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INVERSE PROBLEM FOR THE SYSTEM OF VISCOELASTICITY IN ANISOTROPIC MEDIA WITH TETRAGONAL FORM OF ELASTICITY MODULUS

For the reduced canonical system of integro-differential equations of viscoelasticity, direct and inverse problems of determining the velocity field of elastic waves and the relaxation matrix are considered. The problems are replaced by a closed system of Volterra integral equations of the second kind with respect to the Fourier transform in the variables x_1 and x_2 for the solution of the direct problem and unknowns of the inverse problem. Further, the method of contraction mappings in the space of continuous functions with a weighted norm is applied to this system. Thus, we prove global existence and uniqueness theorems for solutions of the problems.

Keywords: viscoelasticity, resolvent, inverse problem, hyperbolic system, Fourier transform.

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Introduction

During the last few decades, the inverse problems of determining the kernel in a hyperbolic system of integro-differential equations with an integral term of a convolution type have been playing important roles in various fields, such as geophysics, oil prospecting, design of optical devices, as well as in other areas, where the interior of an object is imaged using the response to acoustic waves. There has been an increased interest in hyperbolic systems of integro-differential equations containing integrals of a convolution type. Such equations describe processes with memory or, as they are also called, eriditary processes [1]. Such processes are characterized by the fact that the change in their state at each moment of time depends on the history of the process. Examples of such processes are deformation of a viscoelastic medium [2] and propagation of electromagnetic waves in media with dispersion [3].

Inverse problems for partial differential equations and integro-differential equations have been studied by many authors, we note the works [4–14], which are closest to the topic of this article. In works [4–7], one-dimensional problems of finding the convolution kernel were studied: from the general integro-differential wave equation in [5] to the viscoelasticity equations in others papers. The main results of these works are theorems on the unique solvability of the problems. In [8–10], a new method is proposed, where the solution of the direct problem is a complex-valued function, and to find the coefficients of differential equations only the modulus of solving the direct problem on some special sets is given as information, the solution phase is considered unknown. In [11, 12], the problems of finding memory and coefficients of hyperbolic equations were studied under the assumption that the desired coefficients depend weakly on one of the variables. In the articles [13, 14], the existence and uniqueness theorems for the classical solution of the inverse coefficient and kernel determination problems have been proven.

There are many works devoted to inverse problems for quasilinear or linear systems (see [15–18]). In [15, 16], a theorem on the existence and uniqueness of a solution of such a problem with the use of its reduction to a system of nonlinear integro-functional equations is proved. There are few results for the four spatially varying coefficients with measurements of only one component, given two sets of initial conditions [17]. The inverse problem of the simultaneous identification of two discontinuous diffusion coefficients for a one-dimensional coupled parabolic system with the observation of only one component was investigated in [18].

As a rule, second-order equations are derived from systems of first-order partial differential equations under some additional assumptions. An inverse problem of determining coefficients and kernels of the integral terms from a hyperbolic system of general first-order integro-differential equations with two independent variables was studied in [19–22] and for the system of viscoelasticity integro-differential equations in [23]. In the article [19], the direct and inverse initial boundary-value problems for a first-order system of two hyperbolic equations are considered. In [20], global Lipschitz stability of the inverse source and coefficient problems is proved for a first-order linear hyperbolic system, the coefficients of which depend on both space and time. In the works [21–23], the theorems of local existence and global uniqueness of the solution for systems of first-order partial differential equations with initial-boundary conditions are obtained.

In this paper, we consider a three-dimensional system of first-order viscoelasticity equations written with respect to the displacement and stress tensor. We pose the inverse problem of finding the diagonal memory matrix for a reduced canonical system of integro-differential viscoelsticity equations. The problems are replaced by a closed system of Volterra-type integral equations of the second kind with respect to the Fourier transform in the variables x_1 and x_2 of the solution of the direct problem and unknowns of the inverse problem. To this system, we then apply a reduction method, a mapping in the space of continuous functions with a weighted norm. Thus, we prove global existence and uniqueness theorems to solve the given problems.

The paper is organized as follows. In Section 11, we give formulation of problem for the reduced canonical system of integro-differential equations of viscoelasticity, the initial-boundary direct problem and inverse problems of determining the velocity field of elastic waves and the relaxation matrix, respectively. The derivation of the canonical system of integro-differential equations is given in the Appendix. In this section, we also investigate the solvability of the direct problem. In Section 2, by integrating along the characteristics of the system, integral equations of Volterra type are obtained that are equivalent to direct and inverse problems. Section 3, using the Banach principle, proves the main result of the article, which is the global solvability of the posed problems. Section 4 (Appendix) contains the derivation of the main system of equations (1.1) from the general system of viscoelasticity in the anisotropic media with tetragonal form of elasticity modulus. In the last section, a list of references is given.

§ 1. Problems set up and investigation of the direct problem

Consider the system of integro-differential hyperbolic equations of viscoelasticity

$$\left(I\frac{\partial}{\partial t}+\Lambda\frac{\partial}{\partial z}+B\frac{\partial}{\partial x_1}+C\frac{\partial}{\partial x_2}+F\right)\vartheta(x,t)=\int_0^t R(z,t-\tau)\vartheta(x,\tau)\,d\tau, \tag{1.1}$$

where $(x,t)=(x_1,x_2,z,t)\in D=\{(x,t)\colon (x_1,x_2)\in\mathbb{R}^2,\ z\in(0,H),\ t>0\}$, $H=\text{const}$. The derivation of equation (1.1) from the dynamic differential equations of viscoelasticity for an anisotropic medium with a matrix of the elastic modulus of a tetragonal form and definitions of $\Lambda=\operatorname{diag}(\lambda_i)$, $B=(b_{ij})$, $C=(\widetilde{c}_{ij})$, $F=(p_{ij})$, $R=(\widetilde{r}_{ij})$, $i,j=\overline{1,9}$, in (1.1) are given in the **Appendix** (Section 4).

The purpose of this article is to study the direct and inverse problems for the system (1.1). Moreover, **the direct problem** is an initial-boundary value problem for this system in domain D and in the inverse problem, the elements of the matrix R are assumed to be unknown, which are included in the definition of the matrix R (4.7).

In the direct problem, given matrices B, C, F, and R, it is required, in the domain D to find a vector-function $\vartheta(x,t)$ satisfying equation (1.1) for the following initial and boundary conditions:

$$\vartheta_i\big|_{t=0} = \varphi_i(x),\ (x_1,x_2)\in\mathbb{R}^2,\ z=[0,H],\ i=\overline{1,9}, \tag{1.2}$$

$$\vartheta_i\big|_{z=H} = g_i(x_1,x_2,t),\ i=\overline{1,3},\quad \vartheta_i\big|_{z=0} = g_i(x_1,x_2,t),\ i=\overline{7,9}, \tag{1.3}$$

here $\varphi_i(x)$, $i=\overline{1,9}$, $g_i(x_1,x_2,t)$, $i=1,2,3,7,8,9$, are given functions. It is known that the problem (1.1)–(1.3) is well-posed (see [24,25]).

Remark 1.1. As follows from $U=\Upsilon\vartheta$, the vector function $(u,\sigma)^*$ is expressed in terms of ϑ according to the formula

$$\begin{pmatrix} u_1\\ u_2\\ u_3\\ \sigma_{11}\\ \sigma_{12}\\ \sigma_{13}\\ \sigma_{22}\\ \sigma_{23}\\ \sigma_{33}\end{pmatrix}(x,t) = \begin{pmatrix} \vartheta_2+\vartheta_8\\ \vartheta_3+\vartheta_7\\ \vartheta_1+\vartheta_9\\ c_{13}\sqrt{\frac{\rho}{c_{33}}}\vartheta_9 - c_{13}\sqrt{\frac{\rho}{c_{33}}}\vartheta_1 + c_{16}\sqrt{\frac{\rho}{c_{66}}}\vartheta_7 - c_{16}\sqrt{\frac{\rho}{c_{66}}}\vartheta_3\\ c_{13}\sqrt{\frac{\rho}{c_{33}}}\vartheta_9 - c_{13}\sqrt{\frac{\rho}{c_{33}}}\vartheta_1 + c_{16}\sqrt{\frac{\rho}{c_{66}}}\vartheta_3 - c_{16}\sqrt{\frac{\rho}{c_{66}}}\vartheta_7\\ -\sqrt{c_{33}\rho}\vartheta_1 - \sqrt{c_{33}\rho}\vartheta_9\\ \vartheta_6\\ -\sqrt{c_{44}\rho}\vartheta_2 + \sqrt{c_{44}\rho}\vartheta_8\\ -\sqrt{c_{66}\rho}\vartheta_3 + \sqrt{c_{66}\rho}\vartheta_7 \end{pmatrix}(x,t).$$

where definitions of u_i, σ_{ij}, $i,j=1,2,3$, and Υ in remark are given in Appendix.

From this relation, based on (1.2) and (1.3), the initial-boundary value problem can be formulated for the system of equations (4.1), (4.2) (see Appendix) in terms of the displacement and stress tensor components.

The definition of the elements $\widetilde{r}_{ij}(z,t)$, $i,j=\overline{1,9}$, of the matrix $R(z,t)$ by formula (4.7) includes functions $r_{ij}(t)$, $i,j=\overline{1,9}$, $c_{ij}(z)$ on the modulus of elasticity and $\rho(z)$ on the density of the medium. **The inverse problem** is to determine the nonzero components of the matrix kernel $R(z,t)$, that is $r_{ij}(t)$, $i,j=\overline{1,3}$ (where $c_{ij}(z)$ and $\rho(z)$ are given functions), in (1.1)–(1.3) if the following conditions are known:

$$\vartheta_i\big|_{z=0} = h_i(x_1,x_2,t),\ i=\overline{1,6}, \tag{1.4}$$

where $h_i(x_1,x_2,t)$, $i=\overline{1,6}$, are the given functions.

In the inverse problem, the numbers $r_{ij}(0)$, $i,j=\overline{1,3}$, are also considered to be given.

