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Recovering time dependent function for the fractional diffusion equation in a finite domain

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Abstract. The present study investigates existence and uniqueness of solution to an inverse problem for a one-dimensional time-fractional diffusion equation. This existence and uniqueness result is based on the Fourier method and Laplace transform, the fractional calculus and the Banach fixed point principle. The unknown time dependent coefficient is determined uniquely by the additional data which is an integral condition. Then, the continuous dependence of data is proved.

Keywords: Inverse problem, time-fractional diffusion equation, fractional derivative, Fourier series, Laplace transform, Banach fixed point theorem.

MSC (2010): 80A23, 35R11, 35R30

1 Introduction and problem formulation

Fractional calculus is emerging as an unavoidable tool to model many phenomena in Science and Engineering. Although, there are a number of phenomena in the physical sciences that we associate with the idea of diffusion. Thus, populations of different kinds diffuse; particles in a solvent and other substances diffuse. Besides, heat propagates according to a process that is mathematically similar, and this is a major topic in applied science [5]. The mathematical analysis of initial and boundary value problems (linear or nonlinear) of fractional differential equations has been studied extensively by many authors (see, [11] and references therein).

The problem of determination of temperature at interior points of a region when the initial and boundary conditions alon with heat source term are specified are known as direct heat conduction problems. In many physical problems, determination of coefficients or right hand side (the source term, in case of the heat equation) in a differential equation from some available information is required: these problems are known as inverse problems. A number of articles address the solvability problem of the inverse problems (see, [8], [3]). The inverse problem of determining coefficient was already considered in the literature for parabolic equations, see for example [4], [17], [7], [6], [9].

Here, we consider the so-called one-dimensional fractional diffusion equation

$$\left({}^{C}\mathcal{D}_t^{\alpha} u\right)(x,t) - u_{xx}(x,t) + q(t)u(x,t) = f(x,t), \quad (x,t)\in\Omega, \tag{1.1}$$

with initial

$$u(x,0) = \varphi(x), \qquad x\in(0,l), \tag{1.2}$$

and the homogeneous Dirichlet boundary conditions

$$u(0,t) = u(l,t) = 0, \qquad t\in(0,T], \tag{1.3}$$

where $\Omega := (0,l)\times(0,T]$, ${}^C\mathcal{D}_t^\alpha$ stands for Caputo fractional derivative of order $0<\alpha<1$ in the time variable (see formula (2.3) and $f(x,t)$ is the known source term, $\varphi(x)$ is the initial temperature.

For (1.1)-(1.3) the **direct problem** is the determination of $u(x,t)$ in $\bar{\Omega}$ such that $u(\cdot,t)\in C^2((0,l),\mathbb{R})$ and $\left({}^C\mathcal{D}_t^\alpha u\right)(x,\cdot)\in C((0,T],\mathbb{R})$, when the coefficient $q(t)$, the initial temperature $\varphi(x)$ and the source term $f(x,t)$ are given and continuous.

Inverse problem. Find the couple of functions $\{u(x,t),q(t)\}$ satisfying equation (1.1)-(1.3), under the over-determination conditions

$$\int_0^l u(x,t)dx = g(t), \qquad t\in[0,T], \tag{1.4}$$

where $g(t)\in C([0,T],\mathbb{R})$ is the additional data of the thermal energy. The solvability of inverse problems with such type of integral over-determination has been considered in the paper [19], [6], [9].

We carry out the next converting of the inverse problem (1.1)-(1.4). Denote for this purpose the second derivative of $u(x,t)$ with respect to x, by $v(x,t)$, i.e. $v(x,t)=u_{xx}(x,t)$. Differentiating (1.1)-(1.3) twice in x, we get

$$\left({}^C\mathcal{D}_t^\alpha v\right)(x,t) - v_{xx}(x,t) + q(t)v(x,t) = f_{xx}(x,t), \tag{1.5}$$

$$v(x,0)=\varphi''(x), \quad v(0,t)=v(l,t)=0, \qquad (x,t)\in\Omega. \tag{1.6}$$

Also, we can easily get additional condition for v, by using (1.1):

$$\int_0^l v(x,t)dx = \left({}^C\mathcal{D}_t^\alpha g\right)(t) + q(t)g(t) - F(t), \quad t\in[0,T] \tag{1.7}$$

where

$$F(t) := \int_0^l f(x,t)dx.$$

When the matching conditions

$$\varphi(0)=\varphi(l)=0, \quad \varphi''(0)=\varphi''(l)=0, \quad g(0)=\int_0^l \varphi(x)dx, \tag{1.8}$$

are fulfilled, it is easy to drive from (1.5)-(1.7) to the equations (1.1)-(1.4).

The outline of the paper is as follows. In Section 2, some necessary preliminaries are given. In Section 3, the existence and uniqueness of the solution of the direct problem (1.5)-(1.6) is established by using the Fourier method and Laplace transform. In Section 4, the existence and uniqueness of the solution of the inverse problem (1.5)-(1.7) is established by using the Banach fixed point theorem. In Section 5, the continuous dependence of the solution of the inverse problem upon the data of $\{f(x,t),\varphi(x),g(t)\}$ is shown.

