisoblang. $f(t) = \frac{\sin t}{t}$.
7. Laplas almashtirishidan foydalanib quyidagi integral tenglamani yeching

$$\int_0^t (t-\tau)^{\alpha-1} y(\tau)d\tau = f(t), 0 < \alpha < 1, t > 0.$$

8. Quyidagi funksiyaning teskari Laplas almashtirishini hisoblang.
$\mathcal{L}^{-1}\left[\frac{s}{(s+a)(s+b)}\right]$, bu yerda $a, b \in R; a \neq b$.
9. Laplas almashtirishidan foydalanib quyidagi oddiy differensial tenglamani yeching:

$$y''' + 2y'' + y = sint$$

10. Vaqt bo'yicha Laplas almashtirishini qo'llabcheksiz torning tebranishi uchun Dalamber formulasini keltirib chiqaring:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad (-\infty < x < +\infty, t > 0)$$

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x).$$

11. Laplas almashtirishidan foydalanib quyidagi oddiy differensial tenglamani yeching:

$$y'' + \omega^2 y = \frac{1}{\tau}(t)$$

12. Laplas almashtirishidan foydalanib quyidagi oddiy differensial tenglamani yeching:

$$y''' + 3y'' + 3y' + y = 1$$

13. Laplas almashtirishidan foydalanib quyidagi oddiy differensial tenglamani yeching:

$$y'' + \omega^2 y = Asin\omega t$$

$$y(0) = y_0, \qquad y'(0) = y_1$$

14. Laplas almashtirishidan foydalanib quyidagi oddiy differensial tenglamani yeching:

$$(n+1)y^{(n)} + ty = 0$$

$$y(0) = y_0, \qquad y'(0) = y_1 \dots y^{n-1}(0) = y_{n-1}$$

15.Berilgan funksiya Laplas almashtirishini toping: $f(t) = sin\alpha\sqrt{t}$

16.Laplas almashtirishi xossalarini qo'llab berilgan funksiya Laplas almashtirishini toping

$$\int_1^{-\infty} \frac{e^{-\tau t}}{\tau} d\tau$$

17.Laplas almashtirishi xossalarini qo'llab berilgan funksiya Laplas almashtirishini toping $f(t) = B|sin\omega t|$

18.Laplas almashtirishi xossalarini qo'llab berilgan funksiya Laplas almashtirishini toping

$$\int_1^{-\infty} \frac{cos\tau t}{\tau} d\tau$$

19.Laplas almashtirishi xossalarini qo'llab berilgan funksiya Laplas almashtirishini toping

$$f(t) = \frac{1 - e^{-t}}{t}$$

20.Laplas almashtirishi xossalarini qo'llab berilgan funksiya Laplas almashtirishini toping: $f(t) = A\{t\}$. $\{t\} - t$ ning kasr qismi

21.Laplas almashtirishidan foydalanib quyidagi oddiy differensial tenglamani yeching. $\begin{cases} \frac{d^2}{dt^2}x(t) + x(t) = t, \\ x(0) = x'(0) = 0. \end{cases}$

22.Laplas almashtirishidan foydalanib quyidagi oddiy differensial tenglamani yeching. $\begin{cases} \frac{d^2}{dt^2}x(t) + 5\frac{d}{dt}x(t) + 6x(t) = e^{2t}, \\ x(0) = x'(0) = 0. \end{cases}$

23.Laplas almashtirishidan foydalanib quyidagi oddiy differensial tenglamalar sistemasini yeching.

$$\begin{cases} \frac{d}{dt}x(t) + y(t) = 1, \\ \frac{d}{dt}y(t) - x(t) = -1, \\ x(0) = y(0) = 2. \end{cases}$$

24. Faraz qilaylik , $a \in R, \alpha > 0, \beta > 0, \gamma > 0$ bo'lsin. U holda ushbu tenglik bajarilishini isbotlang.