Let functions $F(x,t)$, $\varphi_i(x)$, $i=\overline{1,9}$, $g_i(x_1,x_2,t)$, $i=1,2,3,7,8,9$, included in the right-hand side of (1.1) and the data (1.2), (1.3) have a compact support in x_1, x_2 for each fixed z, t. The existence for the system (1.1) of a compact support domain of dependence and compact support with respect to x_1, x_2 of the right-hand side (1.1) and data (1.2), (1.3) implies the compact support in x_1, x_2 solutions to the problem (1.1)–(1.3).

Let us study the property of solution to this problem. More precisely, we restrict ourselves to studying the Fourier transform in the variables x_1, x_2 of the solution. Introduce the notation

$$\widehat{\vartheta}(\eta_1,\eta_2,z,t) = \int_{\mathbb{R}^2}\vartheta(x_1,x_2,z,t)e^{i[\eta_1x_1+\eta_2x_2]}\,dx_1\,dx_2,$$

where η_1, η_2 are transformation parameters.

In terms of the function $\widehat{\vartheta}$ we write the problem (1.1)–(1.3) as

$$\left(\frac{\partial}{\partial t}+\lambda_j\frac{\partial}{\partial z}\right)\widehat{\vartheta}_j(z,t)=$$
$$=\sum_{k=1}^{9}\widehat{p}_{jk}\widehat{\vartheta}_k(z,t)+\int_0^t\sum_{k=1}^{9}\widetilde{r}_{jk}(z,\tau)\widehat{\vartheta}_k(\eta_1,\eta_2,z,t-\tau)\,d\tau,\quad j=\overline{1,9},\tag{1.5}$$

where $\widehat{p}_{jk}(z)=\widehat{p}_{jk}=-i\eta_1 b_{jk}-i\eta_2\widetilde{c}_{jk}-p_{jk}$.

We fix η_1, η_2 and for convenience, we introduce the notation $\widehat{\vartheta}(\eta_1,\eta_2,z,t)=\widehat{\vartheta}(z,t)$. We will use similar notations for the Fourier transforms of functions included in the initial, boundary and additional conditions (1.2)–(1.4):

$$\widehat{\vartheta}_i\big|_{t=0}=\widehat{\varphi}_i(z),\ i=\overline{1,9},\tag{1.6}$$
$$\widehat{\vartheta}_i\big|_{z=H}=\widehat{g}_i(t),\ i=\overline{1,3},\quad \widehat{\vartheta}_i\big|_{z=0}=\widehat{g}_i(t),\ i=\overline{7,9},\tag{1.7}$$
$$\widehat{\vartheta}_i\big|_{z=0}=\widehat{h}_i(t),\ i=\overline{1,6},\tag{1.8}$$

where $\widehat{\varphi}_i(z)$, $i=\overline{1,9}$, $\widehat{g}_i(t)$, $i=1,2,3,7,8,9$, are the Fourier transforms of the corresponding functions from (1.2), (1.3) for $\eta=0$. We also denote by D_H the projection of D onto the plane z, t.

For the purpose of further research let us introduce the vector function $\omega(z,t)=\frac{\partial\widehat{\vartheta}}{\partial t}(z,t)$. To obtain a problem for the function $\omega(z,t)$ similar to (1.5)–(1.8), differentiate the equations (1.5) and the boundary conditions (1.7) with respect to the variable t, and the condition for $t=0$ is found using the equations (1.5) and the initial conditions (1.6). In this case, we get

$$\left(\frac{\partial}{\partial t}+\lambda_i\frac{\partial}{\partial z}\right)\omega_i(z,t)=\sum_{k=1}^{9}\widehat{p}_{ik}\omega_k(z,t)+$$
$$+\sum_{k=1}^{9}\widetilde{r}_{ik}(z,t)\widehat{\varphi}_i(z)+\int_0^t\sum_{k=1}^{9}\widetilde{r}_{ik}(z,\tau)\omega_k(z,t-\tau)\,d\tau,\ i=\overline{1,9},\tag{1.9}$$
$$\omega_i\big|_{t=0}=-\lambda_i\frac{d\widehat{\varphi}_i(z)}{dz}+\sum_{j=1}^{9}\widehat{p}_{ji}\widehat{\varphi}_i(z)=:\Phi_i(z),\ i=\overline{1,9},\tag{1.10}$$
$$\omega_i\big|_{z=H}=\frac{d}{dt}\widehat{g}_i(t),\ i=\overline{1,3},\quad \omega_i\big|_{z=0}=\frac{d}{dt}\widehat{g}_i(t),\ i=\overline{7,9}.\tag{1.11}$$

For functions ω_i, additional conditions (1.8) give

$$\omega_i\big|_{z=0}=\frac{d}{dt}\widehat{h}_i(t),\ i=\overline{1,6}.$$

Let us pass from the equalities (1.5)–(1.8) to the integral relations for the components of the vector $\widehat{\vartheta}$ with integration flux along the corresponding characteristics of the equations of the system (1.5). We denote

$$\mu_1(z)=-\mu_9(z)=-\int_0^z\sqrt{\frac{\rho(\beta)}{c_{33}(\beta)}}\,d\beta,\quad \mu_2(z)=-\mu_8(z)=-\int_0^z\sqrt{\frac{\rho(\beta)}{c_{44}(\beta)}}\,d\beta,$$
$$\mu_3(z)=-\mu_7(z)=-\int_0^z\sqrt{\frac{\rho(\beta)}{c_{66}(\beta)}}\,d\beta,\quad \mu_4(z)=\mu_5(z)=\mu_6(z)=0.$$

Inverse functions to $\mu_i(z)$, $i=\overline{1,9}$, will be denoted by $z=\mu_i^{-1}(t)$, $i=\overline{1,9}$. Using the introduced functions, the equations of characteristics passing through the points (z,t) on the plane of variables ξ, τ can be written in the form

$$\tau = t+\mu_i(\xi)-\mu_i(z),\ i=\overline{1,9}. \tag{1.12}$$

Consider an arbitrary point $(z,t)\in D_H$ on the plane of variables ξ, τ and draw through it the characteristic of the i-th equation of the system (1.5), find out intersection in the domain $\tau\le t$. The intersection point is denoted by (z_0^i,t_0^i). Integrating the equations of the system (1.5) along the corresponding characteristics from the point (z_0^i,t_0^i) to the point (z,t), we find

$$\omega_i(z,t)=\omega_i(z_0^i,t_0^i)+\int_{t_0^i}^{t}\sum_{k=1}^{9}\Big(\widehat{p}_{ik}\omega_k(\xi,\tau)+\widetilde{r}_{ik}(\xi,\tau)\widehat{\varphi}_i(\xi)\Big)\Big|_{\xi=\mu_i^{-1}[\tau-t+\mu_i(z)]}d\tau+$$
$$+\int_{t_0^i}^{t}\int_0^{\tau}\sum_{k=1}^{9}\widetilde{r}_{ik}(\xi,\tau-\alpha)\omega_k(\xi,\alpha)\,d\alpha\Big|_{\xi=\mu_i^{-1}[\tau-t+\mu_i(z)]}d\tau,\ i=\overline{1,9}. \tag{1.13}$$

We define t_0^i in (1.13). It depends on the coordinates of the point (z,t). It is not difficult to see that $t_0^i(z,t)$ has the form

$$t_0^i(z,t)=\begin{cases}t-\mu_i(z)+\mu_i(H), & t\ge\mu_i(z)-\mu_i(H),\\ 0, & 0<t<\mu_i(z)-\mu_i(H),\end{cases}\quad i=1,2,3,$$
$$t_0^i(z,t)=0,\ i=4,5,6,\quad t_0^i(z,t)=\begin{cases}t-\mu_i(z), & t\ge\mu_i(z),\\ 0, & 0<t<\mu_i(z),\end{cases}\quad i=7,8,9.$$

Then, from the condition that the pair (z_0^i,t_0^i) satisfies the equation (1.12) it follows

$$z_0^i(z,t)=\begin{cases}H, & t\ge\mu_i(z)-\mu_i(H),\\ \mu_i^{-1}(\mu_i(z)-t), & 0<t<\mu_i(z)-\mu_i(H),\end{cases}\quad i=1,2,3,$$
$$z_0^i(z,t)=z,\ i=4,5,6,\quad z_0^i(z,t)=\begin{cases}0, & t\ge\mu_i(z),\\ \mu_i^{-1}(\mu_i(z)-t), & 0<t<\mu_i(z),\end{cases}\quad i=7,8,9.$$

The free terms of the integral equations (1.13) are defined through the initial and boundary conditions (1.10) and (1.11) as follows:

$$\omega_i(z_0^i,t_0^i)=\begin{cases}\frac{d}{dt}\widehat{g}_i(t-\mu_i(z)+\mu_i(H)), & t\ge\mu_i(z)-\mu_i(H),\\ \Phi_i\big(\mu_i^{-1}(\mu_i(z)-t)\big), & 0<t<\mu_i(z)-\mu_i(H),\end{cases}\quad i=1,2,3,$$
$$\omega_i(z_0^i,t_0^i)=\Phi_i(z),\ i=4,5,6,\quad \omega_i(z_0^i,t_0^i)=\begin{cases}\frac{d}{dt}\widehat{g}_i(t-\mu_i(z)), & t\ge\mu_i(z),\\ \Phi_i\big(\mu_i^{-1}(\mu_i(z)-t)\big), & 0<t<\mu_i(z),\end{cases}\quad i=7,8,9.$$

Let the following conditions hold

$$\widehat{\varphi}_i(H)=\widehat{g}_i(0),\ \text{ and }\ \frac{d\widehat{g}_i(t)}{dt}\Big|_{t=0}=-\lambda_j\,\frac{d\widehat{\varphi}_i(z)}{dz}\Big|_{z=H}+\sum_{j=1}^{9}\widehat{p}_{ij}(H)\widehat{\varphi}_j(H),\ i=\overline{1,3}, \tag{1.14}$$

$$\widehat{\varphi}_i(0)=\widehat{g}_i(0),\ \text{ and }\ \frac{d\widehat{g}_i(t)}{dt}\Big|_{t=0}=-\lambda_j\,\frac{d\widehat{\varphi}_i(z)}{dz}\Big|_{z=0}+\sum_{j=1}^{9}\widehat{p}_{ij}(0)\widehat{\varphi}_j(0),\ i=\overline{7,9}. \tag{1.15}$$

It is easy to see that the conditions for matching the initial and boundary data (1.6), (1.7), (1.10), (1.11) in corner points of the domain D_H coincide with the relations (1.14) and (1.15). If it is clear that for the fulfillment of the matching conditions (1.14) and (1.15), then (1.13) will have unique continuous solutions $\omega_i(z,t)$ (or $\frac{\partial}{\partial t}\widehat{\vartheta}_i(z,t)$).