2 Preliminaries

In this section, we recall some necessary definitions and properties of fractional calculus which can be found in [13], [10].

For an integrable function $f : (0, \infty) \to \mathbb{R}$, the left sided Riemann-Liouville fractional integral of order $0 < \alpha < 1$ is defined by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad \Re\alpha > 0, \tag{2.1}$$

where $\Gamma(\alpha)$ is the Euler Gamma function and $\Re\alpha$ denotes the real part of the complex numbers α.

The left sided Riemann-Liouville fractional derivative of order $\alpha \in (0,1)$ of the continuous function f is defined by

$$D_{0+}^{\alpha} f(t) = \frac{d}{dt} I_{0+}^{1-\alpha}(t). \tag{2.2}$$

The Caputo fractional derivative of order $0 < \alpha < 1$ of a function $f : (0, \infty) \to \mathbb{R}$ is defined by

$${}^{C}\mathcal{D}_t^{\alpha} f(t) = D_{0+}^{\alpha}\big(f(t) - f(+0)\big), \tag{2.3}$$

The Mittag-Leffler function plays an important role in the theory of fractional differential equation; for any $z \in \mathbb{C}$ the Mittag-Leffler function with one parameter is defined as

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)}, \tag{2.4}$$

where $\Re\alpha > 0$.

The Mittag-Leffler function with two parameters is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}, \tag{2.5}$$

where α, β, $z \in \mathbb{C}$ and $\Re\alpha > 0$, $\Re\beta > 0$.

Let us set $e_{\alpha}(t, \mu) := E_{\alpha}(-\mu t^{\alpha})$ where $E_{\alpha}(t)$ is the Mittag-Leffler function with one parameter α as defined in (2.4) and μ is a positive real number.

The Mittag-Leffler functions $e_{\alpha}(t; \mu)$, $e_{\alpha,\beta}(t, \mu) := t^{\beta-1} E_{\alpha,\beta}(-\mu t^{\alpha})$ for $0 < \alpha \le 1$, $0 < \alpha \le \beta < 1$ respectively, and $\mu > 0$ are completely monotone functions; i.e.,

$$(-1)^n [e_{\alpha}(t; \mu)]^n \ge 0, \quad \text{and} \quad (-1)^n [e_{\alpha,\beta}(t; \mu)]^n \ge 0, \quad n \in \mathbb{N} \cup \{0\}.$$

Furthermore, we have

$$E_{\alpha,\beta}(-\mu t^{\alpha}) \le M, \quad t \in [a,b], \tag{2.6}$$

where $[a, b]$ is a finite interval with $a \ge 0$, and

$$\int_0^t (t-s)^{\alpha-1} E_{\alpha,\beta}(\mu \tau^{\alpha}) d\tau < \infty,$$

on $[a,b]$. Furthermore, for $\mu \in \mathbb{R}^+$, $t \in (0,T]$(see [16])

$$\mu t^{\alpha-1}E_{\alpha,\alpha}(-\mu t^{\alpha}) \leq \frac{1}{t}\frac{\mu t^{\alpha}}{1+\mu t^{\alpha}} < \infty.$$

For any $\alpha > 0$, $\beta > 0$ and $\gamma \in \mathbb{C}$, there is

$$\mathfrak{L}\big[t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha})\big] = \frac{s^{\alpha-\beta}}{s^{\alpha}-\lambda}, \tag{2.7}$$

with $\Re s > |\lambda|^{1/\alpha}$, the Laplace transform of a function $f(t)$ is defined by

$$\mathfrak{L}[f](s) = \int_0^{\infty} e^{-st} f(t)dt. \tag{2.8}$$

The initial value problem of fractional differential equation for $\alpha \in (0,1)$,

$$\begin{cases} {}^{C}\mathcal{D}_t^{\alpha}u(t) = \lambda u(t) + f(t), \quad t > 0, \\ u(0) = u_0, \end{cases} \tag{2.9}$$

where ${}^{C}\mathcal{D}_t^{\alpha}$ stands for a Caputo fractional derivative operator, u_0 is a constant number.

Theorem 2.1. *[12] Consider the problem* (2.9)*, then there is a explicit solution which is given in the integral form*

$$u(t) = u_0E_{\alpha,1}(\lambda t^{\alpha}) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^{\alpha})f(s)ds. \tag{2.10}$$

3 Investigation of direct problem (1.5)-(1.6)

Let $\Phi(x,t) := f_{xx}(x,t) - q(t)v(x,t)$. We shall search for a solution $v(x,t)$ as a Fourier series in $\sin \frac{\pi n}{l}x$:

$$v(x,t) = \sum_{n=1}^{\infty} v_n(t)\sin\lambda_n x, \quad \lambda_n = \frac{\pi n}{l}, \, n = 1,2,\dots . \tag{3.1}$$

