RESEARCH ARTICLE | MARCH 11 2024

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AIP Conf. Proc. 3004, 040010 (2024)
https://doi.org/10.1063/5.0199964

The Problem Of Finding The Kernels In The System For Integro-Differential Acoustic Equations

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Abstract. We pose the direct and inverse problem of finding the acoustic wave velocity and pressure, diagonal memory matrix for a reduced canonical system of integro-differential acoustic equations. The problems are replaced by a closed system of Volterra-type integral equations of the second kind with respect to the Fourier transform in the variables x_1 and x_2 of the solution of the unknowns of the direct problem and the inverse problem. To this system, we then apply a reduction method, a mapping in the space of continuous functions with a weighted norm. Thus, we prove global existence and uniqueness theorems to solve the given problems.

INTRODUCTION

Hyperbolic systems of equations of the first order describe many physical processes associated with the propagation of waves of various nature. For example, we can point out the systems of equations of acoustics, electromagnetic oscillations, and the theory of elasticity.

Recently, there has been an increased interest in hyperbolic systems of integro-differential equations with a convolution-type integral. Such equations describe processes with memory (with aftereffect) or, as they are also called, eridite processes (see [1], pp. 180-189). Such processes are characterized by the fact that the change in their state at each moment of time depends on the history of the process. Examples of such processes are the deformation of a viscoelastic medium (see [2], pp. 449-453), the processes of propagation of electromagnetic waves in media with dispersion (see [3], pp. 357-392) and dynamics of coexistence and development of animal and plant populations of various species (see [1], pp. 193-195).An analysis of the dynamic equations describing such processes shows that Volterra operators are added to the right-hand side of the systems of hyperbolic equations operators of the convolution type of some function, depending on the time and the elliptic part of the corresponding hyperbolic operators on the left side.

The theory of inverse problems for hyperbolic systems was developed in the works of L.P. Nizhnik [4], S.P. Belensky [5, 6], V.G. Romanov [7, 8] and others. The study of inverse problems of determining the kernel or coefficient of an integral operator in hyperbolic integrodifferential equations is the object of study by many authors [9, 10, 11, 12, 13].

At present, the inverse problems of determining the kernel from one hyperbolic integro-differential equation of the second order are well studied A. Lorenzi [14], V.G. Romanov [15], Zh. Sh. Safarov [16], J. Janno and L. Von Wolfersdorf [17], V. G. Romanov [18], D. K. Durdiev [19, 20, 21]. In D. K. Durdiev [22, 23], V.G. Romanov [24, 25] results were obtained on the existence and uniqueness of some multidimensional inverse problems for second-order hyperbolic integro-differential equations. The papers [26, 27, 28] discuss the issues of global solvability of one-dimensional memory problems.

The system describing the propagation of acoustic waves in the two-dimensional case is written as follows [29]:

$$\begin{cases} \frac{\partial p}{\partial t}+\rho_0 c_0^2\left(\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y}\right)=\int\limits_0^t \varphi_1(t-\tau)p(x,y,\tau)d\tau, \\ \frac{\partial u}{\partial t}+\frac{1}{\rho_0}\frac{\partial p}{\partial x}=\int\limits_0^t \varphi_2(t-\tau)u(x,y,\tau)d\tau, \\ \frac{\partial v}{\partial t}+\frac{1}{\rho_0}\frac{\partial p}{\partial y}=\int\limits_0^t \varphi_3(t-\tau)v(x,y,\tau)d\tau. \end{cases} \tag{1}$$

International Scientific and Practical Conference on “Modern Problems of Applied Mathematics and Information Technology (MPAMIT2022)”
AIP Conf. Proc. 3004, 040010-1–040010-13; https://doi.org/10.1063/5.0199964
Published by AIP Publishing. 978-0-7354-4876-6/$30.00

We write it in the matrix form of this system:

$$A_0\frac{\partial}{\partial t}\mathscr{U}+A_1\frac{\partial}{\partial x}\mathscr{U}+A_2\frac{\partial}{\partial y}\mathscr{U}=\int\limits_0^t\Phi(t-\tau)\mathscr{U}(x,y,\tau)d\tau, \tag{2}$$

where $\mathscr{U}=(p,u,v)^*-$ is a column vector; $A_j,\ j=\overline{0,2}$ are symmetric matrices, and A_0 is positive definite; $\Phi=diag\,(\varphi_1,\varphi_2,\varphi_3)$. The matrices A_j have a cellular structure:

$$A_0=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad A_1=\begin{pmatrix}0&\rho_0c_0^2&0\\\frac{1}{\rho_0}&0&0\\0&0&0\end{pmatrix},\quad A_2=\begin{pmatrix}0&0&\rho_0c_0^2\\0&0&0\\\frac{1}{\rho_0}&0&0\end{pmatrix}.$$

Let us reduce system (2) to the canonical form. As is known from linear algebra (see [30], pp. 149-153), in the case under consideration there exists a nonsingular matrix T, such that $T^{-1}A_1T=\Lambda$, where $\Lambda-$ is a diagonal matrix with eigenvalues of the matrix B_3 on its diagonal.

Some matrix T, with the above properties was constructed in (see ([31], pp. 5-20)). It looks as

$$T(y)=\begin{pmatrix}\rho_0c_0&-\rho_0c_0&0\\1&1&0\\0&0&1\end{pmatrix}. \tag{3}$$

Note that T is defined not uniquely.

The inverse matrix to T is defined by the formula

$$T^{-1}(y)=\begin{pmatrix}\frac{1}{2\rho_0c_0}&\frac{1}{2}&0\\-\frac{1}{2\rho_0c_0}&\frac{1}{2}&0\\0&0&1\end{pmatrix}. \tag{4}$$

In (2), introduce the new function by the equality

$$\mathscr{U}=TU \tag{5}$$

and multiply by T^{-1} from the left. Then for U we obtain the equation

$$I_3\frac{\partial}{\partial t}U+\Lambda\frac{\partial}{\partial x}U+B_1\frac{\partial}{\partial y}U+B_2U=\int\limits_0^t\bar{\Phi}(t-\tau)U(x,y,\tau)d\tau, \tag{6}$$

where

$$B_1=T^{-1}A_2T,\quad B_2=T^{-1}A_2\frac{\partial}{\partial y}T,\quad \Lambda=diag\,(c_0,-c_0,0),$$

$$\bar{\Phi}(t)=\begin{pmatrix}\frac{\varphi_1(t)+\varphi_2(t)}{2}&\frac{\varphi_2(t)-\varphi_1(t)}{2}&0\\\frac{\varphi_2(t)-\varphi_1(t)}{2}&\frac{\varphi_1(t)+\varphi_2(t)}{2}&0\\0&0&\varphi_3(t)\end{pmatrix}=\left(\bar{\Phi}_{ij}\right)_{i,j=1}^3.$$

In the next section, we consider direct and inverse problems.

STATEMENT OF THE PROBLEM

In the *direct problem*, given matrices $\Lambda,\ B_1,\ B_2,\ \bar{\Phi}$, it is required, in the domain $D=\{(x,y,t):0<x<L,\ t>0,\ y\in\mathbb{R}\}$ find a vector-function $U(x,y,t)$ satisfying equation (6) for the following initial and boundary conditions:

$$U_i(x,y,t)\big|_{t=0}=\psi_i(x,y),\ i=\overline{1,3}, \tag{7}$$

$$U_1(x,y,t)|_{x=0} = g_1(y,t), \quad U_2(x,y,t)|_{x=L} = g_2(y,t), \tag{8}$$

where $\psi_i(x,y) = (\psi_1, \psi_2, \psi_3)(x,y)$, $\quad g(y,t) = (g_1, g_2, g_3)(y,t)$ are some given functions.