Suppose that all given functions included in (1.13) are continuous functions of their arguments in D_H. Then this system of equations is a closed system of Volterra integral equations of the second kind with continuous kernels and free terms. As usual, such a system has a unique solution in the bounded subdomain $D_{HT} = \{(z,t)\colon 0 < z < H,\ 0 < t < T\}$, where $T > 0$ is some fixed number, domain D_H.

Thus, the following statement holds.

Theorem 1.1. *Assume functions $F(x,t)$, $\varphi_i(x)$, $i = \overline{1,9}$, $g_i(x_1,x_2,t)$, $i = 1,2,3,7,8,9$, have compact supports in x_1, x_2 for each fixed z, t. Let $\rho(z)$, $c_{11}(z)$, $c_{12}(z)$, $c_{16}(z)$, $c_{33}(z)$, $c_{44}(z)$, $c_{66}(z) \in C[0,H]$, $\widehat{\varphi}(z) \in C^1[0,H]$, $\widehat{g}(t) \in C^1[0,T]$, $\rho(z) > 0$, $c_{33}(z) > 0$, $c_{44}(z) > 0$, $c_{66}(z) > 0$, $r_{ij}(t) \in C^1[0,T]$, $i,j = 1,2,3$, be given and conditions (1.14), (1.15) be satisfied. Then, there is a unique solution to the problem (1.9)–(1.11) in the domain $\overline{D}_{HT}$.*

§ 2. Derivation of equivalent integral equations

Consider an arbitrary point $(z,0) \in D_{HT}$ and draw through it the characteristics (1.12) for $i = 1,2,3$ up to the intersection with the left lateral boundary of the domain D_H. Integrating the first six components of the equation (1.9), we obtain

$$\omega_i(z,0) = \omega_i(0,t_1^i) - \int_0^{t_1^i} \sum_{k=1}^{9} \Big(\widehat{p}_{ik}\omega_k(\xi,\tau) + \widetilde{r}_{ik}(\xi,\tau)\widehat{\varphi}_i(\xi)\Big)\Big|_{\xi=\mu_i^{-1}[\tau+\mu_i(z)]} d\tau$$
$$- \int_0^{t_1^i}\int_0^{\tau} \sum_{k=1}^{9} \widetilde{r}_{ik}(\xi,\tau-\alpha)\omega_k(\xi,\alpha)\,d\alpha\Big|_{\xi=\mu_i^{-1}[\tau+\mu_i(z)]} d\tau,\ i = \overline{1,6}, \tag{2.1}$$

where $t_1^i = -\mu_i(z)$, $i = 1,2,3$, $t_1^i = 0$, $i = 4,5,6$.

We introduce the following notation for the unknowns:

$$\upsilon_i^1(z,t) = \omega_i(z,t),\ i = \overline{1,9}, \quad \upsilon_1^2(t) = r'_{11}(t), \quad \upsilon_2^2(t) = r'_{12}(t), \quad \upsilon_3^2(t) = r'_{13}(t), \tag{2.2}$$

$$\upsilon_4^2(t) = r'_{22}(t), \quad \upsilon_5^2(t) = r'_{23}(t), \quad \upsilon_6^2(t) = r'_{33}(t), \quad \upsilon_i^3(z,t) = \frac{\partial}{\partial z}\omega_i(z,t),\ i = \overline{4,6}, \tag{2.3}$$

$$\upsilon_i^3(z,t) = \frac{\partial}{\partial z}\omega_i(z,t) - \frac{r'_{33}(t_0^i)}{2}\left(\widehat{\varphi}_1(z_0^i) - \widehat{\varphi}_9(z_0^i)\right)\frac{\partial}{\partial z}t_0^i,\ i = 1,9, \tag{2.4}$$

$$\upsilon_i^3(z,t) = \frac{\partial}{\partial z}\omega_i(z,t) - \frac{r'_{23}(t_0^i)}{2}\left(\widehat{\varphi}_2(z_0^i) - \widehat{\varphi}_8(z_0^i)\right)\frac{\partial}{\partial z}t_0^i,\ i = 2,7, \tag{2.5}$$

$$\upsilon_i^3(z,t) = \frac{\partial}{\partial z}\omega_i(z,t) - \frac{r'_{13}(t_0^i)}{2}\left(\widehat{\varphi}_3(z_0^i) - \widehat{\varphi}_7(z_0^i)\right)\frac{\partial}{\partial z}t_0^i,\ i = 3,8. \tag{2.6}$$

Then, taking into account these notations and the explicit forms of the functions $\widetilde{r}_{ij}(z,t)$ in terms of $r'_{ij}(t)$ by the formula (4.7) (see also formulas below R), we rewrite the equations (1.13) in the form

$$\upsilon_i^1(z,t) = \upsilon_i^{01}(z,t) + \int_{t_0^i}^{t}\left[\sum_{j=1}^{9}\widehat{p}_{ij}\upsilon_j^1(\xi,\tau) - \frac{\upsilon_6^2(\tau)}{2}(\widehat{\varphi}_1 - \widehat{\varphi}_9)(\xi)\right]\Bigg|_{\xi=\mu_i^{-1}[\tau-t+\mu_i(z)]} d\tau$$
$$- \int_{t_0^i}^{t}\int_0^{\tau}\frac{\upsilon_6^2(\alpha)}{2}(\upsilon_1^1 - \upsilon_9^1)(\xi,\tau-\alpha)\,d\alpha\Big|_{\xi=\mu_i^{-1}[\tau-t+\mu_i(z)]} d\tau,\ i = 1,9, \tag{2.7}$$

$$\upsilon_i^1(z,t) = \upsilon_i^{01}(z,t) + \int_{t_0^i}^t \left[\sum_{j=1}^9 \widehat{p}_{ij}\upsilon_j^1(\xi,\tau) - \frac{\upsilon_3^2(\tau)}{2}(\widehat{\varphi}_2 - \widehat{\varphi}_8)(\xi) \right] \Bigg|_{\xi=\mu_i^{-1}[\tau-t+\mu_i(z)]} d\tau$$
$$- \int_{t_0^i}^t \int_0^\tau \frac{\upsilon_3^2(\alpha)}{2}(\upsilon_2^1 - \upsilon_8^1)(\xi,\tau-\alpha)\,d\alpha \Bigg|_{\xi=\mu_i^{-1}[\tau-t+\mu_i(z)]} d\tau,\ i = 2,8, \tag{2.8}$$

$$\upsilon_i^1(z,t) = \upsilon_i^{01}(z,t) + \int_{t_0^i}^t \left[\sum_{j=1}^9 \widehat{p}_{ij}\upsilon_j^1(\xi,\tau) - \frac{\upsilon_5^2(\tau)}{2}(\widehat{\varphi}_3 - \widehat{\varphi}_7)(\xi) \right] \Bigg|_{\xi=\mu_i^{-1}[\tau-t+\mu_i(z)]} d\tau$$
$$- \int_{t_0^3}^t \int_0^\tau \frac{\upsilon_5^2(\alpha)}{2}(\upsilon_3^1 - \upsilon_7^1)(\xi,\tau-\alpha)\,d\alpha \Bigg|_{\xi=\mu_i^{-1}[\tau-t+\mu_i(z)]} d\tau,\ i = 3,7, \tag{2.9}$$

$$\upsilon_4^1(z,t) = \int_0^t \sum_{j=1}^9 \widehat{p}_{4j}\upsilon_j^1(z,\tau)\,d\tau + \int_0^t [\upsilon_1^2(\tau)\widehat{\varphi}_4(z) + (\upsilon_1^2 - \upsilon_2^2)(\tau)\widehat{\varphi}_6(z)]\,d\tau$$
$$+ \int_0^t \big[\frac{c_{13}}{\lambda_1}(\upsilon_1^2 - \upsilon_6^2)(\tau)(\widehat{\varphi}_1 - \widehat{\varphi}_9)(z) + \frac{c_{16}}{\lambda_3}(\upsilon_1^2 - \upsilon_5^2)(\tau)(\widehat{\varphi}_3 - \widehat{\varphi}_7)(z)\big]\,d\tau$$
$$+ \int_0^t \int_0^\tau \big[\frac{c_{13}}{\lambda_1}(\upsilon_1^2 - \upsilon_6^2)(\tau)(\upsilon_1^1 - \upsilon_9^1)(z,\alpha) + \frac{c_{16}}{\lambda_3}(\upsilon_1^2 - \upsilon_5^2)(\tau)(\upsilon_3^1 - \upsilon_7^1)(z,\alpha)\big]\,d\alpha\,d\tau$$
$$+ \int_0^t \int_0^\tau \big[\upsilon_1^2(\alpha)\upsilon_4^1(z,\tau-\alpha) + (\upsilon_1^2(\alpha) - \upsilon_2^2(\alpha))\upsilon_6^1(z,\tau-\alpha)\big]\,d\alpha\,d\tau, \tag{2.10}$$