In order to find the function $v(x,t)$ it is necessary to find the function $v_n(t)$. Let us represent the function $\Phi(x,t)$ as a series

$$\Phi(x,t) = \sum_{n=1}^{\infty} \Phi_n(t)\sin\lambda_n x,$$

where

$$\Phi_n(t) = \frac{2}{l}\int_0^l \Phi(\xi,t)\sin\lambda_n\xi d\xi. \tag{3.2}$$

Substituting the assumed from of the solution in the original equation (1.6), we have:

$$\sum_{n=1}^{\infty}\left\{\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{v_n'(\tau)d\tau}{(t-\tau)^{\alpha}} + \lambda_n^2 v_n(t) - \Phi_n(t)\right\}\sin\lambda_n x = 0, \tag{3.3}$$

or according to the definition of Caputo's derivative (1.2), may rewrite

$$\sum_{n=1}^{\infty}\left\{\left({}^{C}\mathcal{D}_t^{\alpha}v_n\right)(t)+\lambda_n^2 v_n(t)-\Phi_n(t)\right\}\sin\lambda_n x=0, \tag{3.4}$$

This equation will be satisfied if all the coefficients in the bracket equal zero, i.e.

$$\left({}^{C}\mathcal{D}_t^{\alpha}v_n\right)(t)+\lambda_n^2 v_n(t)=\Phi_n(t). \tag{3.5}$$

Making use of the initial condition for $v(x,t)$

$$v(x,0)=\sum_{n=1}^{\infty}v_n(0)\sin\lambda_n x=\varphi''(x),$$

we derive the initial condition for $v_n(t)$:

$$v_n(0)=-\lambda_n^2\varphi_n,\quad \varphi_n=\frac{2}{l}\int_0^l\varphi(\xi)\sin\lambda_n\xi d\xi. \tag{3.6}$$

Equation (3.5) may be solved in the following way. Changing t to τ and τ to s respectively in (3.5), multiplying both sides of the equation by $\frac{1}{\Gamma(\alpha)}(t-\tau)^{\alpha-1}$ and integrating we have

$$\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^t\frac{d\tau}{(t-\tau)^{1-\alpha}}\int_0^{\tau}\frac{v_n'(s)}{(\tau-s)^{\alpha}}ds+\frac{\lambda_n^2}{\Gamma(\alpha)}\int_0^t\frac{v_n(\tau)}{(t-\tau)^{1-\alpha}}d\tau=$$

$$=\frac{1}{\Gamma(\alpha)}\int_0^t\frac{\Phi_n(\tau)}{(t-\tau)^{1-\alpha}}d\tau.$$

Interchanging the order of integration in the left-hand side by Dirichlet formula, we arrive at

$$\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^t v_n'(s)ds\int_s^t\frac{d\tau}{(t-\tau)^{1-\alpha}(\tau-s)^{\alpha}}+\frac{\lambda_n^2}{\Gamma(\alpha)}\int_0^t\frac{v_n(\tau)}{(t-\tau)^{1-\alpha}}d\tau=$$

$$=\frac{1}{\Gamma(\alpha)}\int_0^t\frac{\Phi_n(\tau)}{(t-\tau)^{1-\alpha}}d\tau.$$

The inner integral is easily evaluated after the change of variable $\tau=s+(t-s)y$ and application of the Eualer's beta functions:

$$\int_s^t\frac{d\tau}{(t-\tau)^{1-\alpha}(\tau-s)^{\alpha}}=\int_0^1 y^{-\alpha}(1-y)^{\alpha-1}dy=$$

$$=B(1-\alpha,\alpha)=\Gamma(1-\alpha)\Gamma(\alpha).$$

Therefore

$$\int_0^t v_n'(s)ds+\frac{\lambda_n^2}{\Gamma(\alpha)}\int_0^t\frac{v_n(\tau)}{(t-\tau)^{1-\alpha}}d\tau=\frac{1}{\Gamma(\alpha)}\int_0^t\frac{\Phi_n(\tau)}{(t-\tau)^{1-\alpha}}d\tau.$$

Hence after integration we have:

$$v_n(t) + \lambda_n^2 \varphi_n + \frac{\lambda_n^2}{\Gamma(\alpha)} \int_0^t \frac{v_n(\tau)}{(t-\tau)^\alpha} d\tau = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\Phi_n(\tau)}{(t-\tau)^{1-\alpha}} d\tau, \tag{3.7}$$

According to Definitions (2.7) and (2.8), taking Laplace transform with respect to t in both sides of Eq. (3.7), we obtain

$$\mathfrak{L}[v_n](s) = (s^\alpha + \lambda_n^2)^{-1} \mathfrak{L}[\Phi_n](s) - \lambda_n^2 \varphi_n s^{\alpha-1} (s^\alpha + \lambda_n^2)^{-1}, \tag{3.8}$$

The inverse Laplace transform using formula (2.7) yields

$$\mathfrak{L}^{-1}[s^{\alpha-1}(s^\alpha + \lambda_n^2)^{-1}] = E_{\alpha,1}(-\lambda_n^2 t^\alpha), \tag{3.9}$$

and

$$\mathfrak{L}^{-1}\left[(s^\alpha + \lambda_n^2)^{-1} \mathfrak{L}[\Phi_n](s)\right] = \mathfrak{L}^{-1}[(s^\alpha + \lambda_n^2)^{-1}] * \Phi_n(t) =$$
$$= t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n^2 t^\alpha) * \Phi_n(t) = \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n^2 (t-\tau)^\alpha) \Phi_n(\tau) d\tau. \tag{3.10}$$