We pose the *inverse problem* as follows: find the functions $\varphi_i(t),\ t > 0, i = 1,2,3$ that are involved in the matrix $\bar{\Phi}$ if the extra conditions

$$U_1(x,y,t)|_{x=L} = h_1(y,t), \quad U_2(x,y,t)|_{x=0} = h_2(y,t), \tag{9}$$

are given for a solution to problem (6)-(8). Moreover, we assume that $\varphi_i(0),\ i = 1,2,3$ are given as well.

The inverse problem of finding the kernels of the integral terms from a system of first-order integrodifferential equations of general form with two independent variables was studied in [32, 33]. Some theorem of local existence and global uniqueness was obtained.

Suppose that functions $\psi_i(x,y),\ i = 1,2,3,\ g_j(y,t),\ j = 1,2$ occurring on the right-hand side of (6) and the data (7), (8) have some compact support with respect to y for every fixed x and t. The existence for (6) of a finite dependence domain and the property of having compact support with respect to y of the right-hand side of (6) and the data (7), (8) imply that solutions to problem (6)-(8) have the compact support with respect to y.

We investigate the properties of solutions to this problem. More exactly, we will confine ourselves to the study of the Fourier transform of a solution with respect to y. Put

$$\widehat{U}(x,\eta,t) = \int\limits_{\mathbb{R}} U(x,y,t)e^{i\eta y}dy, \tag{10}$$

where η is the parameters of the transform. Fix η and for convenience introduce the notation $\widehat{U}(x,\eta,t) = \widehat{U}(x,t)$.

In terms of the function $\widehat{U}$, we write problem (6)-(9) as

$$\left(I_3\frac{\partial}{\partial t} + \Lambda\frac{\partial}{\partial x} + B\right)\widehat{U} = \int\limits_0^t \bar{\Phi}(\tau)\widehat{U}(x,t-\tau)d\tau, \tag{11}$$

$$\widehat{U}_i(x,t)\big|_{t=0} = \widehat{\psi}_i(x),\ i = \overline{1,3}, \tag{12}$$

$$\widehat{U}_1(x,t)|_{x=0} = \widehat{g}_1(t), \quad \widehat{U}_2(x,t)|_{x=L} = \widehat{g}_2(t), \tag{13}$$

$$\widehat{U}_1(x,t)|_{x=L} = \widehat{h}_1(t), \quad \widehat{U}_2(x,t)|_{x=0} = \widehat{h}_2(t), \tag{14}$$

where $B = B_1 - i\eta B_0$.

EXAMINATION OF THE DIRECT PROBLEM

Let $\Pi = \{(x,t) : 0 < x < L,\ t > 0\}$ be the projection of the domain D to the plane of the variables $x,\ t$. Consider an arbitrary point $(x,t) \in \Pi$ on the plane of the variables ξ, τ a characteristic of the $i-$ equation of system (11) through (x,t) till the intersection with the boundary of Π in the domain $\tau < t$. The equation looks as

$$\tau = t + \lambda_i(\phi(\xi) - \phi(x)), \tag{15}$$

where $\phi(x) = \int\limits_0^x \frac{d\alpha}{c_0(\alpha)}$.

For $\lambda_1 = 1$ this point lies either on the interval $[0,L]$ of the axis $t = 0$ or on the straight line $x = 0$, and for $\lambda_2 = -1$ either on the interval $[0,L]$ or on the straight line $x = L$.

Integrating the $i-$ component of equations (11) over characteristic (15) from (x_0^i, t_0^i) to (x,t), we find

$$\widehat{U}_i(x,t) = \widehat{U}_i(x_0^i, t_0^i) - \int\limits_{t_0^i}^t \sum_{j=1}^3 b_{ij}(\xi)\widehat{U}_j(\xi,\tau)\bigg|_{\xi=\phi^{-1}[\lambda_i(\tau-t)+\phi(x)]} d\tau$$

$$+\int_{t_0^i}^{t}\int_0^{\tau}\sum_{j=1}^{3}\bar{\Phi}_{ij}(\alpha)\widehat{U}_j(\xi,\tau-\alpha)d\alpha\Big|_{\xi=\phi^{-1}[\lambda_i(\alpha-\tau)+\phi(x)]}d\tau. \tag{16}$$

Find t_0^i in (16). It depends on the coordinates of (x,t). It is easy to observe that $t_0^i(x,t)$ has the form

$$t_0^i(x,t)=\begin{cases}\begin{cases}t-\phi(x),\ t\geq\phi(x),\\ 0,\ 0<t<\phi(x),\quad i=1;\end{cases}\\ \begin{cases}t-\phi(x)+\phi(L),\ t\geq\phi(x),\\ 0,\ 0<t<\phi(x),\quad i=2;\end{cases}\\ 0,\quad i=3.\end{cases}$$

Then the condition that the pair (x_0^i,t_0^i) enjoys (15) implies

$$x_0^i(x,t)=\begin{cases}\begin{cases}0,\ t\geq\phi(x),\\ \phi^{-1}\left(\phi(x)-t\right),\ 0<t<\phi(x),\quad i=1;\end{cases}\\ \begin{cases}L,\ t\geq\phi(x),\\ \phi^{-1}\left(\phi(x)+t\right),\ 0<t<\phi(x),\quad i=2;\end{cases}\\ x,\quad i=3.\end{cases}$$

The free terms of the integral equations (15) are defined through the initial and boundary conditions (12) and (13) as follows:

$$\widehat{U}_i(x_0^i,t_0^i)=\begin{cases}\begin{cases}\widehat{g}_1(t-\phi(x)),\ t\geq\phi(x),\\ \widehat{\psi}_1(\phi^{-1}\left(\phi(x)-t\right)),\ 0<t<\phi(x),\quad i=1;\end{cases}\\ \begin{cases}\widehat{g}_2(t+\phi(x)-\phi(L)),\ t\geq\phi(L)-\phi(x),\\ \widehat{\psi}_2(\phi^{-1}\left(\phi(x)+t\right)),\ 0<t<\phi(L)-\phi(x),\quad i=2;\end{cases}\\ \widehat{\psi}_3(x),\quad i=3.\end{cases}$$

It is required that $\widehat{U}_i(x_0^i,t_0^i)$ be continuous in Π. Note that, for these conditions to be fulfilled, the given functions $\widehat{\psi}_i$ and $\widehat{g}_i$ must satisfy the fitting conditions at the angular points of Π:

$$\widehat{\psi}_1(0)=\widehat{g}_1(0),\quad \widehat{\psi}_2(L)=\widehat{g}_2(0). \tag{17}$$

Here and below, the values of $\widehat{g}_i$ for t = 0 and $\widehat{\psi}_i$ for $x=0$ and $x=L$ are understood as the limit values at these points as the argument tends from the side of the point where these functions are defined.

Suppose that all given functions in (16) are continuous functions of their arguments in Π. Then we have a closed system of Volterra-type integral equations with continuous kernels and free terms. As usual, such a system has a unique solution in the bounded subdomain

$$\Pi_T=\left\{(x,t):0\leq x\leq L,\ 0\leq t\leq T\right\},$$

of Π, where $T>0$ is some fixed number.