$$\upsilon_5^1(z,t) = \int_0^t \sum_{j=1}^9 \widehat{p}_{5j}\upsilon_j^1(z,\tau)\,d\tau + \int_0^t \upsilon_1^2(\tau)\widehat{\varphi}_4(z)\,d\tau$$
$$+ \int_0^t \big[\frac{c_{13}}{\lambda_1}(\upsilon_4^2 - \upsilon_5^2)(\tau)(\widehat{\varphi}_1 - \widehat{\varphi}_9)(z) + \frac{c_{16}}{\lambda_3}(\upsilon_5^2 - \upsilon_4^2)(\tau)(\widehat{\varphi}_3 - \widehat{\varphi}_7)(z)\big]\,d\tau$$
$$+ \int_0^t \int_0^\tau \big[\frac{c_{13}}{\lambda_1}(\upsilon_1^2 - \upsilon_6^2)(\tau)(\upsilon_1^1 - \upsilon_9^1)(z,\alpha) + \frac{c_{16}}{\lambda_3}(\upsilon_1^2 - \upsilon_5^2)(\tau)(\upsilon_3^1 - \upsilon_7^1)(z,\alpha)\big]\,d\alpha\,d\tau$$
$$+ \int_0^t \int_0^\tau \upsilon_1^2(\alpha)\upsilon_4^1(z,\tau-\alpha)\,d\alpha\,d\tau, \tag{2.11}$$

$$\upsilon_6^1(z,t) = \int_0^t \sum_{j=1}^9 \widehat{p}_{6j}\upsilon_j^1(z,\tau)\,d\tau + \int_0^t \int_0^\tau \upsilon_2^2(\alpha)\upsilon_6^1(z,\tau-\alpha)\,d\alpha\,d\tau + \int_0^t \upsilon_2^2(\tau)\widehat{\varphi}_6(z)\,d\tau, \tag{2.12}$$

where $\upsilon_i^{01}(z,t) = \omega_i(z_0^i, t_0^i)$, $i = 1,2,3,7,8,9$.

We consider the Eq. (2.1) with the initial conditions (1.10) and boundary conditions (1.11), we differentiate (2.1) with respect to z for $i = 1,2,3$. After simple calculations, taking into account (2.2)–(2.6), we get the integral equations

$$\upsilon_1^2(t) = \upsilon_1^{02}(t) - M_1 \int_0^t (\upsilon_1^2 - \upsilon_2^2)(\tau)\frac{d}{dt}\widehat{h}_6(t-\tau)\,d\tau + \int_0^t \upsilon_2^2(\tau)\frac{d}{dt}\widehat{h}_6(t-\tau)\,d\tau$$
$$- M_1 \int_0^t \left[\frac{c_{13}}{\lambda_1}(\upsilon_1^1 - \upsilon_6^2)(\tau)\frac{d}{dt}(\widehat{h}_1 - \widehat{g}_9)(t-\tau) + \upsilon_1^2(\tau)\frac{d}{dt}\widehat{h}_4(t-\tau)\right] d\tau$$
$$- M_1 \int_0^t \left[\frac{c_{16}}{\lambda_3}(\upsilon_1^2 - \upsilon_5^2)(\tau)\frac{d}{dt}(\widehat{h}_3 - \widehat{g}_7)(t-\tau)\right] d\tau$$
$$+ 2c_{16}\int_0^t \upsilon_5^2(\tau)\frac{\partial}{\partial z}(\widehat{\varphi}_3 - \widehat{\varphi}_7)(\xi)\Big|_{\xi=\mu_3^{-1}[t-\tau]} d\tau + 2c_{13}\int_0^t \frac{\partial}{\partial z}\sum_{j=1}^9 \widehat{p}_{1j}\upsilon_j^1(\xi,\tau)\Big|_{\xi=\mu_1^{-1}[t-\tau]} d\tau$$

$$- \frac{c_{16}}{\lambda_3}\int_0^t \upsilon_5^2(\tau)\frac{d}{dt}(\widehat{h}_3-\widehat{g}_7)(t-\tau)\,d\tau + 2c_{16}\int_0^t \frac{\partial}{\partial z}\sum_{j=1}^{9}\widehat{p}_{3j}\upsilon_j^1(\xi,\tau)\Big|_{\xi=\mu_3^{-1}[t-\tau]}\,d\tau$$
$$+ c_{16}\int_0^t\int_0^\tau \upsilon_5^2(\alpha)\frac{\partial}{\partial z}(\upsilon_3^1-\upsilon_7^1)(\xi,\tau-\alpha)\,d\alpha\Big|_{\xi=\mu_3^{-1}[t-\tau]}\,d\tau$$
$$+ c_{13}\int_0^t \upsilon_6^2(\tau)\frac{\partial}{\partial z}(\widehat{\varphi}_1-\widehat{\varphi}_9)(\xi)\Big|_{\xi=\mu_1^{-1}[t-\tau]}\,d\tau - \frac{c_{13}}{\lambda_1}\int_0^t \upsilon_6^2(\tau)\frac{d}{dt}(\widehat{h}_1-\widehat{g}_9)(t-\tau)\,d\tau$$
$$+ c_{13}\int_0^t\int_0^\tau \upsilon_6^2(\alpha)\frac{\partial}{\partial z}(\upsilon_1^1-\upsilon_9^1)(\xi,\tau-\alpha)\,d\alpha\Big|_{\xi=\mu_1^{-1}[t-\tau]}\,d\tau, \tag{2.13}$$
$$\upsilon_2^2(t) = \upsilon_2^{02}(t) - [\widehat{\varphi}_6(0)]^{-1}\int_0^t \upsilon_2^2(\tau)\frac{d}{dt}\widehat{h}_6(t-\tau)\,d\tau, \tag{2.14}$$
$$\upsilon_3^2(t) = \upsilon_3^{02}(t) + 2M_2\int_0^t \frac{\partial}{\partial z}\sum_{j=1}^{9}\widehat{p}_{2j}\upsilon_j^1(\xi,\tau)\Big|_{\xi=\mu_2^{-1}[t-\tau]}\,d\tau$$
$$+ M_2\int_0^t \upsilon_3^2(\tau)\frac{\partial}{\partial z}(\widehat{\varphi}_2-\widehat{\varphi}_8)(\xi)\Big|_{\xi=\mu_2^{-1}[t-\tau]}\,d\tau - M_3\int_0^t \upsilon_3^2(\tau)\frac{d}{dt}(\widehat{h}_2-\widehat{g}_8)(t-\tau)\,d\tau$$
$$+ M_2\int_0^t\int_0^\tau \upsilon_3^2(\alpha)\frac{\partial}{\partial z}(\upsilon_2^1-\upsilon_8^1)(\xi,\tau-\alpha)\,d\alpha\Big|_{\xi=\mu_2^{-1}[t-\tau]}\,d\tau, \tag{2.15}$$

where $\upsilon_i^{02}(t)$, $i=\overline{1,3}$, are defined by the formulas

$$\upsilon_1^{02}(t) = M_1\Bigg[M_1\Big(\frac{d^2}{dt^2}\widehat{h}_5(t) - \sum_{j=1}^{9}\widehat{p}_{5j}\omega_j(0,t)\Big) - 2c_{13}\Big(\frac{\partial}{\partial z}\Big(\frac{d}{dt}\widehat{h}_1(t_1^3)\Big)\frac{\partial}{\partial z}\Phi_1(z)\Big)$$
$$+ \frac{2c_{13}}{\lambda_1}\sum_{i=1}^{9}\widehat{p}_{1i}\omega_i(0,t) - 2c_{16}\Big(\frac{\partial}{\partial z}\Big(\frac{d}{dt}\widehat{h}_3(t_1^3)\Big)\frac{\partial}{\partial z}\Phi_3(z) - \frac{1}{\lambda_3}\sum_{i=1}^{9}\widehat{p}_{3i}(0)\omega_i(0,t)\Big)\Bigg],$$
$$\upsilon_2^{02}(t) = \frac{d^2}{dt^2}\widehat{h}_6(t) - \sum_{j=1}^{9}\widehat{p}_{6j}\omega_j(0,t),$$
$$\upsilon_3^{02}(t) = -2M_2\Big[\frac{\partial}{\partial z}\Big(\frac{d}{dt}\widehat{h}_2(t_1^2)\Big)\frac{\partial}{\partial z}\Phi_2(z) - \frac{1}{\lambda_2}\sum_{i=1}^{9}\widehat{p}_{2i}\omega_i(0,t)\Big],$$

here $z=-\mu^{-1}(t)$, and $M_1 = \Big[\frac{c_{13}}{\lambda_1}(\widehat{\varphi}_1(0)-\widehat{\varphi}_9(0)) + \frac{c_{16}}{\lambda_3}(\widehat{\varphi}_3(0)-\widehat{\varphi}_7(0)) + \widehat{\varphi}_4(0)+\widehat{\varphi}_6(0)\Big]^{-1}$, $M_2 = \lambda_2\big[\widehat{\varphi}_2(0)-\widehat{\varphi}_8(0)\big]^{-1}$, $M_3 = \big[\widehat{\varphi}_2(0)-\widehat{\varphi}_8(0)\big]^{-1}$.