Substituting the expressions (3.9), (3.10) and taking into inverse Laplace transform to (3.8), we obtain the final explicit form of the functions

$$v_n(t) = -\lambda_n^2 \varphi_n E_{\alpha,1}(-\lambda_n^2 t^\alpha) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n^2 (t-\tau)^\alpha) \Phi_n(\tau) d\tau. \tag{3.11}$$

Based on the solution (3.1) and Fourier coefficient (3.2), also relation (3.11), one can represent the solution of the considered problem as the sum of the Fourier series

$$v(x,t) = \sum_{n=1}^\infty v_n(t) \sin \lambda_n x, \tag{3.12}$$

where the functions $v_n(t)$ are defined by relation (3.11).

Lemma 3.1. *The estimates*

$$|v_n(t)| \le c_1 \Big(n^2 |\varphi_n| + n^2 |f_n|\Big), \tag{3.13}$$

$$\Big|\big({}^C\mathcal{D}_t^\alpha v_n\big)(t)\Big| \le c_2 \Big(n^4 |\varphi_n| + n^2 |f_n|\Big) \tag{3.14}$$

hold for any $t \in [0,T]$, *where* $f_n(t) = \frac{2}{l} \int_0^l f(\xi,t) \sin \lambda_n \xi d\xi$ *and* c_i, $i = 1,2$ *are positive constants here and throughout the following.*

The validity of the estimates (3.13) and (3.14) follows directly from the representation (3.11), applying the Gronuoll-Bellman inequality [1] and relation (2.3).

Formal termwise differentiation of the series (3.1) yields the series

$$\big({}^C\mathcal{D}_t^\alpha v\big)(x,t) = \sum_{n=1}^\infty \big({}^C\mathcal{D}_t^\alpha v_n\big)(t) \sin \lambda_n x, \tag{3.15}$$

$$v_{xx}(x,t) = -\sum_{n=1}^{\infty} \lambda_n^2 v_n(t) \sin \lambda_n x. \tag{3.16}$$

By lemma 3.1, the series (3.1), (3.15), and (3.16) are dominated by the series

$$c_3 \sum_{n=1}^{\infty} \left(n^4 |\varphi_n| + n^4 |f_n| \right) \tag{3.17}$$

for any $(x,t) \in \bar{\Omega}$.

Lemma 3.2. *If the conditions*

$$\varphi(x) \in C^5[0,l], \quad \varphi^{(k)}(0) = \varphi^{(k)}(l) = 0,$$

$$f(x,t) \in C^{5,0}_{x,t}(\bar{\Omega}), \quad \partial_x^k f(0,t) = \partial_x^k f(l,t) = 0, \quad k = 0,2,4,$$

are satisfied, then the representations

$$\varphi_n = \frac{1}{\lambda_n^5} \varphi_n^{(5)}, \quad f_n(t) = \frac{1}{\lambda_n^5} f_n^{(5)}(t) \tag{3.18}$$

are valid, where

$$\varphi_n^{(5)} := \frac{2}{l} \int_0^l \varphi^{(5)}(x) \sin \lambda_n x dx, \quad f_n^{(5)}(t) := \frac{2}{l} \int_0^l \partial_x^5 f(x,t) \sin \lambda_n x dx,$$

with the following estimates holding ture

$$\sum_{n=1}^{\infty} |\varphi_n^{(5)}|^2 \le c_4 \|\varphi^{(5)}(x)\|^2_{L_2[0,l]}, \quad \sum_{n=1}^{\infty} |f_n^{(5)}|^2 \le c_5 \|\partial_x^5 f(x,t)\|^2_{L_2[0,l]\times C[0,T]}. \tag{3.19}$$

Proof. Suppose $\varphi(x) \in C^5[0,l]$ be such that $\varphi(0) = \varphi(l) = \varphi''(0) = \varphi''(l) = \varphi^{(4)}(0) = \varphi^{(4)}(l) = 0$. As φ_n is the coefficient of the sine Fourier expansion of the function $\varphi(x)$ with respect to basis

$$\sin \frac{\pi n}{l} x, \quad n = 1, 2, ..., \tag{3.20}$$

from (3.6) expression for φ_n, which integrated by parts five times gives

$$\varphi_n = \frac{2}{\lambda_n^5 l} \int_0^l \varphi^{(5)}(x) \sin \lambda_n x dx = \frac{1}{\lambda_n^5} \varphi_n^{(5)},$$

where $\varphi_n^{(5)}$ is the coefficient of the sine Fourier expansion of the function $\varphi^{(5)}(x)$ with respect to the basis (3.20).

Alike, we obtain second part of (3.18) for $f_n(t)$. Inequalities (3.19) are the Bessel inequalities for the coefficients of the Fourier expansions of the functions $\varphi^{(5)}(x)$ and $\partial_x^5 f(x,t)$ in the sine system $\frac{2}{l} \sin \lambda_n x$ on the interval $[0,l]$.

