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INTEGRAL AND INTEGRO-DIFFERENTIAL EQUATIONS

Inverse Problem for a First-Order Hyperbolic System with Memory

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Abstract—For a first-order hyperbolic system of integro-differential equations with a convolution-type integral term, we study the inverse problem of determining the convolution kernel. The direct problem is an initial–boundary value problem for this system on a finite interval $[0, H]$. Under some data consistency conditions, the inverse problem is reduced to a system of Volterra type integral equations. Further, the contraction mapping principle is applied to this system, and a theorem on the unique local solvability of the problem is proved for sufficiently small H.

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INTRODUCTION

Many physical processes are described by first-order hyperbolic systems of partial differential equations, including, for example, hyperbolic systems of equations of acoustics, electromagnetic oscillations, dynamic equations of the theory of elasticity, and many others. As a rule, second-order equations are derived from these systems under some additional restrictions. Therefore, it seems quite natural to study inverse problems for such systems directly in terms of the system itself. Systematic studies of inverse problems for hyperbolic systems began in the 1970s, the papers [1–3] being the first to appear.

Processes or systems with memory (with aftereffect) or, as they are also called, hereditary processes or systems are characterized by the fact that the change in their state at each time depends on the prehistory of the process, i.e., on a continuous or discrete set of their states preceding the current one. An example of such a process is the deformation of a viscoelastic medium, which depends not only on the nature of the forces applied but also on the previous deformations to which the medium was subjected. Such an medium is called a medium with "memory" or with an "aftereffect" [4, p. 11]. Other examples of processes of this kind are the propagation of electromagnetic waves in media with dispersion [5, Ch. 9] and the dynamics of coexistence and development of populations of animals or plants of various species competing in the struggle for existence [6, Ch. 5, Sec. 7].

Mathematically, such processes are usually described by a hyperbolic system of first-order integro-differential equations with an integral term of convolution type with respect to the time variable. Inverse problems, in which the goal is to determine the integral kernels in these systems, play an important role in applied sciences. So far, the problems of determining the integral kernels in one second-order integro-differential equation have been studied quite extensively (see, e.g., the papers [7–14] and the bibliography therein).

In the present paper, we study the inverse problem of determining the integral kernel in a first-order hyperbolic system of integro-differential equations. The kernel is an $n \times n$ matrix depending on the time variable t. In view of the nonlinearity of the problem, we obtain a local theorem on the existence and uniqueness of the solution; i.e., we prove that each component of the kernel can be uniquely reconstructed on a sufficiently small time interval based on the data of the inverse problem.

1. STATEMENT OF THE PROBLEM

In the domain $D = \{(x,t) : 0 < x < H,\, t > 0\}$, consider the system of n equations

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = \int\limits_0^t B(\tau)u(x,t-\tau)\,d\tau + f(x,t) \tag{1}$$

for a vector function $u(x,t) = (u_1(x,t), u_2(x,t), \ldots, u_n(x,t))^{\mathrm{T}}$; here $f(t,x)$ is a vector function, $f(x,t) = (f_1(x,t), f_2(x,t), \ldots, f_n(x,t))^{\mathrm{T}}$, and A and B are $n\times n$ matrices with

$$A = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n),$$

where λ_k, $k = 1, \ldots, n$, are real pairwise distinct constants such that for some $s \in \{0, 1, \ldots, n\}$ one has the inequalities

$$\lambda_k > 0, \quad k = 1, \ldots, s; \quad \lambda_k < 0, \quad k = s+1, \ldots, n. \tag{2}$$

Consider system (1) with the initial conditions

$$u(x,0) = \varphi(x), \quad 0 \le x \le H, \tag{3}$$

and the boundary conditions

$$\begin{aligned} u_i(0,t) &= g_i(t), \qquad i = 1, \ldots, s, \\ u_i(H,t) &= g_i(t), \qquad i = s+1, \ldots, n. \end{aligned} \tag{4}$$

Problem (1)–(4) is well-posed [15, p. 164; 16, Ch. 2, Sec. 4].