Theorem 1. Suppose that $\rho(x), c_0(x), \widehat{\psi}(x)\in C[0,\infty)$, $\widetilde{g}(t)\in C[0,\infty)$, $\bar{\Phi}(t)\in C[0,\infty)$ and conditions (17) are fulfilled. Then there is a unique solution to problem (11)-(13) in Π_T.

Problem (18)-(20) in the field of Π_T is equivalent to a linear integral equation of the second kind of Volterra type with respect to $\widehat{U}$. Based on the theory of linear integral equations, it has unique solutions [34, 35]. So we throw it away.

EXAMINATION OF THE INVERSE PROBLEM. DEDUCTION OF AN EQUIVALENT SYSTEM OF INTEGRAL EQUATIONS

Let us introduce the vector function $V(x,t) = \frac{\partial}{\partial t}\widehat{U}(x,t)$. To obtain a problem for a function $V(x,t)$ similar to (11)-(13), we differentiate equations (11) and boundary conditions (13) with respect to the variable t, and the condition at $t=0$ can be found using equations (11) and initial conditions (12). In doing so, we get following problem

$$\left(I_3\frac{\partial}{\partial t} + \Lambda\frac{\partial}{\partial x} + B\right)V = \bar{\Phi}(t)\widehat{\psi}(x) + \int_0^t \bar{\Phi}(\tau)V(x,t-\tau)d\tau, \tag{18}$$

$$V(x,t)\Big|_{t=0} = -\lambda\frac{\partial}{\partial x}\widehat{\psi}(x) - B(x)\widehat{\psi}(x) := \Psi(x), \tag{19}$$

$$V_1(z,t)\Big|_{x=0} = \frac{d}{dt}\widehat{g}_1(t),\ V_2(z,t)\Big|_{x=L} = \frac{d}{dt}\widehat{g}_2(t), \tag{20}$$

$$V_1(x,t)\Big|_{x=L} = \frac{d}{dt}\widehat{h}_1(t), \quad V_2(x,t)\Big|_{x=0} = \frac{d}{dt}\widehat{h}_2(t), \tag{21}$$

where $\Psi(x) = (\Psi_1,\Psi_2,\Psi_3)(x)$.

Again, integration along the corresponding characteristics will lead problem (18)-(20) to the integral equations

$$V_i(x,t) = V_i(x_0^i,t_0^i) + \int_{t_0^i}^t \left[\sum_{j=1}^3 \bar{\Phi}_{ij}(\tau)\widehat{\psi}_j(\xi) - \sum_{j=1}^3 b_{ij}(\xi)V_j(\xi,\tau)\right]\Bigg|_{\xi=\phi^{-1}[\lambda_i(\tau-t)+\phi(x)]} d\tau +$$

$$+\int_{t_0^i}^t \int_0^\tau \sum_{j=1}^3 \bar{\Phi}_{ij}(\alpha)V_j(\xi,\tau-\alpha)d\alpha\Bigg|_{\xi=\phi^{-1}[\lambda_i(\alpha-\tau)+\phi(x)]} d\tau,\ i=1,2,3. \tag{22}$$

In equations (22), $V_i(x_0^i,t_0^i)$ are defined as follows:

$$V_i(x_0^i,t_0^i) = \begin{cases} \begin{cases} \frac{d}{dt}\widetilde{g}_1(t-\phi(x)),\ t\geq\phi(x), \\ \Psi_1(\phi^{-1}(\phi(x)-t)),\ 0<t<\phi(x), \quad i=1; \end{cases} \\ \begin{cases} \frac{d}{dt}\widetilde{g}_2(t+\phi(x)-\phi(L)),\ t\geq\phi(L)-\phi(x), \\ \Psi_2(\phi^{-1}(\phi(x)+t)),\ 0<t<\phi(L)-\phi(x), \quad i=2; \end{cases} \\ \Psi_3(z), \quad i=3. \end{cases}$$

Let the following conditions be fulfilled

$$-c_0(0)\frac{\partial}{\partial x}\widehat{\psi}_1(0) - \sum_{j=1}^3 b_{1j}(0)\widehat{\psi}_j(0) = \left[\frac{d}{dt}\widehat{g}_1(t)\right]_{t=0},$$

$$c_0(L)\frac{\partial}{\partial x}\widehat{\psi}_1(L) - \sum_{j=1}^3 b_{1j}(L)\widehat{\psi}_j(L) = \left[\frac{d}{dt}\widehat{g}_2(t)\right]_{t=0}. \tag{23}$$

Consider an arbitrary point $(x,0) \in \Pi$ and draw the characteristic (15) through $(x,0)$ till the intersection with the lateral boundaries of Π. Integrating the i th component of equation (18), using the data (21), we get

$$V_i(x,0) = \frac{d}{dt}\widehat{h}_i(t_i(x)) + \int\limits_0^{(t_i(x))} \left[\sum_{j=1}^{3} \bar{\Phi}_{ij}(\tau)\widehat{\psi}_j(\xi) - \sum_{j=1}^{3} b_{ij}(\xi)V_j(\xi,\tau) \right]\Bigg|_{\xi=\phi^{-1}[\lambda_i\tau+\phi(x)]} d\tau +$$

$$+ \int\limits_0^{(t_i(x))} \int\limits_0^{\tau} \sum_{j=1}^{3} \bar{\Phi}_{ij}(\alpha)V_j(\xi,\tau-\alpha)d\alpha \Bigg|_{\xi=\phi^{-1}[\lambda_i(\alpha-\tau)+\phi(x)]} d\tau, \tag{24}$$

where $t_i(x) = \begin{cases} -\phi(x), & i=1, \\ -\phi(x)+\phi(L), & i=2. \end{cases}$

Integrating the 3rd component of equation (18), will lead to the following integral equations

$$V_3(x,t) = \Psi_3(x) + \int\limits_0^t \left[\widehat{\psi}_3(x)\varphi_3(\tau) - \sum_{j=1}^{3} b_{3j}(x)V_j(x,\tau) + \int\limits_0^{\tau} \varphi_3(\alpha)V_3(x,\tau-\alpha)d\alpha \right] d\tau. \tag{25}$$

Reckoning with the initial data (19) and additional conditions (21) rewrite (24), (25) as

$$\int\limits_0^{(t_i(x))} \sum_{j=1}^{3} \bar{\Phi}_{ij}(\tau)\widehat{\psi}_j(\phi^{-1}[\lambda_i\tau+\phi(x)])d\tau = \Psi_i(x) + \int\limits_0^{(t_i(x))} \sum_{j=1}^{3} b_{ij}(\xi)V_j(\xi,\tau)\Bigg|_{\xi=\phi^{-1}[\lambda_i\tau+\phi(x)]} d\tau$$

$$-\frac{d}{dt}\widehat{h}_i(t_i(x)) - \int\limits_0^{(t_i(x))} \int\limits_0^{\tau} \sum_{j=1}^{3} \bar{\Phi}_{ij}(\alpha)V_j(\phi^{-1}[\lambda_i(\alpha-\tau)+\phi(x)],\tau-\alpha)d\alpha d\tau, \ i=1,2;$$

$$\int\limits_0^t \widehat{\psi}_3(x)\varphi_3(\tau)d\tau = V_3(x,t) - \Psi_3(x) + \int\limits_0^t \sum_{j=1}^{3} b_{3j}(x)V_j(x,\tau)d\tau - \int\limits_0^t \int\limits_0^{\tau} \varphi_3(\alpha)V_3(x,\tau-\alpha)d\alpha d\tau.$$