If the functions $\varphi(x)$ and $f(x,t)$ satisfy the assumptions of lemma 3.2, then, by virtue of the representations (3.18) and (3.19), the series (3.17) can be estimated by the convergent numerical series

$$c_6\sum_{n=1}^{\infty}\frac{1}{n}\Big(|\varphi_n^{(5)}|+\|f_n^{(5)}\|_{C[0,T]}\Big)=:\mu_0<\infty. \tag{3.21}$$

Then the series (3.12), (3.15) and (3.16) converge uniformly on $\bar{\Omega}$. Consequently, the sum of the series (3.12) satisfies relations (1.6) and (1.7).

Note that the obtained estimate (3.21) yields the following estimate of the solution to (1.6) and (1.7)

$$\|v(x,t)\|_{L_2[0,l]\times C[0,T]}\le c_0\mu_0 e^{2q_0MT},$$

where $c_0:=\max\{M,T^{\alpha}\}$, $q_0:=\|q\|_{C[0,T]}$.

So, the existence of a unique solution to (3.12) has been proved.

Thus, the theorem is proved.

Theorem 3.3. *If the functions $\varphi(x)$ and $f(x,t)$ satisfy the assumptions of Lemma 3.2, then there exists the unique solution to problem* (1.5) *and* (1.6).

4 Existence and uniqueness of the local solution to the inverse problem (1.5)-(1.7)

For the proof of the main result, i.e., Theorem 3.3 we will use properties of the direct problem and application of the Banach fixed point theorem. This method is widely used by many authors, for example [14], [18], [2], [5].

Let us consider the inverse problem (1.6)-(1.8). First, let rewrite the series (3.12) by integral equation

$$v(x,t)=v_0(x,t)-\int_0^t\int_0^l G(x,\xi,t-s)q(s)v(\xi,s)d\xi ds, \tag{4.1}$$

where

$$v_0(x,t)=-\sum_{n=1}^{\infty}\lambda_n^2\varphi_n e_{\alpha}(t,\lambda_n^2)\sin(\lambda_n x)-\int_0^t K(t,\tau)d\tau\sin\lambda_n x, \tag{4.2}$$

and

$$K(t,\tau)=\sum_{n=1}^{\infty}\lambda_n^2 f_n(t-\tau)e_{\alpha,\alpha}(\tau,\lambda_n^2), \tag{4.3}$$

$$G(x,\xi,t)=\frac{2}{l}\sum_{n=1}^{\infty}e_{\alpha,\alpha}(t,\lambda_n^2)\sin(\lambda_n\xi)\sin(\lambda_n x). \tag{4.4}$$

Let us integrate Eq. (4.1) over the closed interval $[0,l]$ and taking into account (1.7), we get

$$q(t)=q_0(t)-\frac{1}{g(t)}\int_0^t\int_0^l\int_0^l G(x,\xi,t-s)q(s)v(\xi,s)d\xi dxds, \tag{4.5}$$

where

$$q_0(t) := \frac{1}{g(t)}\Big[F(t) - \left({}^C\mathcal{D}_t^\alpha g\right)(t) + \int_0^l v_0(\xi,t)d\xi\Big]. \tag{4.6}$$

We have the following theorem.

Theorem 4.1. *Suppose the following conditions hold:*
(i) $\varphi \in C^5[0,l]$, $\varphi^{(k)}(0) = \varphi^{(k)}(l) = 0$, $k = 0,2,4$;
(ii) $f \in C^{5,0}_{x,t}(\bar{\Omega})$, $\partial_x^k f(0,t) = \partial_x^k f(l,t) = 0$, $k = 0,2,4$;
(iii) $g(t) \in AC[0,T]$, $|g(t)| \geq g_0 > 0$ *and* $g(t)$ *satisfies the matching condition* $g(0) = \int_0^l \varphi(x)dx$.

Then the inverse problem (1.1)-(1.4) *has a unique solution.*

Proof. We rewrite the system formed, respectively, by Eq. (4.1) and (4.5), in the form of the operator equation

$$\psi = A\psi, \tag{4.7}$$

where $\psi = (\psi_1, \psi_2) := (v(x,t), q(t))$ and

$$(A\psi)_1(x,t) = v_0(x,t) - \int_0^t \int_0^l G(x,\xi,t-s)\psi_2(s)\psi_1(\xi,s)d\xi ds,$$

$$(A\psi)_2(t) = q_0(t) - \frac{1}{g(t)} \int_0^t \int_0^l \int_0^l G(x,\xi,t-s)\psi_2(s)\psi_1(\xi,s)d\xi dx ds. \tag{4.8}$$

Let $\psi_0 = (\psi_{10}, \psi_{20})$ be a vector with components

$$\psi_{10} = v_0(x,t), \quad \psi_{20} = q_0(t).$$

Denote by $\mathbf{C}(\bar{\Omega})$ the space of continuous vector functions, with the norm

$$\|\psi\|_{\mathbf{C}(\bar{\Omega})} = \max_{k=1,2} \|\psi_k\|_{C(\bar{\Omega})}.$$