$$\mathcal{L}^{-1}\left[\frac{s^{2\alpha-\beta-\gamma}}{s^{2\alpha}-a^2}\right] = t^{\beta+\gamma-1}E_{2\alpha,\beta+\gamma}(a^2t^{2\alpha}).$$

25. Yuqoridagi misolni maxsus hol ya'ni $\alpha = \beta = \gamma$ uchun Laplas almashtirishini isbotlang.

26. $a \in R, \alpha > 0, \beta > 0$ hol uchun quyidagi funksiyaning Laplas almashtirishini hisoblang.

$$t^{\beta-1}E_{\alpha,\beta}(at^{\alpha}).$$

Mellin almashtirishlari

Biz Mellin almashtirishi va uning teskari almashtirishini kompleks Furye almashtirishi va teskari Furye almashtirishidan keltirib chiqaramiz:

$$F\{g(s)\} = G(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-ik\xi}\, g(\xi)d\xi \qquad (1)$$

$$F^{-1}\{G(k)\} = g(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{ik\xi}\, G(k)dk \qquad (2)$$

$e^{\xi} = x$ va $ik = c - p$ almashtirish bajaramiz va (1) va (2) dan quyidagilarga kelamiz, bu yerda $c = const$

$$G(ip - ic) = \frac{1}{\sqrt{2\pi}}\int_{0}^{+\infty} x^{p-c-1}\, g(lnx)dx \qquad (3)$$

$$g(lnx) = \frac{1}{\sqrt{2\pi}}\int_{c-i\infty}^{c+i\infty} x^{c-p}\, G(ip-ic)dp \qquad (4)$$

Biz $\frac{1}{\sqrt{2\pi}}x^{-c}\, g(lnx) \equiv f(x)$ va $G(ip - ic) \equiv f(p)$ orqali $f(x)$ ning Mellin almashtirishi va Mellin almashtirishining teskarisini belgilaymiz va quyidagicha yozamiz

$$M\{f(x)\} = \tilde{f}(p) = \int_{0}^{\infty} x^{p-1} f(x)dx \qquad (5)$$

$$M^{-1}\{\tilde{f}(p)\} = f(x) = \int_{c-i\infty}^{c+i\infty} x^{-p}\, \tilde{f}(p)dp \qquad (6)$$

Bu yerda $f(x)$ funksiya $(0,\infty)$ da aniqlangan haqiqiy funksiya va Mellin almashtirishining o`zgaruvchisi $p$ kompleks son. Ba`zan $f(x)$ ning Mellin almashtirishi $\tilde{f}(p) = M[f(x),p]$ kabi belgilanadi. M va M^{-1} lar chiziqli integral operatorlar .

Mellin almashtirishlarining xossalari:

Agar $M\{f(x)\} = \tilde{f}(p)$ bo`lsa, u holda quyidagilar o`rinli:

a) $M\{f(ax)\} = a^{-p}\tilde{f}(p), \qquad a > 0,$ (9)

Isbot: $M\{f(ax)\} = \int_0^\infty x^{p-1} f(ax)\, dx =$ $\qquad ax = t \ \ desak$

$$= \frac{1}{a^p}\sum_{n=0}^{\infty} t^{p-1} f(t)dt = \frac{\tilde{f}(p)}{a^p}$$

$b)$ $M[x^\alpha f(x)] = \tilde{f}(p+a)$ (10)

$c)$ $M[f(x^\alpha)] = \frac{1}{a}\tilde{f}\left(\frac{p}{a}\right)$ (11)

$M[\frac{1}{x}f(\frac{1}{x})] = \tilde{f}(1-p)$ (12)

$M[(logx)^n f(x)] = \frac{d^n}{dp^n}\tilde{f}(p)$ n=1,2,3… (13)

5* tenglik $\frac{d}{dp} = (logx)x^{p-1}$ (6*) dan foydalanib isbotlanadi.