Putting the initial conditions (1.10) and boundary conditions (1.11) into the system of integral equations (2.1) for $i=4,5,6$, we differentiate those equations with respect to t. Using the notations (2.2)–(2.6) and after simple transformation, we have the following system of integral equations,

$$\upsilon_4^2(t) = \upsilon_4^{02}(t) + M_5\lambda_1 M_6\int_0^t \upsilon_5^2(\tau)\frac{\partial}{\partial z}(\widehat{\varphi}_3-\widehat{\varphi}_7)(\xi)\Big|_{\xi=\mu_3^{-1}[t-\tau]}\,d\tau$$
$$- M_5\int_0^t \upsilon_5^2(\tau)\frac{d}{dt}(\widehat{h}_3-\widehat{g}_7)(t-\tau)\,d\tau + 2M_6\int_0^t \frac{\partial}{\partial z}\sum_{j=1}^{9}\widehat{p}_{3j}(\xi)\upsilon_j^1(\xi,\tau)\Big|_{\xi=\mu_3^{-1}[t-\tau]}\,d\tau$$
$$+ M_5\lambda_1\int_0^t\int_0^\tau \upsilon_5^2(\alpha)\frac{\partial}{\partial z}(\upsilon_3^1-\upsilon_7^1)(\xi,\tau-\alpha)\,d\alpha\Big|_{\xi=\mu_3^{-1}[t-\tau]}\,d\tau$$
$$+ M_4\int_0^t \upsilon_5^2(\tau)\Big[\frac{c_{13}}{\lambda_1}\frac{d}{dt}(\widehat{h}_1-\widehat{g}_9)(t-\tau) + \frac{c_{16}}{\lambda_3}\frac{d}{dt}(\widehat{h}_3-\widehat{g}_7)(t-\tau)\Big]\,d\tau$$

$$+ M_4\int_0^t \upsilon_6^2(\tau)\Big[\frac{c_{13}}{\lambda_1}\frac{d}{dt}(\widehat{h}_1-\widehat{g}_9)(t-\tau)+\frac{c_{16}}{\lambda_3}\frac{d}{dt}(\widehat{h}_3-\widehat{g}_7)(t-\tau)\Big]\,d\tau$$

$$+ M_5\int_0^t \upsilon_6^2(\tau)\frac{\partial}{\partial z}(\widehat{\varphi}_1-\widehat{\varphi}_9)(\xi)\Big|_{\xi=\mu_1^{-1}[t-\tau]}d\tau + 2M_5\int_0^t \frac{\partial}{\partial z}\sum_{j=1}^{9}\widehat{p}_{1j}\upsilon_j^1(\xi,\tau)\Big|_{\xi=\mu_1^{-1}[t-\tau]}d\tau$$

$$+ M_5\int_0^t\int_0^\tau \upsilon_6^2(\alpha)\frac{\partial}{\partial z}(\upsilon_1^1-\upsilon_9^1)(\xi,\tau-\alpha)\,d\alpha\Big|_{\xi=\mu_1^{-1}[t-\tau]}d\tau, \tag{2.16}$$

$$\upsilon_5^2(t)=\upsilon_5^{02}(t)+M_6\int_0^t \upsilon_5^2(\tau)\frac{\partial}{\partial z}(\widehat{\varphi}_3-\widehat{\varphi}_7)(\xi)\Big|_{\xi=\mu_3^{-1}[t-\tau]}d\tau$$

$$- M_7\int_0^t \upsilon_5^2(\tau)\frac{d}{dt}(\widehat{h}_3-\widehat{g}_7)(t-\tau)\,d\tau + 2M_6\int_0^t \frac{\partial}{\partial z}\sum_{j=1}^{9}\widehat{p}_{3j}\upsilon_j^1(\xi,\tau)\Big|_{\xi=\mu_3^{-1}[t-\tau]}d\tau$$

$$+ M_6\int_0^t\int_0^\tau \upsilon_5^2(\alpha)\frac{\partial}{\partial z}(\upsilon_3^1-\upsilon_7^1)(\xi,\tau-\alpha)\,d\alpha\Big|_{\xi=\mu_3^{-1}[t-\tau]}d\tau, \tag{2.17}$$

$$\upsilon_6^2(t)=\upsilon_6^{02}(t)+2M_8\int_0^t \frac{\partial}{\partial z}\sum_{j=1}^{9}\widehat{p}_{1j}\upsilon_j^1(\xi,\tau)\Big|_{\xi=\mu_1^{-1}[t-\tau]}d\tau$$

$$+ M_8\int_0^t \upsilon_6^2(\tau)\frac{\partial}{\partial z}(\widehat{\varphi}_1-\widehat{\varphi}_9)(\xi)\Big|_{\xi=\mu_1^{-1}[t-\tau]}d\tau - M_9\int_0^t \upsilon_6^2(\tau)\frac{d}{dt}(\widehat{h}_1-\widehat{g}_9)(t-\tau)\,d\tau$$

$$+ M_8\int_0^t\int_0^\tau \upsilon_6^2(\alpha)\frac{\partial}{\partial z}(\upsilon_1^1-\upsilon_9^1)(\xi,\tau-\alpha)\,d\alpha\Big|_{\xi=\mu_1^{-1}[t-\tau]}d\tau, \tag{2.18}$$

where

$$\upsilon_4^{02}(t)=M_4\Bigg[M_4\bigg(\frac{d^2}{dt^2}\widehat{h}_4(t)-\sum_{j=1}^{9}\widehat{p}_{4j}\omega_j(0,t)\bigg)$$

$$-2M_5\lambda_1\bigg(\frac{\partial}{\partial z}\bigg(\frac{d}{dt}\widehat{h}_3(t_1^3)\bigg)\frac{\partial}{\partial z}\Phi_3(z)-\frac{1}{\lambda_3}\sum_{i=1}^{9}\widehat{p}_{3i}\omega_i(0,t)\bigg)\Bigg],$$

$$\upsilon_5^{02}(t)=-2M_6\Bigg[-2M_6\bigg(\frac{\partial}{\partial z}\bigg(\frac{d}{dt}\widehat{h}_3(t_1^3)\bigg)\frac{\partial}{\partial z}\Phi_3(z)-\frac{1}{\lambda_3}\sum_{i=1}^{9}\widehat{p}_{3i}\omega_i(0,t)\bigg)\Bigg],$$

$$\upsilon_6^{02}(t)=-2M_8\Bigg[-2M_8\bigg(\frac{\partial}{\partial z}\bigg(\frac{d}{dt}\widehat{h}_1(t_1^3)\bigg)\frac{\partial}{\partial z}\Phi_1(z)-\frac{1}{\lambda_1}\sum_{i=1}^{9}\widehat{p}_{1i}\omega_i(0,t)\bigg)\Bigg],$$

and

$$M_4=\Big[\frac{c_{13}}{\lambda_1}(\widehat{\varphi}_9(0)-\widehat{\varphi}_1(0))+\frac{c_{16}}{\lambda_3}(\widehat{\varphi}_7(0)-\widehat{\varphi}_3(0))+\widehat{\varphi}_5(0)\Big]^{-1},\quad M_7=\big[\widehat{\varphi}_3(0)-\widehat{\varphi}_7(0)\big]^{-1},$$
$$M_5=(1-\widehat{\varphi}_5(0)M_4)\big[\widehat{\varphi}_3(0)-\widehat{\varphi}_7(0)\big]^{-1},\qquad M_8=\lambda_1\big[\widehat{\varphi}_1(0)-\widehat{\varphi}_9(0)\big]^{-1},$$
$$M_6=\lambda_2\big[\widehat{\varphi}_3(0)-\widehat{\varphi}_7(0)\big]^{-1},\qquad M_9=\big[\widehat{\varphi}_1(0)-\widehat{\varphi}_9(0)\big]^{-1}.$$

In what follows, we will assume that

$$\frac{c_{13}}{\lambda_1}(\widehat{\varphi}_1(0)-\widehat{\varphi}_9(0))+\frac{c_{16}}{\lambda_3}(\widehat{\varphi}_3(0)-\widehat{\varphi}_7(0))+\widehat{\varphi}_4(0)+\widehat{\varphi}_6(0)\neq 0,\quad \widehat{\varphi}_2(0)-\widehat{\varphi}_8(0)\neq 0, \tag{2.19}$$

$$\frac{c_{13}}{\lambda_1}(\widehat{\varphi}_9(0)-\widehat{\varphi}_1(0))+\frac{c_{16}}{\lambda_3}(\widehat{\varphi}_7(0)-\widehat{\varphi}_3(0))+\widehat{\varphi}_5(0)\neq 0,\quad \widehat{\varphi}_1(0)-\widehat{\varphi}_9(0)\neq 0. \tag{2.20}$$

The equations (2.13)–(2.18) contain unknown functions $\frac{\partial}{\partial z}\omega_i(z,t)$, $i=\overline{1,9}$. Therefore, differentiating the equations (1.13) with respect to the variable z, using the notation (2.2)–(2.6), we obtain the integral equations for them

$$\upsilon_i^3(z,t)=\upsilon_i^{03}(z,t)+\int_{t_0^i}^t\frac{\partial}{\partial z}\left[\sum_{k=1}^9\widehat{p}_{ik}\upsilon_k^1(\xi,\tau)+\frac{1}{2}\upsilon_6^2(\xi,\tau)\left(\widehat{\varphi}_1-\widehat{\varphi}_9\right)(\xi)\right]\Bigg|_{\xi=\mu_i^{-1}[\tau-t+\mu_i(z)]}d\tau$$
$$+\frac{\partial}{\partial z}t_0^i\int_0^{t_0^i}\upsilon_6^2(\xi,t_0^i-\tau)(\upsilon_1^1-\upsilon_9^1)(\xi,\tau)\Big|_{\xi=\mu_i^{-1}[t_0^i-t+\mu_i(z)]}d\tau$$
$$+\int_{t_0^i}^t\int_0^\tau\frac{\partial}{\partial z}\left[\upsilon_6^2(\xi,\tau-\alpha)(\upsilon_1^1-\upsilon_9^1)(\xi,\alpha)\right]d\alpha\Big|_{\xi=\mu_i^{-1}[\tau-t+\mu_i(z)]}d\tau,\ i=1,9, \tag{2.21}$$

$$\upsilon_i^3(z,t)=\upsilon_i^{03}(z,t)+\int_{t_0^i}^t\frac{\partial}{\partial z}\left[\sum_{k=1}^9\widehat{p}_{ik}\upsilon_k^1(\xi,\tau)+\frac{1}{2}\upsilon_3^2(\xi,\tau)(\widehat{\varphi}_2-\widehat{\varphi}_8)(\xi)\right]\Bigg|_{\xi=\mu_i^{-1}[\tau-t+\mu_i(z)]}d\tau$$
$$+\frac{\partial}{\partial z}t_0^i\int_0^{t_0^i}\upsilon_3^2(\xi,t_0^i-\tau)(\upsilon_2^1-\upsilon_8^1)(\xi,\tau)\Big|_{\xi=\mu_i^{-1}[t_0^i-t+\mu_i(z)]}d\tau$$
$$+\int_{t_0^i}^t\int_0^\tau\frac{\partial}{\partial z}\left[\upsilon_3^2(\xi,\tau-\alpha)(\upsilon_2^1-\upsilon_8^1)(\xi,\alpha)\right]d\alpha\Big|_{\xi=\mu_i^{-1}[\tau-t+\mu_i(z)]}d\tau,\ i=2,8, \tag{2.22}$$