If $g(t) \neq 0$, $\forall t \in [0,T]$ and $g(t), F(t) \in C[0,T]$, then all vector functions defined by (4.8) are evidently elements of $\mathbf{C}(\bar{\Omega})$. We introduce in this Banach space the closed ball

$$\mathbf{B}_T := \{\psi \in \mathbf{C}(\bar{\Omega}) : \ \|\psi - \psi_0\|_{\mathbf{C}(\bar{\Omega})} \leq \|\psi_0\|_{\mathbf{C}(\bar{\Omega})}\}, \tag{4.9}$$

of radius $\|\psi_0\|_{\mathbf{C}(\bar{\Omega})} > 0$ centered at $\psi_0 \in \mathbf{C}(\bar{\Omega})$. Evidently,

$$\|\psi_0\|_{\mathbf{C}(\bar{\Omega})} \leq c_7 := \max\left(\|v_0\|_{C(\bar{\Omega})}, \|q_0\|_{C[0,T]}\right), \tag{4.10}$$

where $q_0(t)$ and $v_0(x,t)$ are defined by (4.2) and (4.5), respectively.

Hereafter, we assume that $q_0(t)$ and $v_0(x,t)$ are given fixed functions. Then their norms $\|v_0\|_{C(\bar{\Omega})}, \|q_0\|_{C[0,T]}$ depend on T and α. Taking it into account we have used in (4.10) the notation c_7 for the maximum of these two norms. The similar notations for some values we shall use and later on in order indicate on a dependence of these values on T and α.

Now we are going to prove that the operator A, defined by (4.7) and (4.8) is a contraction on the Banach space $\mathbf{B}_T$, if the final time $T > 0$ is small enough. Recall that an operator is named contracting one on $\mathbf{B}_T$, if the following two conditions hold:

(c1) $A\psi \in \mathbf{B}_T$, for all $\psi \in \mathbf{B}_T$;

(c2) for all ψ^1, $\psi^2 \in \mathbf{B}_T$, the condition

$$\|A\psi^1 - A\psi^2\|_{\mathbf{C}(\bar{\Omega})} \leq \rho \|\psi^1 - \psi^2\|_{\mathbf{C}(\bar{\Omega})}$$

holds with some $\rho \in (0, 1)$.

We verify the first condition **(c1)**. Let $\psi \in \mathbf{B_T}$. Then

$$\|\psi\|_{\mathbf{C}(\bar{\Omega})} \leq 2c_7,$$

by (4.10). Using this in (4.8) we estimate the norms $|(A\psi)_k - \psi_{k0}|$, $k = 1, 2$ as follows:

$$|(A\psi)_1 - \psi_{10}| \leq \int_0^t \Big| \int_0^l G(x,\xi,t-s)\psi_2(s)\psi_1(\xi,s)d\xi \Big| ds \leq$$

$$\leq 2c_1 M T^\alpha \sum_{n=1}^{\infty} n^2 \big[|\varphi_n| + \|f_n(t)\|_{C[0,T]}\big] \|\psi_0\| =: c_7^{(1)} \|\psi_0\|,$$

$$|(A\psi)_2 - \psi_{20}| \leq \int_0^t \Big| \frac{1}{g(t)} \int_0^l \int_0^l G(x,\xi,t-s)\psi_2(s)\psi_1(\xi,s)d\xi dx \Big| ds \leq$$

$$\leq \frac{2c_1}{g_0} M T^\alpha \sum_{n=1}^{\infty} \frac{n^2}{2n+1} \big[|\varphi_n| + \|f_n(t)\|_{C[0,T]}\big] \|\psi_0\| =: c_7^{(2)} \|\psi_0\|.$$

Therefore $A\psi \in \mathbf{B}_T$, if the following condition holds:

$$\max_{k=1,2} c_7^{(k)}(T) \leq 1. \tag{4.11}$$

We verify the second condition **(c2)**. Let $\psi^k := (\psi_1^k, \psi_2^k)$ and $\psi^k \in \mathbf{B}_T$, $k = 1, 2$. Then one has

$$|(A\psi^1 - A\psi^2)_1| \leq \int_0^t \Big| \int_0^l G(x,\xi,t-s)\big(\psi_2^1(s)\psi_1^1(\xi,s) - \psi_2^2(s)\psi_1^2(\xi,s)\big)d\xi \Big| ds \leq$$

$$\leq \int_0^t \bigg| \int_0^l G(x,\xi,t-s) \Big[\psi_2^1(s)\big(\psi_1^1(\xi,s) - \psi_1^2(\xi,s)\big) +$$

$$+ \psi_1^2(\xi,s)\big(\psi_2^1(s) - \psi_2^2(s)\big)\Big] d\xi \bigg| ds \leq 2c_1 M T^\alpha \sum_{n=1}^{\infty} n^2 \big(|\varphi_n| + \|f_n\|_{C[0,T]}\big) \times$$

$$\times \Big(1 + \frac{T}{g_0(\alpha+1)(2n+1)}\Big) \|\psi^1 - \psi^2\| =: c_7^{(3)} \|\psi^1 - \psi^2\|.$$