Differentiate the first equations with respect to x, and the second, with respect to t. Then

$$\frac{\partial t_i(x)}{\partial x} \sum_{j=1}^{3} \bar{\Phi}_{ij}(t_i(x))\widehat{\psi}_j(\phi^{-1}[\lambda_i t_i(x)+\phi(x)]) + \int\limits_0^{(t_i(x))} \sum_{j=1}^{3} \bar{\Phi}_{ij}(\tau)\frac{d}{dx}\widehat{\psi}_j(\phi^{-1}[\lambda_i\tau+\phi(x)])d\tau$$

$$= \frac{d}{dx}\Psi_i(x) + \frac{\partial t_i(x)}{\partial x} \sum_{j=1}^{3} b_{ij}(\phi^{-1}[\lambda_i t_i(x)+\phi(x)])V_j(\phi^{-1}[\lambda_i t_i(x)+\phi(x)],t_i(x)) - \frac{\partial t_i(x)}{\partial x}\frac{d^2}{dt^2}\widehat{h}_i(t_i(x))$$

$$+ \int\limits_0^{(t_i(x))} \sum_{j=1}^{3} \frac{\partial}{\partial x}\Big(b_{ij}(\phi^{-1}[\lambda_i\tau+\phi(x)])V_j(\phi^{-1}[\lambda_i\tau+\phi(x)],\tau) \Big) d\tau$$

$$-\frac{\partial t_i(x)}{\partial x} \int\limits_0^{(t_i(x))} \sum_{j=1}^{3} \bar{\Phi}_{ij}(\tau)V_j(\phi^{-1}[\lambda_i(\tau-t_i(x))+\phi(x)],t_i(x)-\tau)d\tau$$

$$-\int\limits_0^{(t_i(x))}\int\limits_0^{\tau}\sum_{j=1}^{3}\bar{\Phi}_{ij}(\alpha)\frac{\partial}{\partial x}V_j(\phi^{-1}\left[\lambda_i(\alpha-\tau)+\phi(x)\right],\tau-\alpha)d\alpha d\tau,\ i=1,2; \tag{26}$$

$$\widehat{\psi}_3(x)\varphi_3(t)=\frac{\partial}{\partial t}V_3(x,t)+\sum_{j=1}^{3}b_{3j}(x)V_j(x,t)-\int\limits_0^{t}\sum_{j=1}^{3}\bar{\Phi}_{ij}(\tau)V_j(x,t-\tau)d\tau. \tag{27}$$

Now, replace $t_i(z)$ by t in (26). We infer

$$\sum_{j=1}^{3}\bar{\Phi}_{1j}(t)\widehat{\psi}_j(0)-c_0(x)\int\limits_0^{t}\sum_{j=1}^{3}\bar{\Phi}_{1j}(\tau)\frac{d}{dx}\widehat{\psi}_j(t-\tau)d\tau=-c_0(x)\int\limits_0^{t}\sum_{j=1}^{3}\frac{\partial}{\partial x}\big(b_{1j}(0)V_j(0,t-\tau)\big)d\tau$$

$$-c_0(x)P(t)-\int\limits_0^{t}\sum_{j=1}^{3}\bar{\Phi}_{1j}(\tau)V_j(0,t-\tau)d\tau+c_0(x)\int\limits_0^{t}\int\limits_0^{\tau}\sum_{j=1}^{3}\bar{\Phi}_{1j}(\alpha)\frac{\partial}{\partial x}V_j(-(t-\tau),\tau-\alpha)d\alpha d\tau, \tag{28}$$

$$\sum_{j=1}^{3}\bar{\Phi}_{2j}(t)\widehat{\psi}_j(L)+c_0(x)\int\limits_0^{t}\sum_{j=1}^{3}\bar{\Phi}_{2j}(\tau)\frac{d}{dx}\widehat{\psi}_j(L-(t-\tau))d\tau=c_0(x)\int\limits_0^{t}\sum_{j=1}^{3}\frac{\partial}{\partial x}\big(b_{2j}(L)V_j(L,L-t+\tau)\big)d\tau$$

$$+c_0(x)P(L-t)-\int\limits_0^{t}\sum_{j=1}^{3}\bar{\Phi}_{2j}(\tau)V_j(L,L-t+\tau)d\tau-c_0(x)\int\limits_0^{t}\int\limits_0^{\tau}\sum_{j=1}^{3}\bar{\Phi}_{2j}(\alpha)\frac{\partial}{\partial x}V_j(L-(t-\tau),\tau-\alpha)d\alpha d\tau, \tag{29}$$

where $P_i(z)$ are defined by the formulas

$$P_i(z)=\frac{d}{dx}\Psi_i(x)-\frac{\partial t_i(x)}{\partial x}\frac{d^2}{dt^2}\widetilde{h}_i(t_i(z))$$

.

$$+\frac{\partial t_i(x)}{\partial x}\sum_{j=1}^{3}b_{ij}(\phi^{-1}\left[\lambda_i t_i(x)+\phi(x)\right])V_j(\phi^{-1}\left[\lambda_i t_i(x)+\phi(x)\right],t_i(x)),\ i=1,2.$$

Let us introduce the following notation:

$$Q(\nu_i,\widehat{\psi}(\nu_i)):=\begin{pmatrix}\frac{1}{2}\left(\widehat{\psi}_1(0)-\widehat{\psi}_2(0)\right) & \frac{1}{2}\left(\widehat{\psi}_1(0)+\widehat{\psi}_2(0)\right)\\ \frac{1}{2}\left(-\widehat{\psi}_1(L)+\widehat{\psi}_2(L)\right) & \frac{1}{2}\left(\widehat{\psi}_1(L)+\widehat{\psi}_2(L)\right)\end{pmatrix}=\left(Q_{ij}(\nu_i,\widehat{\psi}(\nu_i))\right)_{i,j=1}^{3}, \tag{30}$$

where $\nu_i:=\begin{cases}0,\ i=1\\ L,\ i=2\end{cases}$

Reckoning with (30), rewrite (22) as follows:

$$V_i(x,t)=V_i(x_0^i,t_0^i)+\int\limits_{t_0^i}^{t}\left[\sum_{j=1}^{2}Q_{ij}(\xi,\widehat{\psi})\varphi_j(\tau)-\sum_{j=1}^{3}b_{ij}(\xi)V_j(\xi,\tau)\right]\Bigg|_{\xi=\phi^{-1}[\lambda_i(\tau-t)+\phi(x)]}d\tau$$

$$+\int\limits_{t_0^i}^{t}\int\limits_0^{\tau}\sum_{j=1}^{2}Q_{ij}(\xi,V(\xi,\tau-\alpha))\varphi_j(\alpha)d\alpha\Bigg|_{\xi=\phi^{-1}[\lambda_i(\alpha-\tau)+\phi(x)]}d\tau,\ i=1,2. \tag{31}$$