$$\upsilon_i^3(z,t)=\upsilon_i^{03}(z,t)+\int_{t_0^i}^t\frac{\partial}{\partial z}\left[\sum_{k=1}^9\widehat{p}_{ik}\upsilon_k^1(\xi,\tau)+\frac{1}{2}\upsilon_5^2(\xi,\tau)(\widehat{\varphi}_3-\widehat{\varphi}_7)(\xi)\right]\Bigg|_{\xi=\mu_i^{-1}[\tau-t+\mu_i(z)]}d\tau$$
$$+\frac{\partial}{\partial z}t_0^i\int_0^{t_0^i}\upsilon_5^2(\xi,t_0^i-\tau)(\upsilon_3^1-\upsilon_7^1)(\xi,\tau)\Big|_{\xi=\mu_i^{-1}[t_0^i-t+\mu_i(z)]}d\tau$$
$$+\int_{t_0^i}^t\int_0^\tau\frac{\partial}{\partial z}\left[\upsilon_5^2(\xi,\tau-\alpha)(\upsilon_3^1-\upsilon_7^1)(\xi,\alpha)\right]d\alpha\Big|_{\xi=\mu_i^{-1}[\tau-t+\mu_i(z)]}d\tau,\ i=3,7, \tag{2.23}$$

$$\upsilon_4^3(z,t)=\int_0^t\frac{\partial}{\partial z}\left[\sum_{j=1}^9\widehat{p}_{4j}\upsilon_j^1(z,\tau)\right]d\tau+\int_0^t\frac{\partial}{\partial z}\left[\upsilon_1^2(\tau)\widehat{\varphi}_4(z)+(\upsilon_1^2-\upsilon_2^2)(\tau)\widehat{\varphi}_6(z)\right]d\tau$$
$$+\int_0^t\frac{\partial}{\partial z}\left[\frac{c_{13}}{\lambda_1}(\upsilon_1^2-\upsilon_6^2)(\tau)(\widehat{\varphi}_1-\widehat{\varphi}_9)(z)+\frac{c_{16}}{\lambda_3}(\upsilon_1^2-\upsilon_5^2)(\tau)(\widehat{\varphi}_3-\widehat{\varphi}_7)(z)\right]d\tau$$
$$+\int_0^t\int_0^\tau\frac{\partial}{\partial z}\left[\frac{c_{13}}{\lambda_1}(\upsilon_1^2-\upsilon_6^2)(\tau)(\upsilon_1^1-\upsilon_9^1)(z,\alpha)+\frac{c_{16}}{\lambda_3}(\upsilon_1^2-\upsilon_5^2)(\tau)(\upsilon_3^1-\upsilon_7^1)(z,\alpha)\right]d\alpha\,d\tau$$
$$+\int_0^t\int_0^\tau\frac{\partial}{\partial z}\left[\upsilon_1^2(\alpha)\upsilon_4^1(z,\tau-\alpha)+(\upsilon_1^2-\upsilon_2^2)(\alpha)\upsilon_6^1(z,\tau-\alpha)\right]d\alpha\,d\tau, \tag{2.24}$$

$$\upsilon_5^3(z,t)=\int_0^t\frac{\partial}{\partial z}\left[\sum_{j=1}^9\widehat{p}_{5j}\upsilon_j^1(z,\tau)\right]d\tau+\int_0^t\frac{\partial}{\partial z}\upsilon_1^2(\tau)\widehat{\varphi}_4(z)d\tau$$
$$+\int_0^t\frac{\partial}{\partial z}\left[\frac{c_{13}}{\lambda_1}(\upsilon_4^2-\upsilon_5^2)(\tau)(\widehat{\varphi}_1-\widehat{\varphi}_9)(z)+\frac{c_{16}}{\lambda_3}(\upsilon_5^2-\upsilon_4^2)(\tau)(\widehat{\varphi}_3-\widehat{\varphi}_7)(z)\right]d\tau$$
$$+\int_0^t\int_0^\tau\frac{\partial}{\partial z}\left[\frac{c_{13}}{\lambda_1}(\upsilon_1^2-\upsilon_6^2)(\tau)(\upsilon_1^1-\upsilon_9^1)(z,\alpha)+\frac{c_{16}}{\lambda_3}(\upsilon_1^2-\upsilon_5^2)(\tau)(\upsilon_3^1-\upsilon_7^1)(z,\alpha)\right]d\alpha\,d\tau$$
$$+\int_0^t\int_0^\tau\frac{\partial}{\partial z}\upsilon_1^2(\alpha)\upsilon_4^1(z,\tau-\alpha)\,d\alpha\,d\tau, \tag{2.25}$$

$$\upsilon_6^3(z,t) = \int_0^t \frac{\partial}{\partial z}\left[\sum_{j=1}^{9} \widehat{p}_{6j}\upsilon_j^1(z,\tau)\right] d\tau$$

$$+ \int_0^t \frac{\partial}{\partial z}\upsilon_2^2(\tau)\widehat{\varphi}_6(z)\,d\tau + \int_0^t\int_0^\tau \frac{\partial}{\partial z}\upsilon_2^2(\alpha)\upsilon_6^1(z,\tau-\alpha)\,d\alpha\,d\tau, \tag{2.26}$$

where $\upsilon_i^{03}(z,t) = \frac{\partial}{\partial z}\omega_i(z_0^i,t_0^i) - \frac{\partial}{\partial z}t_0^i \sum\limits_{j=1}^{9} \widehat{p}_{ij}\omega_j(z_0^i,t_0^i)$, $i = 1,2,3,7,8,9$.

We require the fulfillment of the matching conditions

$$-\lambda_j \frac{d\widehat{\varphi}_i(z)}{dz}\bigg|_{z=0} + \sum_{j=1}^{9}\widehat{p}_{ij}\widehat{\varphi}_j(0) = \frac{d}{dt}\widehat{h}_i\bigg|_{t=0}, \quad i=\overline{1,6}. \tag{2.27}$$

§ 3. Main result and its proof

The main result of this work is the following theorem.

Theorem 3.1. *Let the conditions of Theorem 1.1 be satisfied, and function $h(x_1,x_2,t)$ have compact support in x_1, x_2 for each fixed t, $\varphi_i(z) \in C^2[0,H]$, $i=\overline{1,9}$, $g_i(t) \in C^2[0,H]$, $i=1,2,3,7,8,9$, $h_i(t) \in C^2[0,H]$, $i=\overline{1,6}$, equality (2.19), (2.20) and matching condition (2.27) hold. Then, for any $H>0$ on the segment $[0,H]$, there is a unique solution to the inverse problem (1.1)–(1.4).*

P r o o f. Consider now a square $D_0 := \{(z,t)\colon 0 \le z \le H,\ 0 \le t \le H\}$. Equations (2.7)–(2.12), (2.13)–(2.18) and (2.21)–(2.26) show that the values of the functions $\omega_i(z,t)$, $i=\overline{1,9}$, $r'_{ij}(t)$, $i,j=\overline{1,3}$, $\frac{\partial}{\partial z}\omega_i(z,t)$, $i=\overline{1,9}$ at $(z,t)\in D_0$ are expressed through integrals of some combinations of the same functions over segments lying in D_0.

The system of equations (2.7)–(2.12), (2.13)–(2.18) and (2.21)–(2.26) form a complete system of equalities for unknown functions in the domain D_0. According to the introduced notation for the vector function $\upsilon(z,t) = (\upsilon_i^1(z,t), \upsilon_j^2(t), \upsilon_i^3(z,t))$, $i=\overline{1,9}$, $j=\overline{1,6}$, we write this system in the operator form

$$\upsilon = A\upsilon, \tag{3.1}$$

where the operator $A = (A_i^1, A_j^2, A_i^3)$, $i=\overline{1,9}$, $j=\overline{1,6}$, the components of the operator A are determined by the right-hand sides of the equations (2.7)–(2.12), (2.13)–(2.18) and (2.21)–(2.26), respectively.

Let $C_s(D_0)$ $(s \ge 0)$, be the Banach space of continuous functions with the ordinary norm, denoted by $\|\cdot\|_s$,

$$\|\upsilon\|_s = \max\left\{\max_{1\le i\le 9,\ (z,t)\in D_0}\left|\upsilon_i^1(z,t)e^{-st}\right|,\ \max_{1\le i\le 6,\ t\in[0,H]}\left|\upsilon_i^2(t)e^{-st}\right|,\ \max_{1\le i\le 9,\ (z,t)\in D_0}\left|\upsilon_i^3(z,t)e^{-st}\right|\right\}.$$

Obviously, C_s with $s=0$ is the usual space of continuous functions with the ordinary norm, denoted by $\|\cdot\|$ in what follows, because

$$e^{-sH}\|\upsilon\| \le \|\upsilon\|_s \le \|\upsilon\|.$$

The norms $\|\upsilon\|_s$ and $\|\upsilon\|$ are equivalent for any $H\in(0,\infty)$, where $s\in(0,1)$ and we will choose that number later.

Next, consider the set of functions $S(\upsilon^0, r) \subset C_s(D_0)$, satisfying the inequality

$$\|\upsilon - \upsilon^0\|_s \le r, \tag{3.2}$$

where r is a known number, the vector function $\upsilon^0(z,t) = (\upsilon_i^{01}(z,t),\ i = \overline{1,9},\quad \upsilon_i^{02}(t),\ i = \overline{1,6},$ $\upsilon_i^{03}(z,t),\ i = \overline{1,9})$, as defined by the free terms of the operator equation (3.1). It is easy to see that for $\upsilon \in S(\upsilon^0, r)$ the estimate $\|\upsilon\|_s \le \|\upsilon^0\|_s + r \le \|\upsilon^0\| + r := r_0$ holds. Thus, r_0 is known.