Similarly,

$$|(A\psi^1 - A\psi^2)_2| \leq \int_0^t \Big| \frac{1}{g(t)} \int_0^l \int_0^l G(x,\xi,t-s)\big(\psi_2^1(s)\psi_1^1(\xi,s) -$$

$$-\psi_2^2(s)\psi_1^2(\xi,s)\big)d\xi dx\Big|ds \leq \frac{2c_1}{g_0}MT^\alpha\sum_{n=1}^{\infty}\frac{n^2}{2n+1}\big(|\varphi_n|+\|f_n\|_{C[0,T]}\big)\times$$

$$\times\Big(1+\frac{T}{g_0(\alpha+1)(2n+1)}\Big)\|\psi^1-\psi^2\| =: c_7^{(4)}\|\psi^1-\psi^2\|.$$

Hence, $\|A\psi^1 - A\psi^2\| \leq \rho\|\psi^1-\psi^2\|$ with $\rho < 1$, if T satisfies the conditions

$$\max_{k=3,4} c_7^{(k)} \leq \rho < 1. \tag{4.12}$$

Thus, if the final time $T > 0$ is chosen so (small) that both conditions (4.11) and (4.12) hold, then the operator A is contracting on $\mathbf{B}_T$. Then, according to Banach contraction mapping principle, there exists a unique solution of the operator Eq. (4.7) in $\mathbf{B}_T$. This completes the proof of the theorem.

5 Continuous dependence on the data

Let Ψ be the set of triples $\{f,\varphi,g\}$ where the functions f,φ,g satisfy the assumptions of Theorem 4.1 and

$$\|f\|_{C^5([0,l])\times C([0,T])} \leq M_1, \quad \|\varphi\|_{C^5([0,l])} \leq M_2, \quad \|g\|_{AC([0,T])} \leq M_3.$$

For $\psi \in \Psi$, we define the norm

$$\|\Psi\| = \|f\|_{C^5([0,l])\times C([0,T])} + \|\varphi\|_{C([0,l])} + \|g\|_{AC([0,T])}.$$

Before presenting the result about the stability of the solution of the inverse problem let us mention that the series

$$\sum_{n=1}^{\infty}\frac{l^5}{\pi^5 n^5}|f_n^{(5)}| \leq M_4,$$

is uniformly convergent, where $f_n^{(5)}$ are the coefficients of the sine Fourier expansion of the function $f^{(5)}(\cdot,t)$. The functions $\{f_n^{(5)}\}_{n=1}^{\infty}$ are bounded by virtue of the Bessel's inequality.

Setting T such that

$$T < \min\left\{\frac{1}{2c_7\sqrt[\alpha]{1+\alpha}}, \frac{g_0}{l^2 c_7\sqrt[\alpha]{1+\alpha}},\right\} \tag{5.1}$$

where c_7 is from (4.10). Then we have the following theorem.

Theorem 5.1. *The solution $(u(x,t),q(t))$ of the inverse problem* (1.1)-(1.4)*, under the assumptions of Theorem 4.1, depends continuously upon the data of $\Psi = \{f(x,t),\varphi(x),g(t)\}$ for satisfying* (5.1).

Proof. Let $(u(x,t),q(t))$, $(\bar{u}(x,t),\bar{q}(t))$ be the solutions of the inverse problem (1.1)-(1.4), corresponding to the data Ψ and $\bar{\Psi}$ respectively. From (4.3) we have

$$\|K\|_{C([0,T])\times C([0,T])}\leq M\sum_{n=1}^{\infty}\frac{l^5}{\pi^5 n^5}|f_n^{(5)}|$$

or

$$\|K\|_{C([0,T])\times C([0,T])}\leq MM_4.$$

First, we estimate each term of $u(x,t)-\bar{u}(x,t)$ in $C([0,l]\times[0,T])$. We will use this from $|qu-\bar{q}\bar{u}|\leq|q||u-\bar{u}|+|\bar{u}||q-\bar{q}|$.

From, (4.2) we have

$$v_0(x,t)-\bar{v}_0(x,t)=$$

$$=-\sum_{n=1}^{\infty}\lambda_n^2E_{\alpha,1}(-\lambda_n^2t^\alpha)\sin(\lambda_n x)[\varphi_n-\bar{\varphi}_n]-\int_0^t[K(t,\tau)-\bar{K}(t,\tau)]d\tau\sin(\lambda_n x).$$

Notice that we can consider $\varphi_n-\bar{\varphi}_n$ as the Fourier coefficient of the function $\varphi-\bar{\varphi}$; i.e.,

$$\varphi_n-\bar{\varphi}_n=\frac{2}{l}\int_0^l(\varphi-\bar{\varphi})(x)\sin(\lambda_n x)dx.$$

The following estimates for the Mittag-Leffler type function

$$|\lambda_n^2E_\alpha(-\lambda_n^2t^\alpha)|\leq\frac{\lambda_n^2}{1+\lambda_n^2t^\alpha}\leq M,$$

$$|t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n^2t^\alpha)|\leq\frac{1}{t}\frac{t^\alpha\lambda_n^2}{1+\lambda_n^2t^\alpha}\leq M_5,$$

leads to the estimate

$$\|v_0-\bar{v}_0\|_{C([0,l])\times C([0,T])}\leq M\|\varphi-\bar{\varphi}\|_{C^5([0,l])}+M_5\|f-\bar{f}\|_{C^5([0,l])\times C([0,T])},\tag{5.2}$$

where $M_5:=\frac{1}{\sqrt[\alpha]{1+\alpha}}$.