$d)$ $M[f'(x)] = -(p-1)\tilde{f}(p-1)$

$M\left[f''(x)\right] = (p-1)(p-2)\tilde{f}(p-2);$

$M\left[f^{(n)}(x)\right] = (-1)^n \frac{\Gamma(p)}{\Gamma(p-n)}\tilde{f}(p-n) = (-1)^n \frac{\Gamma(p)}{\Gamma(p-n)} M[f(x,p-n)];$

Misol: Agar $f(x) = e^{-nx}$ bo`lsa, bunda $n > 0$

$$M\{e^{-nx}\} = \tilde{f}(p) = \int_0^\infty x^{p-1}e^{-nx}\, dx$$

$nx = t$ desak,

$$= \frac{1}{n^p}\int_0^\infty t^{p-1}e^{-t}\,dt = \frac{\Gamma(p)}{n^p} \qquad (7)$$

Misol: $f(x) = \frac{1}{1+x}$ funksiyaning Mellin alshtirishini toping.

$$M\left\{\frac{1}{1+x}\right\} = \tilde{f}(p) = \int_0^\infty x^{p-1}\frac{1}{1+x}dx =$$

$$x = \frac{t}{1-t} \qquad yoki \qquad t = \frac{x}{1+x} \qquad desak$$

$$= \int_0^\infty t^{p-1}(1-t)^{1-p-1}\,dt = B(p, 1-p) = \Gamma(p)\Gamma(1-p) \qquad (8)$$

Misol: $f(x) = (e^x - 1)^{-1}$

$$M\{(e^x-1)^{-1}\} = \tilde{f}(p) = \int_0^\infty x^{p-1}(e^x-1)^{-1}\,dx=$$

Biz $\sum_{n=0}^{\infty} e^{-nx} = \frac{1}{1-e^{-x}}$ dan foydalanamiz, bunnda $\sum_{n=1}^{\infty} e^{-nx} = \frac{1}{e^x-1}$ bo`ladi.

$$= \sum_{n=1}^{\infty} x^{p-1}e^{-nx}dx = \sum_{n=1}^{\infty}\frac{\Gamma(p)}{n^p} = \Gamma(p)\cdot\varsigma(p)$$

$$\varsigma(p) = \sum_{n=1}^{\infty}\frac{1}{n^p}; \quad (Rep > 1)$$

Isbot: $M[f'(x)] = \int_0^\infty x^{p-1}f'(x)dx = [x^{p-1}f(x)]_0^\infty - (p-1)\int_0^\infty x^{p-2}f(x)dx =$

$$-(p-1)\tilde{f}(p-1)$$

e) Agar $M[f(x)] = \tilde{f}(p)$ bo`lsa, quyidagilar o`rinli:

$M[xf'(x)] = -p\tilde{f}(p)$

$M\left[x^2 f''(x)\right] = (-1)^2 p(p+1)\tilde{f}(p);$

$M\left[x^n f^{(n)}(x)\right] = (-1)^n \frac{\Gamma(p+n)}{\Gamma(p)} \tilde{f}(p);$

Isbot:

$$M[xf'(x)] = \int_0^\infty x^p f'(x)dx = [x^p f(x)]_0^\infty - p\int_0^\infty x^{p-1} f(x)dx =$$

$$-p\tilde{f}(p)$$

f) Agar $M[f(x)] = \tilde{f}(p)$ bo`lsa, quyidagilar o`rinli:

$$M\left[\left(x\frac{d}{dx}\right)^2 f(x)\right] = M\left[x^2 f''(x) + xf'(x)\right] = (-1)^2 p^2 \tilde{f}(p)\ ;$$

$$M\left[\left(x\frac{d}{dx}\right)^n f(x)\right] = (-1)^n p^n \tilde{f}(p);$$

Isbot: $M\left[\left(x\frac{d}{dx}\right)^2 f(x)\right] = M\left[x^2 f''(x) + xf'(x)\right] = M\left[x^2 f''(x)\right] + M\left[xf'(x)\right] =$

$$= -p\tilde{f}(p) + p(p+1)\tilde{f}(p) = (-1)^2 p^2 \tilde{f}(p)$$

g) $M\left\{\int_0^x f(t)dt\right\} = \frac{-1}{p}\tilde{f}(p+1)\ ;$

$$M\{I_n f(x)\} = M\left\{\int_0^x I_{n-1} f(t)dt\right\} = (-1)^n \frac{\Gamma(p)}{\Gamma(p+n)} \tilde{f}(p+n);$$

$$I_n f(x) = \int_0^x I_{n-1} f(t)dt.$$

Ba'zi funksiyalr Mellin almashtirishlarini jadvalda keltiramiz.