Note that the operator A maps the space $C_s(D_0)$ into itself. Let us show that for a suitable choice of s (recall that $H > 0$ is an arbitrary fixed number) it is a contraction operator on the set $S(\upsilon^0, r)$. First, let us make sure that the operator A takes the set $S(\upsilon^0, r)$ into itself, that is, from the condition $\upsilon(z,t) \in S(\upsilon^0, r)$, it follows that $A\upsilon \in S(\upsilon^0, r)$, if s satisfies some constraints. In fact, for any $(z,t) \in D_0$ and $\upsilon \in S(\upsilon^0, r)$ the following inequalities hold:

$$\|A\upsilon - \upsilon^0\|_s \le \frac{r_0}{s}\alpha_i,\ i = \overline{1,24},$$

where $\alpha_i := 9M^0 + \varphi_0 + r_0,\ i = 1,2,3,7,8,9,\ \alpha_4 := 9M^0 + 8M^0\varphi_0 + 8M^0 r_0 + 3r_0 + 3\varphi_0,$ $\alpha_5 := 9M^0 + \varphi_0 + r_0 + 8M^0\varphi_0 + 8M^0 r_0,\ \alpha_6 := 9M^0 + r_0 + \varphi_0,\ \alpha_{10} := 8M^0 h_0 + 5M^0 g_0 +$ $+ 6M^0\varphi_0 + h_0 + 72M^0 + 4M^0 r_0,\ \alpha_{11} := r_0 M^0 h_0,\ \alpha_i := M^0(18 + 2\varphi_0 + +h_0 + g_0 + 2r_0),$ $i = 12, 14, 15,\ \alpha_{21} := 18M^0 + r_0 + \varphi_0,\ \alpha_{13} := 2M^0\big((M^0)^2\varphi_0 + g_0 + h_0 + \varphi_0 + 18 + M^0 r_0 +$ $+ r_0 + 2M^0 + 2M_0 h_0 + 2M^0 g_0\big),\ \alpha_i := 18M^0 + 2\varphi_0 + 6r_0 + 2M^0 r_0,\ i = 16, 17, 18, 22, 23, 24,$ $\alpha_{19} := 9M^0 + 16M^0 r_0 + 3\varphi_0(16M^0 + 3) + 6r_0,\ \alpha_{20} := 18M^0 + r_0 + \varphi_0 + 16M^0(\varphi_0 + r_0)$, and $\varphi_0 :=$ $:= \max_{i=\overline{1,9}} \|\widehat{\varphi}_i\|_{C^2[0,H]},\ g_0 := \max_{i=1,2,3,7,8,9} \|\widehat{g}_i\|_{C^2[0,H]},\ h_0 := \max_{i=\overline{1,6}} \|h_i\|_{C^2[0,H]},\ M := \max_{i=\overline{1,9}} \|M_i\|_{C[0,H]},$ $p_0 := \max_{i,j=\overline{1,9}} \|\widehat{p}_{ij}\|_{C^1[0,H]},\ c_0 := \max_{i,j=\overline{1,6}} \|c_{ij}\|_{C^1[0,H]},\ M^0 = \max\Big\{M; p_0; c_0; \|\lambda_i\|; \|\frac{\partial t_1^i}{\partial z}\|; \big[\widehat{\varphi}_6(0)\big]^{-1}\Big\}.$

Choosing $s > (1/r)\alpha_0$ $\big(\alpha_0 = \max\{\alpha_i,\ i = \overline{1,24}\}\big)$ we get that the operator A maps the set $S(\upsilon^0, r)$ into itself.

Now, let υ and $\widetilde{\upsilon}$ be two arbitrary elements in $S(\upsilon^0, r)$. Using the obvious inequality

$$|\upsilon_i^k \upsilon_i^l - \widetilde{\upsilon}_i^k \widetilde{\upsilon}_i^l| e^{-st} \le |\upsilon_i^k - \widetilde{\upsilon}_i^k||\upsilon_i^l| e^{-st} + |\widetilde{\upsilon}_i^k||\upsilon_i^l - \widetilde{\upsilon}_i^l| e^{-st} \le 2r_0\|\upsilon - \widetilde{\upsilon}\|_s, \quad (z,t) \in D_0,$$

after some easy estimations, we find that for $(z,t) \in D_0$,

$$\|A\upsilon - A\widetilde{\upsilon}\|_s \le \frac{\|\upsilon - \widetilde{\upsilon}\|}{s}\gamma_i,\ i = \overline{1,24},$$

where, $\gamma_i := 12M^0 + M^0\varphi_0 + r_0,\ i = 1,2,3,7,8,9,\ \gamma_4 := 12M^0 + 8M^0\varphi_0 + 10M^0 r_0 + 3r_0 + 3\varphi_0,$ $\gamma_5 := 9M^0 + \varphi_0 + 4r_0 + 8M^0\varphi_0 + 8M^0 r_0,\ \gamma_6 := 9M^0\varphi_0 + r_0 + \varphi_0,\ \gamma_{10} := 8M^0 h_0 + 10M^0 g_0 +$ $+ 6M^0\varphi_0 + h_0 + 72M^0 + 4M^0 r_0,\ \gamma_{11} := r_0 M^0 h_0,\ \gamma_i := M^0(20 + 2\varphi_0 + +h_0 + g_0 + 2r_0),$ $i = 12, 14, 15,\ \gamma_{13} := 2M^0\big((M^0)^2\varphi_0 + g_0 + h_0 + \varphi_0 + 18 + M^0 r_0 + r_0 + 2M^0 + 2M_0 h_0 + 2M^0 g_0\big),$ $\gamma_{21} := 24M^0 + r_0 + \varphi_0,\ \gamma_i := 18M^0 + 4\varphi_0 + 6r_0 + 2M^0 r_0,\ i = 16, 17, 18, 22, 23, 24,\ \gamma_{19} :=$ $:= 12M^0 + 16M^0 r_0 + 3\varphi_0(16M^0 + 3) + 6r_0,\ \gamma_{20} := 18M^0 + r_0 + \varphi_0 + 16M^0\varphi_0 + 16M^0 r_0.$

Choosing now $s > \gamma_0$ $\big(\gamma_0 = \max\{\gamma_i,\ i = \overline{1,24}\}\big)$, we get, that the operator A compresses the distance between the elements $\upsilon, \widetilde{\upsilon}$ to $S(\upsilon^0, r)$.

As follows from the performed estimates, if the number s is chosen from conditions $s > s^* := \max\{\alpha_0, \gamma_0\}$, then the operator A is contracting on $S(\upsilon^0, r)$. In this case, according to the Banach principle [26], the equation (3.1) has the only solution in $S(\upsilon^0, r)$ for any fixed $H > 0$. Theorem 3.1 is proved. □

By the found functions $r'_{ij}(t),\ i,j = \overline{1,3}$, the functions $r_{ij}(t),\ i,j = \overline{1,3}$, are found by the formulas

$$r_{ij}(t) = r_{ij}(0) + \int_0^t r'_{ij}(\tau)\,d\tau,\ i,j = \overline{1,3}.$$

§ 4. Appendix

Let us denote by σ_{ij} the projection onto the x_i-axis of the stress acting on the area with the normal parallel to the x_j-axis, and let $\overline{u}_i$ be the projection onto the x_i-axis of the vector particle displacement. In viscoelastic anisotropic media, the stress tensor has the following representation:

$$\begin{gathered}\sigma_{ij}(x,t)=\sum_{k,l=1}^{3}c_{ijkl}\Big[S_{kl}+\int_0^t K_{ij}(t-\tau)S_{kl}(x,\tau)\,d\tau\Big],\ i,j=1,2,3,\\ S_{kl}=\frac{1}{2}\bigg(\frac{\partial\overline{u}_k}{\partial x_l}+\frac{\partial\overline{u}_l}{\partial x_k}\bigg),\ x\in\mathbb{R}^3,\ k,l=1,2,3,\end{gathered}\tag{4.1}$$

here $c_{ijkl}=c_{ijkl}(x_3)$ are moduli of elasticity, $K_{ij}(t)$ are functions responsible for the viscosity of the medium and $K_{ij}=K_{ji}$, $i,j=\overline{1,3}$.

The equations of motion of viscoelastic body particles in the absence of external forces have the form [27]:

$$\rho\frac{\partial^2\overline{u}_i}{\partial t^2}=\sum_{j=1}^{3}\frac{\partial\sigma_{ij}}{\partial x_j},\ i=\overline{1,3},\tag{4.2}$$

where $\rho=\rho(x_3)$ is medium density and $\rho>0$, $\overline{u}(x,t)=(\overline{u}_1(x,t),\overline{u}_2(x,t),\overline{u}_3(x,t))$ is displacement vector.

Note that (4.1) can be considered as integral Volterra equations of the second kind with respect to the expression $\sum\limits_{k,l=1}^{3}c_{ijkl}S_{kl}$. For each fixed pair (i,j) solving these equations, we get

$$\sigma_{ij}(x,t)=\sum_{k,l=1}^{3}c_{ijkl}(x_3)S_{kl}(x,t)+\int_0^t r_{ij}(t-\tau)\sigma_{ij}(x,\tau)\,d\tau,\tag{4.3}$$

where r_{ij} are the resolvents of the kernels K_{ij} and they are related by the following integral relations:

$$r_{ij}(t)=-K_{ij}(t)-\int_0^t K_{ij}(t-\tau)r_{ij}(\tau)\,d\tau,\ i,j=\overline{1,3}.\tag{4.4}$$

The condition $K_{ij}=K_{ji}$ implies that $r_{ij}=r_{ji}$.