Now assume that we consider n problems of the form (1)–(4), each with its own tuple of functions $f = f^l$, $\varphi = \varphi^l$, and $g_i = g_i^l$ $(l = 1, \ldots, n)$ but with the same matrices A and B,

$$\begin{aligned} \left(E\frac{\partial}{\partial t} + A\frac{\partial}{\partial x}\right)u^l &= \int\limits_0^t B(\tau)u^l(x,t-\tau)\,d\tau + f^l(x,t), \\ u^l|_{t=0} &= \varphi^l(x), \qquad 0 \le x \le H, \\ u_i^l(0,t) &= g_i^l(t), \qquad i = 1, \ldots, s, \\ u_i^l(H,t) &= g_i^l(t), \qquad i = s+1, \ldots, n, \quad l = 1, \ldots, n, \end{aligned} \tag{5}$$

where E is the $n\times n$ identity matrix and $u^l = u^l(x,t) = (u_1^l(x,t), u_2^l(x,t), \ldots, u_n^l(x,t))^{\mathrm{T}}$.

Inverse problem. Find the matrix $B(t) = (b_{ij})(t)$, $t > 0$, $i,j = 1, \ldots, n$, given the following information about the solutions of problem (5):

$$\begin{aligned} u_i^l(0,t) &= h_i^l(t), \qquad t > 0, \qquad i = s+1, \ldots, n, \\ u_i^l(H,t) &= h_i^l(t), \qquad t > 0, \qquad i = 1, \ldots, s, \quad l = 1, \ldots, n. \end{aligned} \tag{6}$$

Note that for each l the initial–boundary value problem (5) has a unique continuous solution in the domain D if $\varphi^l \in C[0,H]$, $g_i^l \in C[0,\infty)$, and $B \in C[0,\infty)$ and the following consistency conditions for the initial and boundary data are satisfied:

$$\begin{aligned} \varphi_i^l(0) &= g_i^l(0), \qquad i = 1, \ldots, s, \\ \varphi_i^l(H) &= g_i^l(0), \qquad i = s+1, \ldots, n, \quad l = 1, \ldots, n, \end{aligned}$$

where φ_i^l is the ith component of the vector φ^l; i.e., $\varphi^l = \varphi^l(x) = (\varphi_1^l(x), \varphi_2^l(x), \ldots, \varphi_n^l(x))^{\mathrm{T}}$.

It can readily be seen that the matrix $B(t)$ contains as many unknown functions as there are functions given in the additional conditions (6) (their number is n^2). Further investigation shows (see Sec. 3) that the entries $b_{il}(t)$ of the matrix $B(t)$ are uniquely determined on the interval $t \in [0, H/|\lambda_i|]$ for a sufficiently small H based on the data (6) of the inverse problem, i.e., the functions $h_i^l(t)$, $i,l = 1, \ldots, n$, defined only on the interval $t \in [0, H/|\lambda_i|]$. This local nature of the dependence between the unknown and given functions follows from the hyperbolicity of system (1).

Remark 1. Instead of system (1), we could consider a more general Petrovskii hyperbolic (see [17, Ch. 2, Sec. 11]) system of equations, because such a system can be reduced by a nonsingular transformation to the form (1) [18, Ch. 5, Sec. 5].

2. AUXILIARY INITIAL–BOUNDARY VALUE PROBLEM

To study the inverse problem, it is convenient to introduce the functions

$$w^l(x,t)=\frac{\partial^2}{\partial t^2}u^l(x,t),\quad l=1,\ldots,n.$$

First, let us study the properties of the functions $v^l(x,t)=\dfrac{\partial}{\partial t}u^l(x,t)$. To produce a problem satisfied by the vector functions v^l, $l=1,\ldots,n$, we differentiate the equation in problem (5) with respect to t, and then, by differentiating the boundary condition of problem (5) with respect to t, we find boundary conditions for the functions v^l, with the initial conditions (at $t=0$) for the function v^l obtained using the equations and the initial conditions of problem (5). As a result, we conclude that each function v^l, $l=1,\ldots,n$, is a solution of the problem

$$\begin{gathered}\left(E\frac{\partial}{\partial t}+A\frac{\partial}{\partial x}\right)v^l=B(t)\varphi^l(x)+\int\limits_0^t B(\tau)v^l(x,t-\tau)\,d\tau+\frac{\partial}{\partial t}f^l(x,t),\\ \begin{aligned} v^l(x,0)&=f^l(x,0)-A\frac{d}{dx}\varphi^l(x), && 0\le x\le H,\\ v_i^l(0,t)&=\frac{d}{dt}g_i^l(t), && i=1,\ldots,s,\\ v_i^l(H,t)&=\frac{d}{dt}g_i^l(t), && i=s+1,\ldots,n.\end{aligned}\end{gathered}\tag{7}$$