Using (30), we can also rewrite (28), (29) so that:

$$\sum_{j=1}^{2} Q_{ij}(\nu_i;\widehat{\psi}(\nu_i))\varphi_j(t) = -\lambda_i c_0(x)P_i(\bar{t}_i(t)) - \lambda_i c_0(x)\int_0^t \sum_{j=1}^{3} \frac{\partial}{\partial x}\big(b_{ij}(\nu_i)V_j(\nu_i,\nu_i+\lambda_i(t-\tau))\big)d\tau$$

$$+\lambda_i c_0(x)\int_0^t \sum_{j=1}^{2} Q_{ij}(\nu_i;\frac{d}{dx}\widehat{\psi}_j(\nu_i+\lambda_i(t-\tau)))\varphi_j(\tau)d\tau - \int_0^t \sum_{j=1}^{2} Q_{ij}(\nu_i;V_j(\nu_i,\nu_i+\lambda_i(t-\tau)))\varphi_j(\tau)d\tau$$

$$+\lambda_i c_0(x)\int_0^t\int_0^\tau \sum_{j=1}^{2} Q_{ij}(\nu_i;\frac{\partial}{\partial x}V_j(\nu_i,\nu_i+\lambda_i(t-\tau),\tau-\alpha)))\varphi_j(\alpha)d\alpha d\tau, \tag{32}$$

where

$$\bar{t}_i(t) = \begin{cases} t, & i=1, \\ L-t, & i=2. \end{cases}$$

Let $\varphi(t) = (\varphi_1(t),\varphi_2(t),\varphi_3(t))$ be the vector-function composed of the derivatives of the unknown functions of the inverse problem, where $\varphi_i(t)$ are the entries of this vector-function.

$$\widehat{\psi}_1(0)\widehat{\psi}_2(L) \neq \widehat{\psi}_1(L)\widehat{\psi}_2(0). \tag{33}$$

Now, solving (32) with respect to $\varphi_i(t)$, we obtain

$$\varphi_i(t) = \frac{1}{\det Q(\nu_i;\widetilde{\phi})}\sum_{j=1}^{2}\Big[-\lambda_{ij}c_0(x)P_j(\bar{t}_j(t)) - \lambda_j c_0(x)\int_0^t \sum_{l=1}^{3}\frac{\partial}{\partial x}\big(b_{jl}(\nu_j)V_l(\nu_j,\nu_j+\lambda_j(t-\tau))\big)d\tau$$

$$+\lambda_j c_0(x)\int_0^t \sum_{l=1}^{2} Q_{jl}(\nu_j;\frac{d}{dx}\widehat{\psi}_l(\nu_j+\lambda_j(t-\tau)))\varphi_l(\tau)d\tau - \int_0^t \sum_{l=1}^{2} Q_{jl}(\nu_j;V_l(\nu_j,\nu_j+\lambda_j(t-\tau)))\varphi_l(\tau)d\tau$$

$$+\lambda_j c_0(x)\int_0^t\int_0^\tau \sum_{l=1}^{2} Q_{jl}(\nu_j;\frac{\partial}{\partial x}V_l(\nu_j,\nu_j+\lambda_j(t-\tau),\tau-\alpha)))\varphi_l(\alpha)d\alpha d\tau\Big]\mathscr{Q}_{ji}(\nu_i;\widetilde{\phi}), \tag{34}$$

where $\mathscr{Q}_{ji}-$ are the algebraic complements to the entries Q_{ji} of Q, $i,j=1,2$.

Equations (34) contain the unknown functions $\frac{\partial}{\partial x}V_j,\ j=\overline{1,3}$. For them we obtain integral equations from (25) and (31) by differentiating in x. Moreover,

$$\frac{\partial}{\partial x}V_i(x,t) = \frac{\partial}{\partial x}V_i(x_0^i,t_0^i) - \frac{\partial}{\partial x}t_0^i\Big[\sum_{j=1}^{3} Q_{ij}(x_0^i;\widehat{\psi}(x_0^i))\varphi_j(t_0^i) - \sum_{j=1}^{3} b_{ij}(x_0^i)V_j(x_0^i,t_0^i)\Big]$$

$$+\int_{t_0^i}^{t}\Big[\sum_{j=1}^{3}\frac{\partial}{\partial x}Q_{ij}(\xi;\widehat{\psi})\varphi_j(\tau) - \sum_{j=1}^{3}\frac{\partial}{\partial x}b_{ij}(\xi)V_j(\xi,\tau) - \sum_{j=1}^{3} b_{ij}(\xi)\frac{\partial}{\partial x}V_j(\xi,\tau)\Big]\Big|_{\xi=\phi^{-1}[\lambda_i(\tau-t)+\phi(x)]}d\tau$$

$$-\frac{\partial}{\partial x}t_0^i\int_0^{t_0^i}\sum_{j=1}^{3} Q_{ij}(x_0^i;H_j(x_0^i,t_0^i-\tau))\varphi_j(\alpha)d\alpha$$

$$+\int_{t_0^i}^{t}\int_0^{\tau}\sum_{j=1}^{2}\frac{\partial}{\partial x}Q_{ij}(\xi,V(\xi,\tau-\alpha))\varphi_j(\alpha)d\alpha\Big|_{\xi=\phi^{-1}[\lambda_i(\alpha-\tau)+\phi(x)]}d\tau,\ i=1,2, \tag{35}$$

$$\frac{\partial}{\partial x}V_3(x,t)=\frac{\partial}{\partial x}\Psi_3(x)+\int_0^t\left[\frac{\partial}{\partial x}\widehat{\psi}_3(x)\varphi_3(\tau)\right.$$

$$\left.-\sum_{j=1}^{3}\frac{\partial}{\partial x}b_{3j}(x)V_j(x,\tau)-\sum_{j=1}^{3}b_{3j}(x)\frac{\partial}{\partial x}V_j(x,\tau)+\int_0^{\tau}\frac{\partial}{\partial x}V_3(x,\tau-\alpha)\varphi_3(\alpha)d\alpha\right]d\tau, \tag{36}$$

where

$$H_j\left(x_0^i,t_0^i-\tau\right)=\begin{cases}\frac{d}{dt}h_j\left(t+\lambda_j(\phi(\xi)-\phi(x))\right),\ j=1,\\ \frac{d}{dt}g_j\left(t+\lambda_j(\phi(\xi)-\phi(x))\right),\ j=2.\end{cases}$$

THE MAIN RESULT AND THE PROOF

The main result of the present article is as follows:

Theorem 2. Suppose the fulfillment of the conditions of Theorem 1 and also the conditions $\widehat{\psi}(z)\in C^2[0,\ L]$, $\widehat{g}(t)\in C^2[0,\ \infty)$, $\widehat{h}(t)\in C^2(0,\infty)$, condition (33), and the fitting conditions (17), (23). Then, for every $L>0$, on the interval $\left[0,L\right]$, there exists a unique solution to the inverse problem (18)-(21) of the class $\varphi(t)\in C^1\left[0,L\right]$.

Proof. Equations (25), (31), (34), (35)-(36), supplemented with the initial and boundary value conditions from (18) constitute the closed system of equations on the unknown $V_i(x,t)$, $\varphi_j(t)$, $\frac{\partial}{\partial x}V_i(x,t)$, $i,j=1,2,3$. Now, consider the square

$$\Pi_L:=\{(x,t):0\leq x\leq L,0\leq t\leq L\}.$$

Equations (25), (31), (34)-(36) show that the values of $V_i(x,t)$, $\varphi_j(t)$, $\frac{\partial}{\partial x}V_i(x,t)$ for $(x,t)\in\Pi_L$ are expressed in terms of the integrals of some combinations of these functions over segments lying in Π_L.