Differentiating (4.3) with respect to t and introducing the notation $u_i=\frac{\partial}{\partial t}\overline{u}_i$, we get

$$\frac{\partial}{\partial t}\sigma_{ij}(x,t)=\sum_{k,l=1}^{3}c_{ijkl}\bigg(\frac{\partial u_k}{\partial x_l}+\frac{\partial u_l}{\partial x_k}\bigg)+r_{ij}(0)\sigma_{ij}(x,t)+\int_0^t r'_{ij}(t-\tau)\sigma_{ij}(x,\tau)\,d\tau.\tag{4.5}$$

Let $c_{ijkl}=c_{jikl}=c_{ijlk}=c_{klij}$. The symmetry of the stress tensor reduces the number of independent elastic moduli from 81 to 21. If we assume that $c_{mn}=c_{ijkl}$, where $m=(ij)$ and $n=(kl)$, in accordance with the notation $(11)\to 1$, $(22)\to 2$, $(33)\to 3$, $(23)=(32)\to 4$, $(13)=(31)\to 5$, $(12)=(21)\to 6$, then the matrix of independent elastic moduli can be given in the form of a 6×6 symmetrical matrix. We will consider anisotropic media with a matrix of independent elastic moduli of the following form [28]:

$$c_{\alpha\beta}(x_3)=\begin{pmatrix}c_{11}&c_{12}&c_{13}&0&0&c_{16}\\ c_{12}&c_{11}&c_{13}&0&0&-c_{16}\\ c_{13}&c_{13}&c_{33}&0&0&0\\ 0&0&0&c_{44}&0&0\\ 0&0&0&0&c_{44}&0\\ c_{16}&-c_{16}&0&0&0&c_{66}\end{pmatrix}.$$

Then, the system of equations (4.1) and (4.2) for the velocity u_i and strain $\sigma_{ij}(\sigma_{ij}=\sigma_{ji})$ in view of (4.3)–(4.5) can be written as a system of first-order integro-differential equations. For convenience, let $x_3 = z$,

$$\Big(A_1\frac{\partial}{\partial t} - B_1\frac{\partial}{\partial x_1} - C_1\frac{\partial}{\partial x_2} - D_1\frac{\partial}{\partial z} - F_1\Big)U(x_1,x_2,z,t) = \int_0^t R_1(t-\tau)U(x_1,x_2,z,\tau)\,d\tau, \tag{4.6}$$

where $U = (u_1,u_2,u_3,\sigma_{11},\sigma_{12},\sigma_{13},\sigma_{22},\sigma_{23},\sigma_{33})^*$, $*$ is the transposition sign,

$$A_1 = \begin{pmatrix} \rho\mathbf{I}_{3\times3} & \mathbf{O}_{3\times6} \\ \mathbf{O}_{6\times3} & I_{6\times6} \end{pmatrix}, \quad B_1 = \begin{pmatrix} \mathbf{O}_{3\times3} & \begin{matrix} 1&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0 \end{matrix} \\ \begin{matrix} c_{11}&0&0\\ c_{12}&0&0\\ c_{13}&0&0\\ 0&c_{44}&0\\ 0&0&c_{44}\\ c_{16}&0&0 \end{matrix} & \mathbf{O}_{6\times6} \end{pmatrix},$$

$$C_1 = \begin{pmatrix} \mathbf{O}_{3\times3} & \begin{matrix} 0&0&0&1&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&1 \end{matrix} \\ \begin{matrix} 0&c_{12}&c_{16}\\ 0&c_{11}&-c_{16}\\ 0&c_{13}&0\\ c_{44}&0&0\\ 0&0&0\\ 0&-c_{16}&c_{66} \end{matrix} & \mathbf{O}_{6\times6} \end{pmatrix},$$

$$D_1 = \begin{pmatrix} \mathbf{O}_{3\times3} & \begin{matrix} 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&1&0&0&0 \end{matrix} \\ \begin{matrix} 0&c_{16}&c_{13}\\ 0&-c_{16}&c_{13}\\ 0&0&c_{33}\\ 0&0&0\\ c_{44}&0&0\\ 0&c_{66}&0 \end{matrix} & \mathbf{O}_{6\times6} \end{pmatrix},$$

$$F_1 = \begin{pmatrix} \mathbf{0}_{3\times3} & \mathbf{O}_{3\times6} \\ \mathbf{0}_{6\times3} & \operatorname{diag}(r_{11}(0), r_{22}(0), r_{33}(0), r_{12}(0), r_{13}(0), r_{23}(0)) \end{pmatrix},$$

$$R_1 = R_1(t) = \begin{pmatrix} \mathbf{O}_{3\times3} & \mathbf{O}_{3\times6} \\ \mathbf{O}_{6\times3} & \operatorname{diag}(r'_{11}, r'_{22}, r'_{33}, r'_{12}, r'_{13}, r'_{23})(t) \end{pmatrix}.$$

The system (4.6) can be reduced to a symmetric hyperbolic system [25].

Let us reduce system (4.6) to the canonical form with respect to the variables t and z. As is known from linear algebra, in the case under consideration there exists a nonsingular matrix T, where Λ is a diagonal matrix with eigenvalues of the matrix D_1 on its diagonal. Such conditions can be satisfied, for example, by a matrix:

$$\Upsilon(z)=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\ -c_{13}\sqrt{\frac{\rho}{c_{33}}} & 0 & -c_{16}\sqrt{\frac{\rho}{c_{66}}} & 1 & 0 & 1 & c_{16}\sqrt{\frac{\rho}{c_{66}}} & 0 & c_{13}\sqrt{\frac{\rho}{c_{33}}}\\ -c_{13}\sqrt{\frac{\rho}{c_{33}}} & 0 & c_{16}\sqrt{\frac{\rho}{c_{66}}} & 0 & 1 & 0 & -c_{16}\sqrt{\frac{\rho}{c_{66}}} & 0 & c_{13}\sqrt{\frac{\rho}{c_{33}}}\\ -\sqrt{c_{33}\rho} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{c_{33}\rho}\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & -\sqrt{c_{44}\rho} & 0 & 0 & 0 & 0 & 0 & \sqrt{c_{44}\rho} & 0\\ 0 & 0 & -\sqrt{c_{66}\rho} & 0 & 0 & 0 & \sqrt{c_{66}\rho} & 0 & 0 \end{pmatrix}.$$

We introduce a new function in equation (4.6) using the equality

$$U=\Upsilon\vartheta,$$

and multiply this equation on the left by the matrix Υ. Then, for the function ϑ, after obvious transformations, we obtain the equation

$$\Big(I\frac{\partial}{\partial t}+\Lambda\frac{\partial}{\partial z}+B\frac{\partial}{\partial x_1}+C\frac{\partial}{\partial x_2}+F\Big)\vartheta(x_1,x_2,z,t)=\int_0^t R(z,t-\tau)\vartheta(x,\tau)\,d\tau,$$

where

$$\vartheta=(\vartheta_1,\vartheta_2,\dots,\vartheta_9)^*,\quad B=\Upsilon^{-1}A_1^{-1}B_1\Upsilon=(b_{ij}),\quad C=\Upsilon^{-1}A_1^{-1}C_1\Upsilon=(\widetilde c_{ij}),$$
$$\Lambda=\mathrm{diag}(\lambda_i)=\mathrm{diag}\Big(-\sqrt{\frac{c_{33}}{\rho}},-\sqrt{\frac{c_{44}}{\rho}},-\sqrt{\frac{c_{66}}{\rho}},0,0,0,\sqrt{\frac{c_{66}}{\rho}},\sqrt{\frac{c_{44}}{\rho}},\sqrt{\frac{c_{33}}{\rho}}\Big),\ \ i=\overline{1,9},$$
$$F=\Upsilon^{-1}A_1^{-1}D_1\frac{\partial\Upsilon}{\partial z}+\Upsilon^{-1}A_1^{-1}F_1\Upsilon=(p_{ij}),\quad R=\Upsilon^{-1}A_1^{-1}R_1\Upsilon=(\widetilde r_{ij}).\tag{4.7}$$

and $\widetilde r_{ij}(z,t)=\widetilde r_{ij}$, $\widetilde r_{11}=\widetilde r_{99}=-\widetilde r_{19}=-\widetilde r_{91}=\frac{r'_{33}}{2}$, $\widetilde r_{22}=\widetilde r_{88}=-\widetilde r_{28}=-\widetilde r_{82}=\frac{r'_{13}}{2}$, $\widetilde r_{33}=\widetilde r_{77}=-\widetilde r_{37}=-\widetilde r_{73}=\frac{r'_{23}}{2}$, $\widetilde r_{41}=-\widetilde r_{49}=\frac{c_{13}}{\lambda_1}(r'_{11}-r'_{33})$, $\widetilde r_{43}=-\widetilde r_{47}=\frac{c_{16}}{\lambda_3}(r'_{11}-r'_{23})$, $\widetilde r_{44}=r'_{11}$, $\widetilde r_{46}=r'_{11}-r'_{12}$, $\widetilde r_{51}=-\widetilde r_{59}=\frac{c_{13}}{\lambda_1}(r'_{22}-r'_{23})$, $\widetilde r_{53}=-\widetilde r_{57}=\frac{c_{16}}{\lambda_3}(r'_{23}-r'_{22})$, $\widetilde r_{55}=r'_{22}$, $\widetilde r_{66}=r'_{12}$. All other elements of the matrix $\widetilde r_{ij}$, which are not mentioned in this equality, are zero.

Conclusion

In this work, the inverse problem was considered for determining the kernel $R(t)$ included in the equation (1.1) by using additional condition (1.4) of the solution of the problem with initial and boundary conditions (1.2), (1.3). Sufficient conditions for given functions are obtained, under which the inverse problem has unique solutions on a sufficiently small interval.

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Обратная задача для системы вязкоупругости в анизотропных средах с тетрагональной формой модуля упругости

Ключевые слова: вязкоупругость, резольвента, обратная задача, гиперболическая система, преобразование Фурье.

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Для приведенной канонической системы интегро-дифференциальных уравнений вязкоупругости рассмотрены прямая и обратная задачи определения поля скоростей упругих волн и матрицы релаксации. Задачи заменены замкнутой системой интегральных уравнений типа Вольтерра второго рода относительно преобразования Фурье по переменным x_1 и x_2 для решения прямой и обратной задачи. Далее к этой системе применяется метод сжимающих отображений в пространстве непрерывных функций с весовой нормой. В работе доказаны теоремы о глобальные существования и единственности решений задач.

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