By $C^{0,k}(D)$ we denote the class of functions continuous in the domain D and having the first k continuous partial derivatives with respect to t. It is obvious that for the solution of problem (7) to be continuous in the domain D for each l, it suffices that the conditions

$$f^l\in C^{0,1}(D),\quad \varphi^l\in C^1[0,H],\quad g_i^l\in C^1[0,\infty),\quad B\in C[0,\infty)$$

and the consistency conditions

$$\begin{aligned} f_i^l(0,0)-\lambda_i\left[\frac{d\varphi_i^l(x)}{dx}\right]\bigg|_{x=0}&=\left[\frac{dg_i^l(t)}{dt}\right]\bigg|_{t=0}, && i=1,\ldots,s,\\ f_i^l(H,0)-\lambda_i\left[\frac{d\varphi_i^l(x)}{dx}\right]\bigg|_{x=H}&=\left[\frac{dg_i^l(t)}{dt}\right]\bigg|_{t=0}, && i=s+1,\ldots,n,\quad l=1,\ldots,n,\end{aligned}$$

be satisfied.

Let us produce a problem satisfied by the functions w^l, $l=1,\ldots,n$. To this end, we differentiate the equation in problem (7) with respect to t and take its initial condition into account in this solution. Then, differentiating the boundary conditions of problem (7) with respect to t, we obtain the boundary conditions for the function w^l, with the initial conditions (at $t=0$) for the function w^l found using the equation in problem (7) and differentiating its initial conditions with respect to x. As a result, we arrive at the problem

$$\begin{aligned}\left(E\frac{\partial}{\partial t}+A\frac{\partial}{\partial x}\right)w^l={}&B'(t)\varphi^l(x)+B(t)\left[f^l(x,0)-A\frac{d}{dx}\varphi^l(x)\right]\\ &+\int\limits_0^t B(\tau)w^l(x,t-\tau)\,d\tau+\frac{\partial^2}{\partial t^2}f^l(x,t),\end{aligned}\tag{8}$$

$$w^l(x,0) = \frac{\partial}{\partial t} f^l(x,0) + B(0)\varphi^l(x) - A\frac{\partial}{\partial x} f^l(x,0) + A^2 \frac{d^2}{dx^2}\varphi^l(x),$$
$$w_i^l(0,t) = \frac{d^2}{dt^2} g_i^l(t), \qquad i = 1,\dots,s,$$
$$w_i^l(H,t) = \frac{d^2}{dt^2} g_i^l(t), \qquad i = s+1,\dots,n.$$

Consider an arbitrary point $(x,t) \in D$ in the plane of the variables ξ, τ (where ξ and τ are the current coordinates of the point), and draw the characteristic L_i corresponding to the value λ_i through this point, which we continue to the intersection at $\tau \le t$ with the boundary of the domain D. We denote the points of intersection by (x_0^i, t_0^i). For the values $i = 1,\dots,s$, the point (x_0^i, t_0^i) lies on the bottom base or on the left lateral side of the half-strip D, and for the values $i = s+1,\dots,n$, on the bottom base or on the right lateral side of D, depending on the numbers λ_i and the point (x,t). We integrate the ith component of the equation in problem (8) along the characteristic L_i from the point (x_0^i, t_0^i) to the point (x,t) and, by calculating the integral by parts in the second term of the resulting equality, find

$$\begin{aligned} w_i^l(x,t) = {} & w_i^l(x_0^i,t_0^i) + \sum_{j=1}^{n}[b_{ij}(t)\varphi_j^l(x) - b_{ij}(t_0^i)\varphi_j^l(x+\lambda_i(t_0^i - t))] \\ & + \int_{t_0^i}^{t} \Bigg[\frac{d^2}{d\tau^2} f_i^l(\xi,\tau) + \sum_{j=1}^{n} b_{ij}(\tau)\bigg(f_j^l(\xi,0) - 2\lambda_i \frac{d}{d\xi}\varphi_j^l(\xi)\bigg) \\ & + \int_0^{\tau} b_{ij}(\eta) w_j^l(\xi, \tau-\eta)\, d\eta\Bigg]_{\xi = x+\lambda_i(\tau - t)} d\tau, \end{aligned} \tag{9}$$