Write (25), (31), (34)-(36) as a closed system of Volterra-type integral equations. For this introduce the vector-functions $\vartheta(x,t)=\left(\vartheta_i^1,\vartheta_j^2,\vartheta_i^3\right)$, $i=1,2,3,\ j=1,2,3$ by defining their components by the equalities

$$\vartheta_i^1(x,t)=V_i(x,t),\ \vartheta_i^2(x,t)=\varphi_i(t),$$

$$\vartheta_i^3(x,t)=\frac{\partial}{\partial x}V_i(x,t)+\beta_i\frac{\partial}{\partial x}t_0^i\sum_{j=1}^{3}Q_{ij}(x_0^i;\widehat{\psi}(x_0^i))\varphi_j(t_0^i),$$

where $\beta_i=\begin{cases}1,\ \ i=1,2,\\ 0,\ \ i=3.\end{cases}$

Then the system (25), (31), (34)-(36) takes the operator form

$$\vartheta=A\vartheta, \tag{37}$$

where A is the operator $A=\left(A_i^1,\ A_j^2,\ A_i^3\right)$, $i=1,2,3,\ j=1,2,3$ that is defined in accordance with the right-hand sides of (25),(31), (34) -(36) by the equalities

$$A_i^1\vartheta(x,t)=\vartheta_i^{01}(x,t)+\int_{t_0^i}^{t}\left[\sum_{j=1}^{2}Q_{ij}(\xi,\widehat{\psi})\vartheta_j^2(\tau)-\sum_{j=1}^{3}b_{ij}(\xi)\vartheta_j^1(\xi,\tau)\right]\Bigg|_{\xi=\phi^{-1}[\lambda_i(\tau-t)+\phi(x)]}d\tau$$

$$+\int_{t_0^i}^{t}\int_{0}^{\tau}\sum_{j=1}^{2}Q_{ij}(\xi,\vartheta_j^1(\xi,\tau-\alpha))\vartheta_j^2(\alpha)d\alpha\Big|_{\xi=\phi^{-1}[\lambda_i(\alpha-\tau)+\phi(x)]}d\tau,\ i=1,2, \tag{38}$$

$$A_3^1\vartheta(x,t)=\vartheta_3^{01}(x,t)+\int_0^t\left[\widehat{\psi}_3(x)\vartheta_3^2(\tau)-\sum_{j=1}^{3}b_{3j}(x)\vartheta_j^1(x,\tau)+\int_0^\tau\vartheta_3^2(\alpha)\vartheta_3^1(x,\tau-\alpha)d\alpha\right]d\tau, \tag{39}$$

$$A_i^2\vartheta(x,t)=\frac{1}{\det Q(\nu_i;\widetilde{\phi})}\sum_{j=1}^{2}\left[-\lambda_j c_0(x)\int_0^t\sum_{l=1}^{3}\frac{\partial}{\partial x}\big(b_{jl}(\nu_j)\vartheta_l^1(\nu_j,\nu_j+\lambda_j(t-\tau))\big)d\tau\right.$$

$$+\lambda_j c_0(x)\int_0^t\sum_{l=1}^{2}Q_{jl}(\nu_j;\frac{d}{dx}\widehat{\psi}_l(\nu_j+\lambda_j(t-\tau)))\vartheta_l^2(\tau)d\tau-\int_0^t\sum_{l=1}^{2}Q_{jl}(\nu_j;\vartheta_l^1(\nu_j,\nu_j+\lambda_j(t-\tau)))\vartheta_l^2(\tau)d\tau$$

$$\left.+\lambda_j c_0(x)\int_0^t\int_0^\tau\sum_{l=1}^{2}Q_{jl}(\nu_j;\vartheta_l^3(\nu_j,\nu_j+\lambda_j(t-\tau),\tau-\alpha)))\vartheta_l^2(\alpha)d\alpha d\tau\right]\mathscr{Q}_{ji}(\nu_i;\widetilde{\phi}),\ i=1,2, \tag{40}$$

$$A_i^3\vartheta(x,t)=\int_{t_0^i}^{t}\left[\sum_{j=1}^{3}\frac{\partial}{\partial x}Q_{ij}(\xi;\widehat{\psi})\vartheta_j^2(\tau)-\sum_{j=1}^{3}\frac{\partial}{\partial x}b_{ij}(\xi)\vartheta_j^1(\xi,\tau)-\sum_{j=1}^{3}b_{ij}(\xi)\vartheta_j^3(\xi,\tau)\right]\Bigg|_{\xi=\phi^{-1}[\lambda_i(\tau-t)+\phi(x)]}d\tau$$

$$-\frac{\partial}{\partial x}t_0^i\int_0^{t_0^i}\sum_{j=1}^{3}Q_{ij}(x_0^i;H_j(x_0^i,t_0^i-\tau))\vartheta_j^2(\alpha)d\alpha$$

$$+\int_{t_0^i}^{t}\int_0^\tau\sum_{j=1}^{2}\frac{\partial}{\partial x}Q_{ij}(\xi,\vartheta^1(\xi,\tau-\alpha))\vartheta_j^2(\alpha)d\alpha\Big|_{\xi=\phi^{-1}[\lambda_i(\alpha-\tau)+\phi(x)]}d\tau,\ i=1,2, \tag{41}$$

$$A_3^3\vartheta(x,t)=\int_0^t\left[\frac{\partial}{\partial x}\widehat{\psi}_3(x)\vartheta_3^2(\tau)-\sum_{j=1}^{3}\frac{\partial}{\partial x}b_{3j}(x)\vartheta_j^1(x,\tau)-\sum_{j=1}^{3}b_{3j}(x)\vartheta_j^3(x,\tau)+\int_0^\tau\vartheta_3^3(x,\tau-\alpha)\vartheta_3^2(\alpha)d\alpha\right]d\tau. \tag{42}$$

In these formulas, we used the notations

$$\vartheta_i^{01}(z,t)=V_i(x_0^i,t_0^i),\ i=1,2,\ \vartheta_3^{01}(z,t)=\Psi_3(x),\ \vartheta_i^{02}(z,t)=\frac{1}{\det Q(\nu_i;\widetilde{\phi})}\sum_{j=1}^{2}-\lambda_j c_0(x)P_j(\bar{t}_j(t)),\ i=1,2,$$

$$\vartheta_i^{03}(z,t)=\frac{\partial}{\partial x}V_i(x_0^i,t_0^i)-\frac{\partial}{\partial x}t_0^i\left[\sum_{j=1}^{3}Q_{ij}(x_0^i;\widehat{\psi}(x_0^i))\varphi_j(t_0^i)-\sum_{j=1}^{3}b_{ij}(x_0^i)V_j(x_0^i,t_0^i)\right],\ i=1,2,\ \vartheta_3^{03}(z,t)=\frac{\partial}{\partial x}\Psi_3(x).$$

Endow the set of continuous functions $C(\Pi_L)$ with the norm

$$\|\vartheta\|_s=\max_{1\le i\le 3,\ 1\le l\le 3}\sup_{(x,t)\in\Pi_L}\left|\vartheta_i^l(x,t)e^{-st}\right|,$$

where $s \geq 0-$ is a number to be chosen below. Obviously, for $s = 0$ this space coincides with the set of continuous functions with the norm $\|\vartheta\|_s$. By the inequality,

$$e^{-sL}\|\vartheta\| \leq \|\vartheta\|_s \leq \|\vartheta\|,$$

the norms $\|\vartheta\|_s$ and $\|\vartheta\|$ are equivalent for any fixed $L \in (0,\infty)$.