	$f(x)$	$g(p) = \int_0^\infty f(x)\, x^{p-1} dx$
1	$f(ax)$	$a^{-p} g(p)$
2	$x^\alpha f(x)$	$g(p+\alpha)$
3	$f\left(\frac{1}{x}\right)$	$g(-p)$

4	$f(x^h), h>0$	$h^{-1}g\left(\frac{p}{h}\right)$
5	$f(x^{-h}), h>0$	$h^{-1}g\left(-\frac{p}{h}\right)$
6	$f'(x)$	$(1-p)g(p-1)$
7	$f^{(n)}(x)$	$(-1)^n(p-n)g(p-n)$
8.	$e^{-\alpha x^h}\quad Re\alpha>0, h>0$	$h^{-1}\alpha^{-\frac{p}{h}}\Gamma\left(\frac{p}{h}\right), Rep>0$
9	$e^{-\alpha x^{-h}}\quad Re\alpha>0, h>0$	$h^{-1}\alpha^{-\frac{p}{h}}\Gamma\left(-\frac{p}{h}\right), Rep<0$
10	$sinax,\quad a>0$	$a^{-p}\Gamma(p)\sin\left(\frac{\pi p}{2}\right), -1<Rep<1$

Mustaqil ishlash uchun misollar:

1. $f(x)=\frac{2}{e^{2x}-1}$
2. $f(x)=\frac{1}{e^x+1}$
3. $f(x)=\frac{1}{(1+x)^n}$
4. $f(x)=coskx$
5. $f(x)=sinkx$
6. $f(x)=\frac{siax}{1+x^2}. a>0$

Mellin almashtirishining qo`llanishi

Misol: Ushbu Chegaraviy masalaning yechimini toping.

$$x^2u_{xx}+xu_x+u_{yy}=0 \qquad 0\le x<\infty, \qquad 0<y<1;$$

$u(x,0)=0, \qquad u(x,1)=\begin{cases} A, & 0\le x\le 1 \\ 0, & x>1 \end{cases};$

Bu yerda $A=const$

x bo`yicha $u(x,y)$ ning Mellin almashtirishini quyidagicha bajaramiz.

$$\tilde{u}(p,y)=\int_0^{\infty} x^{p-1}\,u(x,y)dx$$

$$\tilde{u}_{yy}+p^2\tilde{u}=0, \qquad 0<y<1$$

$\tilde{u}(p,0)=0, \qquad \tilde{u}(p,1)=A\int_0^{\infty} x^{p-1}\,dx=\frac{A}{p}$

$$\tilde{u}(p,y)=\frac{A}{p}\cdot\frac{sinpy}{sinp}, \qquad 0<Rep<1$$

$$u(x,y)=\frac{A}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^p}{p}\frac{sinpy}{sinp}dp;$$

$$u(x,y)=\frac{A}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(-1)^n x^{-n\pi}sinn\pi y\,;$$

Mellin almashtirishlarining Integral tenglamalarni yechishga tatbiqi.

$$M\left\{\int_0^{+\infty} f(t)\,\varphi\left(\frac{x}{t}\right)\frac{dt}{t}\right\}=F(S)\Phi(S);$$

$$F(S)=M|f(t)|, \qquad \Phi=M|\varphi(t)|.$$

Yuqoridagi xossa

$$\varphi(x)=f(x)+\int_0^{\infty}K\left(\frac{x}{t}\right)\varphi(t)\frac{dt}{t}$$

ko'rinishdagi integral tenglamani yechishda foydalaniladi.