$(x,t) \in D$, $i = 1,\dots,n$, where $w_i^l(x_0^i,t_0^i)$ assumes either the initial or the boundary value in (8), depending on the part of the boundary of the domain D where the point (x_0^i,t_0^i) lies. These equations, together with the initial and boundary conditions, determine a closed system of equations in any domain $G(x_1,t_1) = \{(x,t) : (x,t) \in D,\ \tau \le t_1 - |x - x_1|/\lambda_0\}$. Here (x_1,t_1) is an arbitrary point of the domain D, and $\lambda_0 = \max\{|\lambda_i| : 1 \le i \le n\}$.

Let $\Phi(x) = (\varphi^1, \varphi^2, \dots, \varphi^n)(x)$ be the matrix formed by the columns $\varphi^l(x)$, $l = 1,\dots,n$. In what follows, we assume the condition

$$\det\Phi(0) \ne 0, \quad \det\Phi(H) \ne 0 \tag{10}$$

to be satisfied. To find the entries of the matrix $B(t)$ at $t = 0$, which is in particular required in the inverse problem, we use the conditions of continuity of the functions w_i^l, $i = 1,\dots,s$, at the point $(0,0)$ and the functions w_i^l, $i = s+1,\dots,n$, at the point $(H,0)$. To this end, from the initial and boundary conditions in (8) we obtain the relations

$$\begin{gathered}\big[B(0)\varphi^l(0)\big]_i = \frac{d^2}{dt^2} g_i^l(0) + \lambda_i f_i^l(0,0) - \lambda_i \frac{d^2}{dx^2}\varphi_i^l(0) - \frac{\partial}{\partial t} f_i^l(0,0), \\ i = 1,\dots,s,\end{gathered} \tag{11}$$

$$\begin{gathered}\big[B(0)\varphi^l(0)\big]_i = \frac{d^2}{dt^2} g_i^l(0) + \lambda_i f_i^l(H,0) - \lambda_i \frac{d^2}{dx^2}\varphi_i^l(H) - \frac{\partial}{\partial t} f_i^l(H,0), \\ i = s+1,\dots,n; \quad i = 1,\dots,n,\end{gathered} \tag{12}$$

where $[B(0)\varphi^l(0)]_i$ is the ith component of the product $B(0)\varphi^l(0)$; that is, $[B(0)\varphi^l(0)]_i = \sum_{j=1}^{n} b_{ij}(0)\varphi_j^l(0)$. Under conditions (10), relations (11) permit determining the values $b_{il}(0)$ for $i = 1,\dots,s$ and $l = 1,\dots,n$, and the relations (12), the values $b_{il}(0)$ for $i = s+1,\dots,n$ and $l = 1,\dots,n$. They have the form

$$\begin{gathered} b_{il}(0) = \frac{1}{\det\Phi(0)} \sum_{j=1}^{n}\left[\frac{d^2}{dt^2} g_i^j(0) + \lambda_i f_i^j(0,0) - \lambda_i \frac{d^2}{dx^2}\varphi_i^j(0) - \frac{\partial}{\partial t} f_i^j(0,0)\right]\Psi_j^l(0), \\ i = 1,\dots,s; \end{gathered} \tag{13}$$

$$b_{il}(0)=\frac{1}{\det\Phi(H)}\sum_{j=1}^{n}\left[\frac{d^2}{dt^2}g_i^j(0)+\lambda_i f_i^j(H,0)-\lambda_i\frac{d^2}{dx^2}\varphi_i^j(H)-\frac{\partial}{\partial t}f_i^j(H,0)\right]\Psi_j^l(H),\qquad(14)$$
$$i=s+1,\dots,n,\quad l=1,\dots,n,$$

where $\Psi_j^l(\cdot)$ is the cofactor of the entry $\varphi_l^j(\cdot)$ in the matrix $\Phi(\cdot)$.