Further, consider the set of functions $S\left(\vartheta^0, r\right) \subset C_s\left(\Pi_0\right)$, satisfying the inequality

$$\|\vartheta - \vartheta^0\|_s \leq r, \tag{43}$$

where the vector-function $\vartheta^0(z,t) = \left(\vartheta_i^{01}(x,t),\ \vartheta_i^{02}(t),\ \vartheta_i^{03}(x,t),\ i = \overline{1,6}\right)$ is defined by the free terms of the operator equation (37). It is not hard to observe that the following estimate holds for $\vartheta \in S\left(\vartheta^0, r\right)$ $\|\vartheta\|_s \leq \|\vartheta^0\|_s + r \leq \|\vartheta^0\| + r := r_0$.

Thus, r_0-is known.

Introduce the notations

$$\widetilde{\phi}_0 := \max_{1\leq i\leq 3}\left\|\widetilde{\phi}_i\right\|_{C^2[0,L]},\ g_0 := \max_{1\leq i\leq 3}\|g_i\|_{C^2[0,L]},\ h_0 := \max_{1\leq i\leq 3}\|h_i\|_{C^2[0,L]},$$

$$\Gamma_0 := \max\left\{g_0,\ \widetilde{\phi}_0\right\},\ P_0 := \min\left\{|\mathscr{Q}(0)|, |\mathscr{Q}(L)|\right\},$$

$$\Upsilon_0\widetilde{\phi}_0 = \max_{1\leq i\leq 3}\left\|Q_{ij}(z+\gamma_i(\tau-t);\widetilde{\phi})\right\|_{C^1[0,L]},\ Q_0 := \max\left\{\max_{1\leq i\leq 3}|Q_i(0)|,\ \max_{1\leq i\leq 3}|Q_i(L)|\right\}.$$

The operator A takes $C_s\left(\Pi_L\right)$ into itself. Show that for a suitable choice of s (note that $L > 0$ is an arbitrary fixed number) it is a contraction operator on $S\left(\vartheta^0, r\right)$. Let us first verify that A takes the set $S\left(\vartheta^0, r\right)$ into itself; i.e., the condition $\vartheta(z,t) \in S\left(\vartheta^0, r\right)$ implies that $A\vartheta \in S\left(\vartheta^0, r\right)$, if s satisfies some constraints. Indeed, given $(z,t) \in \Pi_L$ and $\vartheta \in S\left(\vartheta^0, r\right)$, we have

$$\left|\left(A_i^1\vartheta - \vartheta_i^{01}\right)e^{-st}\right| \leq \left[(\Upsilon_0\widetilde{\phi}_0 + b_0)\|\vartheta\|_s + \Upsilon_0\|\vartheta\|_s^2\tau\right]\int_0^t e^{-s(t-\tau)}d\tau \leq \frac{1}{s}\left((\Upsilon_0\widetilde{\phi}_0 + b_0) + \Upsilon_0 L r_0\right)r_0 := \frac{1}{s}\alpha_1.$$

$$\left|\left(A_i^2\vartheta - \vartheta_i^{02}\right)e^{-st}\right| \leq \frac{Q_0}{sP_0}\left(c_0\left(6b_0 + 2\Upsilon_0\widetilde{\phi}_0\right) + \Upsilon_0\Gamma_0 + \Upsilon_0 L r_0\right)r_0 := \frac{1}{s}\alpha_2,$$

$$\left|\left(A_i^3\vartheta - \vartheta_i^{03}\right)e^{-st}\right| \leq \frac{1}{s}\left[6b_0 + 3\Upsilon_0\Gamma_0 + \Upsilon_0 h_0 + \Upsilon_0 L r_0\right]r_0 := \frac{1}{s}\alpha_3.$$

These together with (37) and (38)–(42) imply the estimates

$$\|A\vartheta - \vartheta^0\|_s = \max\left\{\max_{1\leq i\leq 3}\sup_{(x,t)\in\Pi_L}\left|\left(A_i^1\vartheta - \vartheta_i^{01}\right)e^{-st}\right|,\ \max_{1\leq i\leq 2}\sup_{t\in[0,L]}\left|\left(A_i^2\vartheta - \vartheta_i^{02}\right)e^{-st}\right|,\right.$$

$$\left.\max_{1\leq i\leq 3}\sup_{(x,t)\in\Pi_L}\left|\left(A_i^3\vartheta - \vartheta_i^{03}\right)e^{-st}\right|\right\} \leq \frac{1}{s}\alpha_0,$$

where $\alpha_0 := \max\left(\alpha_1, \alpha_2,\ \alpha_3\right)$. Choosing $s > (1/r)\alpha_0$, we obtain that A takes $S\left(\vartheta^0, \rho\right)$ into itself.

Now, take $\vartheta, \widetilde{\vartheta} \in S\left(\vartheta^0, r\right)$ and estimate the norm of the difference $U\vartheta - U\widetilde{\vartheta}$. Using the obvious inequality

$$\left|\vartheta_i^k\vartheta_i^l - \widetilde{\vartheta}_i^k\widetilde{\vartheta}_i^l\right|e^{-st} \leq \left|\vartheta_i^k - \widetilde{\vartheta}_i^k\right|\left|\vartheta_i^l\right|e^{-st} + \left|\widetilde{\vartheta}_i^k\right|\left|\vartheta_i^l - \widetilde{\vartheta}_i^l\right|e^{-st} \leq 2r_0\|\vartheta - \widetilde{\vartheta}\|_s$$

and estimates for the integrals analogous to those above, we arrive at

$$\left|\left(A_i^1\vartheta - A_i^1\widetilde{\vartheta}\right)e^{-st}\right| \leq \frac{1}{s}\left((\Upsilon_0\widetilde{\phi}_0 + b_0) + 2\Upsilon_0 L r_0\right)\|\vartheta - \widetilde{\vartheta}\|_s := \frac{1}{s}\beta_1\|\vartheta - \widetilde{\vartheta}\|_s,$$

$$\left|\left(A_i^2\vartheta - A_i^2\widetilde{\vartheta}\right)e^{-st}\right| \leq \frac{Q_0}{sP_0}\left(c_0\left(6b_0 + 2\Upsilon_0\widetilde{\phi}_0\right) + \Upsilon_0\Gamma_0 + 2\Upsilon_0 L r_0\right)\|\vartheta - \widetilde{\vartheta}\|_s := \frac{1}{s}\beta_2\|\vartheta - \widetilde{\vartheta}\|_s,$$

$$\left|\left(A_i^3\vartheta - A_i^3\widetilde{\vartheta}\right)e^{-st}\right| \leq \frac{1}{s}\left[6b_0 + 3\Upsilon_0\Gamma_0 + \Upsilon_0 h_0 + 2\Upsilon_0 L r_0\right)\|\vartheta - \widetilde{\vartheta}\|_s := \frac{1}{s}\beta_3\|\vartheta - \widetilde{\vartheta}\|_s.$$

Hence,

$$\|A\vartheta - A\widetilde{\vartheta}\|_s = \max\left\{\max_{1\leq i\leq 3}\sup_{(x,t)\in\Pi_L}\left|\left(A_i^1\vartheta - A_i^1\widetilde{\vartheta}\right)e^{-st}\right|,\ \max_{1\leq i\leq 2}\sup_{t\in[0,L]}\left|\left(A_i^2\vartheta - A_i^2\widetilde{\vartheta}\right)e^{-st}\right|,\right.$$

$$\left.\max_{1\leq i\leq 3}\sup_{(x,t)\in\Pi_L}\left|\left(A_i^3\vartheta - A_i^3\widetilde{\vartheta}\right)e^{-st}\right|\right\} \leq \frac{1}{s}\beta_0\|\vartheta - \widetilde{\vartheta}\|_s,$$

where $\beta_0 := \max(\beta_1,\ \beta_2,\ \beta_3)$.