Mustaqil bajarish uchun mashqlar:

Quyidagi funksiyalarning Kaputo kasr hosilasini toping.

1. $e^{\lambda x}$ 2. sin(λx) 3. cos(λx)
2. $f(x)=e^{-x^2}$, $x\epsilon\mathbb{R}$ funksiyaning Furye almashtirishini hisoblang.
3. $f(t)=e^{2t}cost$ funksiyaning Laplas almashtirishini aniqlang.

4. $f(t) = 4tcos^2t$ funksiyaning Laplas almashtirishini aniqlang.

5. $E^1_{\alpha,\beta}(x) = E_{\alpha,\beta}(x)$ ni isbotlang.

Foydalanilgan adabiyotlar

1. T.M. Atanackovi´c, S. Pilipovi´c, B. Stankovi´c and D. Zorica (2014): Fractional Calculus with Applications in Mechanics, 2 Vol l., Wiley, New York (2014).
2. Yu.I. Babenko (1986): Heat and Mass Transfer, Chimia, Leningrad 1986 [in Russian].
3. R.T. Baillie and M.L. King (Editors)(1996): Fractional Differencing and Long Memory Processes, Journal of Econometrics 73(1)(1996), 1-324.
4. A.V. Balakrishnan (1959): Operational calculus for infinitesimal generators of semi-groups, Trans. Amer. Math. Soc. 91(2)(1959), 330-353.
5. A.V. Balakrishnan (1960): Fractional powers of closed operators and the semi-groups generated by them, Pacific J. Math. 10(2)(1960). 419-437.
6. D. Baleanu, K. Diethelm, E. Scalas and J.J. Trujillo (2012): Fractional Calculus: Models and Numerical Methods. World Scientific, Singapore (2012).
7. R. Balescu (2007): V-Langevin equations, continuous time random walks and fractional diffusion, Chaos, Solitons and Fractals 34(2007), 62-80.
8. J.H. Barret, (1954): Differential equations of non-integer order, Canad. J. Math., 6(1954), 529-541.
9. C.M. Bender and S.A. Orszag (1987): Advanced Mathematical Methods for Scientists and Engineers, McGraw-Hill, Singapore 1987, Ch 3.
10. L. Blank (1996): Numerical treatment of differential equations of fractional order, MCCM Numerical Analysis Report No. 287, The University of Manchester 1996.
11. S. Bochner (1949): Diffusion equation and stochastic processes, Proc. Nat. Acad. Sciences, USA, 35(1949), 368-370.
12. P. Butzer and U. Westphal (2000): Introduction to fractional calculus, in: H. Hilfer (Editor), Fractional Calculus, Applications in Physics, World Scientific, Singapore 2000, pp. 1-85.
13. R. Gorenflo (1997): Fractional calculus: some numerical methods, in: A. Carpinteri and F. Mainardi (Editors), Fractals and Fractional Calculus in Continuum Mechanics, Springer Verlag, Wien, 1997, pp. 277-290.

14. A.A. Kilbas, H.M. Srivastava and J.J. Trujillo (2006): Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam 2006 [North-Holland Mathematics Studies No 204].
15. A.V. Letnikov (1868): On historical development of differentiation theory with an arbitrary index, Mat. Sb., 3(1868), 85-112 [In Russian].
16. K.S. Miller and B. Ross (1993): An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York (1993).
17. I. Podlubny (1999): Fractional Differential Equations, Academic Press, San Diego (1999).
18. B. Ross (Editor)(1975): Fractional Calculus and its Applications, Springer Verlag, Berlin (1975), [Lecture Notes in Mathematics # 457].
19. B. Rubin (1996): Fractional Integrals and Potentials, Addison Wesley and Longman, Harlow 1996, [Pitman Monographs and Surveys in Pure and Applied Mathematics # 82].

MUNDARIJA