In what follows, we assume that the numerical matrix $B(0)$ is known and its entries are determined by relations (13) and (14). Note that Eqs. (9) are integral equations of Volterra type of the second kind with continuous kernels and free terms. It is well known that such equations have continuous solutions.

We have thus proved the following assertion.

Lemma. *Let the matrix $B(t)$ be continuously differentiable on the interval $[0,\infty)$, and let the inclusions $f^l(x,t)\in C^2(D)$, $\varphi^l(x)\in C^2[0,H]$, and $g^l(t)\in C^2(0,\infty)$ hold. In addition, assume that conditions (10) are satisfied and so are relations (13) and (14). Then for each l there exists a unique solution of the initial–boundary value problem (8) continuous in the closed domain $\bar{D}$.*

3. STUDYING THE INVERSE PROBLEM. MAIN RESULT

Let us proceed to the analysis of the inverse problem.

It can readily be seen that for the inverse problem (5), (6) to be solvable, under the assumptions of the lemma, it is necessary that the functions $h_i^l(t)\in C^1(0,\infty)$ satisfy the conditions of consistency with the initial data,

$$h_i^l(0)=\varphi_i^l(0),\quad \left[\frac{dh_i^l(t)}{dt}\right]_{t=0}=f_i^l(0,0)-\lambda_i\left[\frac{d\varphi_i^l(x)}{dx}\right]_{x=0},\qquad(15)$$
$$i=s+1,\dots,n,$$

$$h_i^l(0)=\varphi_i^l(H),\quad \left[\frac{dh_i^l(t)}{dt}\right]_{t=0}=f_i^l(H,0)-\lambda_i\left[\frac{d\varphi_i^l(x)}{dx}\right]_{x=H},\qquad(16)$$
$$i=1,\dots,s,\quad l=1,\dots,n.$$

The main result of the present paper is given by the following assertion.

Theorem. *Let the assumptions of the lemma, the inclusions $h_i^l(t)\in C^2(0,\infty)$, and relations (15) and (16) be satisfied. Then there exists an $H^*>0$ such that for $H\in(0,H^*)$ there exists a unique solution of the inverse problem (5), (6) on the interval $[0,H/\mu]$, where $\mu=\min_{1\le i\le n}|\lambda_i|$, and this solution is determined by the functions $h_i^l(t)$ on the intervals $t\in[0,H/|\lambda_i|]$.*

Proof. Consider an arbitrary point $(x,0)\in D$ and draw the characteristic L_i through this point until the intersection with the lateral boundaries of the half-strip D. By integrating the ith component of the first equation in (8) and by differentiating the data in (6) twice, we find that

$$w_i^l(x,0)-\frac{d^2}{dt^2}h_i^l(t_i(x))=\sum_{j=1}^{n}\int_0^{t_i(x)}\frac{d}{d\tau}b_{ij}(\tau)\varphi_j^l(x+\lambda_i\tau)\,d\tau+\int_0^{t_i(x)}\Bigg\{\frac{\partial^2}{\partial\tau^2}f_i^l(\tau,\xi)$$
$$+\sum_{j=1}^{n}\left[b_{ij}(\tau)\left(f_j^l(\xi,0)-\lambda_i\frac{d}{d\xi}\varphi_j^l(\xi)\right)+\int_0^{\tau}b_{ij}(\eta)w_j^l(\xi,\tau-\eta)\,d\eta\right]_{\xi=x+\lambda_i\tau}\Bigg\}\,d\tau,\qquad(17)$$
$$x\in[0,H],\quad i,l=1,\dots,n,$$

where

$$t_i(x)=\frac{1}{|\lambda_i|}\begin{cases}H-x, & i=1,\dots,s,\\ x, & i=s+1,\dots,n.\end{cases}$$