Now, choosing $s > \beta_0$, we conclude that A contracts the distance between $\vartheta, \widetilde{\vartheta}$ by $S\left(\vartheta^0, \rho\right)$.

As follows from the estimates above, if s is chosen so that $s > s^* := \max\{\alpha_0, \beta_0\}$, then A is a contraction on $S\left(\vartheta^0, \rho\right)$. In this event, by the Banach Principle (see [36], pp. 87–97), equation (37) has a unique solution in $S\left(\vartheta^0, \rho\right)$ for every fixed $L > 0$. Theorem 2 is proved.

Knowing $\varphi_i'(t),\ i = 1,2$ we can find the functions $\varphi_i(t),\ i = 1,2$:

$$\varphi_i(t) = \varphi_i(0) + \int_0^t \varphi_i'(\tau)d\tau,\ i = 1,2.$$

CONCLUSION

The system of two-dimensional integro-differential acoustic equations is reduced to normal form, the inverse problem of determining the kernels for the system of two-dimensional integro-differential acoustic equations in normal form is posed and studied, and existence and uniqueness theorems for the solution of the inverse problem are also proved in the class of continuous functions with exponential weight.

ACKNOWLEDGMENTS

We would like to thank our colleagues at Bukhara State University and the V.I.Romanovskiy Institute of Mathematics for making convenient research facilities.

REFERENCES

1. V. Volterra, *Theory of functionals and of integral and integro-differential equations* (London, Dover publication, 1959).
2. M. Toshio, *Micromechanics of defects in solids, Second, Revised Edition* (IL, USA, Northwestern University, Evanston, 1987).
3. L. D. Landau and E. M. Lifshitz, *Electrodynamics of continuous media* (New York, Pergamon Press, 1984).
4. L. P. Nizhnik and V. G. Tarasov, "The inverse nonstationary scattering problem for a hyperbolic system of equations," Dokl. Akad. Nauk SSSR **233:3**, 300–303 (1977).
5. S. P. Belinskii, "A certain inverse problem for linear symmetric t-hyperbolicsystems with n+1 independent variables," Differential Equations **12:1**, 15–23 (1976).

6. S. P. Belinskii and V. Romanov, "Determination of the coefficients of a t-hyperbolic system," Math. Notess **28:4**, 723–727 (1980).
7. V. G. Romanov, "Uniqueness of the solution of an inverse problem for first-order hyperbolic systems," Math. Notes **24:3**, 692–695 (1978).
8. V. G. Romanov, "On the problem of determining the right hand side of a hyperbolic system," Math. Notes **13:3**, 343–348 (1977).
9. Z. Totieva, "The problem of determining the matrix kernel of the anisotropic viscoelasticity equations system," Vladikavkaz Math. Jour. **21:2**, 58–66 (2019).
10. D. K. Durdiev and Z. D. Totieva, "The problem of determining the one-dimensional kernel of the viscoelas- ticity equation," Sib. Zh. Ind. Mat. **16:2**, 72–82 (2013).
11. D. K. Durdiev and A. A. Rakhmonov, "Inverse problem for a system of integro-differential equations for sh waves in a visco-elastic porous medium: Global solvability," Theoret. and Math. Phys. **195:3**, 923–937 (2018).
12. U. Durdiev, "Problem of determining the reaction coefficient in a fractional diffusion equation," Differential Equations **59**, 1195–1204 (2021).
13. U. Durdiev, "Inverse problem of determining an unknown coefficient in the beam vibration equation," Differential Equations **58**, 36–43 (2022).
14. A. Lorenzi, "An identification problem related to a nonlinear hyperbolic integrodifferential equation," Nonlinear Anal., Theory, Methods Appl. **22:1**, 21–44 (1994).
15. V. Romanov, "Problem of determining the permittivity in the stationary system of maxwell equations," Dokl. Math. **95:3**, 230–234 (2017).
16. Z. S. Safarov and D. K. Durdiev, "Inverse problem for an integro-differential equation of acoustics," Differential Equations **54:1**, 134–142 (2018).
17. J. Janno and L. V. Wolfersdorf, "Inverse problems for identification of memory kernels in viscoelasticity," Math. Methods Appl. Sci. **20:4**, 291–314 (1997).
18. V. G. Romanov, "Stability estimates for the solution to the problem of determining the kernel of a viscoelastic equation," Journal of Applied and Industrial Mathematics **6:3**, 60–370 (2012).
19. D. K. Durdiev and Z. D. Totieva, "The problem of determining the one-dimensional matrix kernel of the system of viscoelasticity equations," Mathematical Methods in the Applied Sciences **41:17**, 8019–8032 (2018).
20. Z. D. Totieva and D. K. Durdiev, "The problem of finding the one-dimensional kernel of the thermoviscoelasticity equation," Mathematical Methods in the Applied Sciences **103:1-2**, 118–132 (2018).
21. D. K. Durdiev and Z. S. Safarov, "Inverse problem of determining the one-dimensional kernel of the viscoelasticity equation in a bounded domain," Math. Notes **97:6**, 867–877 (2015).
22. D. K. Durdiev and Z. D. Totieva, "The problem of determining the multidimensional kernel of the viscoelasticity equation," Vladikavkaz. Mat. Zh. **17:4**, 18–43 (2015).
23. D. K. Durdiev, "Some multidimensional inverse problems of memory determination in hyperbolic equations," Journal of Applied and Industrial Mathematics **3:4**, 411–423 (2007).
24. V. G. Romanov, "Problem of kernel recovering for the viscoelasticity equation," Doklady Mathematics **86:2**, 608–610 (2012).
25. V. Romanov, "Inverse problems for equation with a memory," Eurasian Jour. of Math. and Computer Applications **V. 2, 4**, 51 – 80 (2014).
26. D. K. Durdiev and Z. R. Bozorov, "A problem of determining the kernel of integrodifferential wave equation with weak horizontal properties," Dal'nevost. Mat.Zh. **13:2**, 209–221 (2013).
27. D. K. Durdiev and A. A. Rahmonov, "A 2d kernel determination problem in a visco-elastic porous medium with a weakly horizontally inhomogeneity," Mathematical Methods in the Applied Sciences **43:15**, 8776–8796 (2020).
28. D. K. Durdiev and A. A. Rakhmonov, "The problem of determining the 2d-kernel in a system of integro-differential equations of a viscoelastic porous medium," J. Appl. Industr. Math. **14:2**, 281–295 (2020).
29. S. Godunov, *Equations of Mathematical Physics* (Moscow:Nauka, 1979).
30. F. Gantmacher, *Theory of Matrices* (Nauka, Moscow, 1988).
31. V. Romanov, *Inverse problems of mathematical physics* (Utrecht, The Netherlands, 1987).
32. D. K. Durdiev and K. K. Turdiev, "The problem of finding the kernels in the system of integro-differential maxwell's equations," J. Appl. Industr. Math. **15:2**, 190–211 (2021).
33. D. K. Durdiev and K. K. Turdiev, "Inverse problem for a first-order hyperbolic system with memory," Differential Equations **56**, 1634–1643 (2020).
34. A. Kilbas, *Integral Equations: Course of Lectures* (Minsk, Belarusian State University, 2005).
35. A. Bukhgeym, *Volterra equations and inverse problems* (DeGruyter, 1999).
36. A. N. Kolmogorov and S. V. Fomin, *Elements of the Theory of Functions and Functional Analysis* (Nauka, Moscow, 1989).