From (4.1) and (5.2), we obtain

$$\|v-\bar{v}\|_{C([0,l])\times C([0,T])}\leq MM_6\|\varphi-\bar{\varphi}\|_{C^5([0,l])}+$$

$$+M_5M_6\|f-\bar{f}\|_{C^5([0,l])\times C([0,T])}+2c_7M_5M_6\|q-\bar{q}\|_{C([0,T])},\tag{5.3}$$

where $M_6:=\left(1-2c_7M_5T\right)^{-1}$.

Now, we estimate each term of $q(t)-\bar{q}(t)$ in $C([0,T])$. From (4.6), we have

$$q_0(t)-\bar{q}_0(t)=\frac{1}{g(t)}\Big[\int_0^l f(x,t)dx-\left({}^C\mathcal{D}_t^\alpha g\right)(t)+\int_0^l v_0(\xi,t)d\xi\Big]-$$

$$-\frac{1}{\bar{g}(t)}\Big[\int_0^l\bar{f}(x,t)dx-\left({}^C\mathcal{D}_t^\alpha\bar{g}\right)(t)+\int_0^l\bar{v}_0(\xi,t)d\xi\Big]=$$

$$=(g(t)\bar{g}(t))^{-1}\Big[\bar{g}(t)\int_0^l\left(f(x,t)-\bar{f}(x,t)\right)dx-\left(g(t)-\bar{g}(t)\right)\int_0^l\bar{f}(x,t)dx+$$

$$+\bar{g}(t)\Big(\left({}^C\mathcal{D}_t^\alpha g\right)(t)-\left({}^C\mathcal{D}_t^\alpha \bar{g}\right)(t)\Big)-\left({}^C\mathcal{D}_t^\alpha \bar{g}\right)(t)\Big(g(t)-\bar{g}(t)\Big)+$$

$$+\bar{g}(t)\int_0^l\big(v_0(x,t)-\bar{v}_0(x,t)\big)dx-\big(g(t)-\bar{g}(t)\big)\int_0^l\bar{v}_0(x,t)dx\Big].$$

We use (2.3) and (5.2), the estimate of $q_0(t)-\bar{q}_0(t)$ in $C([0,T])$

$$\|q_0-\bar{q}_0\|_{C[0,T]}\leq M_8\|f-\bar{f}\|_{C^5([0,l])\times C([0,T])}+$$

$$+M_9\|\varphi-\bar{\varphi}\|_{C([0,l])}+M_{10}\|g-\bar{g}\|_{AC[0,T]}, \tag{5.4}$$

where $M_8 := \frac{g_0 l+M_5}{g_0}$, $M_9 := \frac{lM}{g_0}$, $M_{10} := \frac{g_0+M_1+M_7+lc_7}{g_0^2}$, M_7 is a bound of $\left({}^C\mathcal{D}_t^\alpha g\right)(t)=D_{0+}^\alpha\big(g(t)-g(+0)\big)$. In addition, we have

$$\Bigg|-\frac{1}{g(t)}\int_0^t\int_0^l\int_0^l G(x,\xi,t-s)q(s)v(\xi,s)d\xi dxds+$$

$$+\frac{1}{\bar{g}(t)}\int_0^t\int_0^l\int_0^l G(x,\xi,t-s)\bar{q}(s)\bar{v}(\xi,s)d\xi dxds\Bigg|\leq$$

$$\leq M_{11}\|g-\bar{g}\|_{AC[0,T]}+M_{12}\|q-\bar{q}\|_{C([0,T])}+M_{12}\|v-\bar{v}\|_{C([0,l])\times C([0,T])}, \tag{5.5}$$

where $M_{11} := \frac{l^2c_7^2}{g_0^2}TM_5$, $M_{12} := \frac{l^2c_7}{g_0}TM_5$. From (5.4) and (5.5), we get estimate of $q-\bar{q}$, i.e.,

$$\|q-\bar{q}\|_{C([0,T])}\leq M_8M_{13}\|f-\bar{f}\|_{C^5([0,l])\times C([0,T])}+M_9M_{13}\|\varphi-\bar{\varphi}\|_{C([0,l])}+$$

$$+(M_{10}+M_{11})M_{13}\|g-\bar{g}\|_{AC[0,T]}+M_{12}M_{13}\|v-\bar{v}\|_{C(\bar{\Omega})}, \tag{5.6}$$

where $M_{13} := (1-M_{12})^{-1}$. From (5.3) and (5.6), we can obtain stability estimate for the inverse problem (1.1)-(1.4) for some positive constant C:

$$\|q-\bar{q}\|_{C([0,l])}\leq C\|\Psi-\bar{\Psi}\|.$$

The theorem 5.1 is proved.

Conclusion. The purpose of this paper is to determine the pair of functions $\{u(x,t),q(t)\}$ for the fractional diffusion equation (1.1)-(1.4). The inverse problem regarding the simultaneous identification of the time-dependent coefficient in a one-dimensional equation with nonlocal boundary and integral overdetermination conditions has been considered.

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