Integrating by parts in the first term on the right-hand side in Eq. (17), after simple transformations we obtain

$$\sum_{j=1}^{n} b_{ij}(t)\varphi_j^l\big(\bar t_i(t)+\lambda_i t\big) = w_i^l\big(\bar t_i(t),0\big) - \frac{d^2}{dt^2}h_i^l(t) + \sum_{j=1}^{n} b_{ij}(0)\varphi_j^l\big(\bar t_i(t)\big) - \int_0^t \bigg\{ \frac{\partial^2}{\partial\tau^2} f_i^l(\tau,\xi) + \sum_{j=1}^{n}\bigg[b_{ij}(\tau)\bigg(f_j^l(\xi,0) - 2\lambda_i \frac{d}{d\xi}\varphi_j^l(\xi)\bigg) + \int_0^\tau b_{ij}(\eta) w_j^l(\xi,\tau-\eta)\,d\eta \bigg]\bigg\}_{\xi=\bar t_i(t)+\lambda_i\tau} d\tau, \quad i,l=1,\dots,n, \tag{18}$$

where

$$\bar t_i(t) = \begin{cases} H-\lambda_i t, & i=1,\dots,s,\\ -\lambda_i t, & i=s+1,\dots,n.\end{cases}$$

Now, solving system (18) for the unknowns $b_{ij}(t)$, we have

$$\begin{aligned} b_{il}(t) = {} & \frac{1}{\det\Phi(\nu_i)} \sum_{j=1}^{n}\bigg[w_i^j\big(\bar t_i(t),0\big) - \frac{d^2}{dt^2}h_i^j(t) \\ & + \sum_{k=1}^{n} b_{ik}(0)\varphi_k^j\big(\bar t_i(t)\big) - \int_0^t \frac{\partial^2}{\partial\tau^2} f_i^j\big(\bar t_i(t)+\lambda_i\tau,\tau\big)\,d\tau\bigg]\Psi_j^l(\nu_i) \\ & - \frac{1}{\det\Phi(\nu_i)} \int_0^t \sum_{j=1}^{n}\sum_{k=1}^{n}\bigg[b_{ik}(\tau)\bigg(f_k^j(\xi,0) - 2\lambda_i\frac{d}{d\xi}\varphi_k^j(\xi)\bigg) \\ & + \int_0^\tau b_{ik}(\eta) w_k^j(\xi,\tau-\eta)\,d\eta\bigg]\bigg|_{\xi=\bar t_i(t)+\lambda_i\tau} d\tau\, \Psi_j^l(\nu_i), \quad i=1,\dots,n; \quad l=1,\dots,n, \end{aligned} \tag{19}$$

where

$$\nu_i = \begin{cases} H, & i=1,\dots,s,\\ 0, & i=s+1,\dots,n.\end{cases}$$

Equations (9) and (19) supplemented with the initial and boundary conditions inrelations (8) form a closed system of equations for the unknowns $w_i^l(x,t)$ and $b_{il}(t)$, $i,l=1,\dots,n$. Consider the rectangular domain

$$D(\mu) := \big\{(x,t) : 0\le x\le H,\ \ 0\le t\le H/\mu\big\}.$$

Equations (9) and (19) show that the values of the functions $w_i^l(x,t)$ and $b_{il}(t)$ for $(x,t)\in D(\mu)$ can be expressed via integrals of some combinations of the same functions over segments lying in $D(\mu)$.

Let us write Eqs. (9) and (19) in the form of a closed system of Volterra type integral equations of the second kind. To this end, we introduce vector functions $\psi(x,t) = (\psi_{il}^1, \psi_{il}^2,\ i,l=1,\dots,n)$, defining their components by the relations

$$\begin{aligned} \psi_{il}^1(x,t) &= w_i^l(x,t) - \sum_{j=1}^{n}\Big[b_{ij}(t)\varphi_j^l(x) - b_{ij}(t_0^i)\varphi_j^l\big(x+\lambda_i(t_0^i - t)\big)\Big], \\ \psi_{il}^2(x,t) &= \psi_{il}^2(t) = b_{il}(t). \end{aligned}$$

Then system (9), (19) acquires the operator-vector form

$$\psi = U\psi, \tag{20}$$

where the operator $U = (U^1_{il}, U^2_{il},\, i,l = 1,\dots,n)$, in accordance with the right-hand sides of Eqs. (9) and (19), is defined by the relations

$$
\begin{aligned}
U^1_{il}\psi &= \psi^{10}_{il}(x,t) \\
&\quad + \int_{t_i}^{t} \sum_{j=1}^{n} \Bigg\{ \psi^2_{ij}(\tau)\Big[f^l_j(\xi,0) - 2\lambda_i \frac{d}{d\xi}\varphi^l_j(\xi)\Big] + \int_0^{\tau} \psi^2_{ij}(\eta)\Big\{\psi^1_{jl}(\xi,\tau-\eta) \\
&\quad + \sum_{k=1}^{n}\Big[\psi^2_{jk}(\tau-\eta)\varphi^l_k(\xi) - \psi^2_{jk}(t_i)\varphi^l_k\big(\xi + \lambda_i\big(t_i - (\tau-\eta)\big)\big)\Big]\Big\}\,d\eta\Bigg\}\Bigg|_{\xi = x+\lambda_i(\tau - t)} d\tau, \\
U^2_{il}\psi &= \psi^{20}_{il}(x,t) \\
&\quad - \frac{1}{\det\Phi(\nu_i)} \int_0^{t} \sum_{j=1}^{n}\sum_{k=1}^{n} \Bigg[\psi^2_{ik}(\tau)\Big(f^j_k(\xi,0) - 2\lambda_i \frac{d}{d\xi}\varphi^j_k(\xi)\Big) + \int_0^{\tau}\psi^2_{ik}(\eta)\Big(\psi^1_{kj}(\xi,\tau-\eta) \\
&\quad + \sum_{p=1}^{n}\Big[\psi^2_{ip}(\tau-\eta)\varphi^j_p(\xi) - \psi^2_{ip}(t^i_0)\varphi^j_p\big(\xi + \lambda_i\big(t^i_0 - (\tau-\eta)\big)\big)\Big]\Big)\,d\eta\Bigg]\Bigg|_{\xi = \overline{t_i}(t)+\lambda_i\tau} d\tau\, \Psi^l_j(\nu_i).
\end{aligned}
\tag{21}
$$

In these formulas, we have introduced the notation

$$
\psi^{10}_{il}(x,t) = w^l_i\big(x + \lambda_i(t^i_0 - t), t^i_0\big) + \int_{t^i_0}^{t} \frac{d^2}{d\tau^2} f^l_i\big(x - \lambda_i(t-\tau),\tau\big)\,d\tau,
$$

$$
\begin{aligned}
\psi^{20}_{il}(t) = \frac{1}{\det\Phi(\nu_i)} \sum_{j=1}^{n} \Bigg[& w^j_i\big(\bar t_i(t),0\big) - \frac{d^2}{dt^2}h^j_i(t) + \sum_{k=1}^{n} b_{ik}(0)\varphi^j_k\big(\bar t_i(t)\big) \\
& - \int_0^{t} \frac{\partial^2}{\partial\tau^2} f^j_i\big(\bar t_i(t) + \lambda_i\tau,\tau\big)\,d\tau\Bigg]\Psi^l_j(\nu_i).
\end{aligned}
$$

We define a norm on the set $C(D(\mu))$ of continuous functions by the formula

$$
\|\psi\| = \max_{i,l,s}\ \sup_{(x,t)\in D(\mu)} \big|\psi^s_{il}(x,t)\big| \quad (i,l = 1,\dots,n, \quad s = 1,2)
$$

and consider the set $S(\psi^0,\rho) \subset C(D(\mu))$ of functions satisfying the inequality

$$
\|\psi - \psi^0\| \le \rho, \tag{22}
$$

where the vector function $\psi^0(x,t) = (\psi^{10}_{il}(x,t), \psi^{20}_{il}(t), i,l = 1,\dots,n)$ is determined by the free terms of the operator equation (20).

Let

$$
\begin{gathered}
\varphi_0 := \max_{1\le i,l\le n}\|\varphi^l_i\|_{C^2[0,H]}, \quad g_0 := \max_{1\le i,l\le n}\|g^l_i\|_{C^2[0,H/\mu]}, \quad f_0 := \max_{1\le i,l\le n}\|f^l_i\|_{C^2[D(\mu)]}, \\
h_0 := \max_{1\le j,l\le n}\|h^l_i\|_{C^2[0,H/\mu]}, \quad \Phi_0 := \min\Big\{\big|\Phi(0)\big|, \big|\Phi(H)\big|\Big\}, \\
\Psi_0 := \max\Big\{\max_{1\le i,l\le n}\big|\Psi^l_j(0)\big|, \max_{1\le j,l\le n}\big|\Psi^l_j(H)\big|\Big\}.
\end{gathered}
$$

The operator U takes the space $C(D(\mu))$ into itself. Let us show that for a sufficiently small H it is a contraction operator on the set $S(\psi^0,\rho)$. First, let us verify that the condition $\psi \in S(\psi^0,\rho)$ implies the inclusion $U\psi \in S(\psi^0, \rho)$ for a sufficiently small H. Indeed, as follows from inequality (22), for each $\psi \in S(\psi^0, \rho)$ one has the estimate $\|\psi\| \le \|\psi^0\| + \rho =: R$. At the same time, for any $(x,t) \in D(\mu)$ and each $\psi \in S(\psi^0, \rho)$, the following inequalities hold:

$$|U_{il}^1\psi - \psi_{il}^{10}| = \Bigg| \int\limits_{t_i(x)}^{t} \sum_{j=1}^{n} \Big\{ \psi_{ij}^2(\tau)\Big[f_j^l(\xi,0) - 2\lambda_i \frac{d}{d\xi}\varphi_j^l(\xi)\Big] + \int\limits_0^\tau \psi_{ij}^2(\eta)\Big\{\psi_{jl}^1(\xi,\tau-\eta)$$
$$+ \sum_{k=1}^{n}\Big[\psi_{jk}^2(\tau-\eta)\varphi_k^l(\xi) - \psi_{jk}^2\big(t_i(x)\big)\varphi_k^l\Big(\xi + \lambda_i\big(t_i(x) - (\tau-\eta)\big)\Big)\Big]\Big\}\,d\eta\Big\}\Big|_{\xi = x+\lambda_i(\tau-t)} d\tau\Bigg|$$
$$\le n\int\limits_0^t \big[R(f_0 + 2\lambda_0\varphi_0) + R^2(1+2n\varphi_0)\tau\big]\,d\tau \le n\left[\frac{f_0 + 2\lambda_0\varphi_0}{\mu} + \frac{HR(1+2n\varphi_0)}{2\mu^2}\right]HR,$$

$$|U_{il}^2\psi - \psi_{il}^{20}| = \Bigg| -\frac{1}{\det\Phi(\nu_i)} \int\limits_0^t \sum_{j=1}^{n}\sum_{k=1}^{n}\Big[\psi_{ik}^2(\tau)\Big(f_k^j(\xi,0) - 2\lambda_i\frac{d}{d\xi}\varphi_k^j(\xi)\Big) + \int\limits_0^\tau \psi_{ik}^2(\eta)\Big(\psi_{kj}^1(\xi,\tau-\eta)$$
$$+ \sum_{p=1}^{n}\Big[\psi_{ip}^2(\tau-\eta)\varphi_p^j(\xi) - \psi_{ip}^2(t_0^i)\varphi_p^j\Big(\xi + \lambda_i\big(t_0^i - (\tau-\eta)\big)\Big)\Big]\Big)\,d\eta\Big]\Big|_{\xi=\overline{t_i}(t)+\lambda_i\tau} d\tau\, \Psi_j^l(\nu_i)\Bigg|$$
$$\le \frac{n^2\Psi_0}{\Phi_0}\int\limits_0^t \big[R(f_0 + 2\lambda_0\varphi_0) + R^2(1+2n\varphi_0)\tau\big]\,d\tau \le \frac{n^2\Psi_0}{\Phi_0}\left[\frac{f_0+2\lambda_0\varphi_0}{\mu} + \frac{HR(1+2n\varphi_0)}{2\mu^2}\right]HR.$$

Based on this and formulas (21) and (22), we conclude that

$$\|U\psi - \psi^0\| = \max\left\{\max_{1\le i,l\le n}\sup_{(x,t)\in D(\mu)}|U_{il}^1\psi - \psi_{il}^{10}|,\ \max_{1\le i,l\le n}\sup_{t\in[0,H/\mu]}|U_{il}^2\psi - \psi_{il}^{20}|\right\}$$
$$\le \max\left(n, \frac{n^2\Psi_0}{\Phi_0}\right)\left[\frac{f_0+2\lambda_0\varphi_0}{\mu} + \frac{HR(1+2n\varphi_0)}{2\mu^2}\right]HR.$$

Therefore if by H_1 we denote the positive root of the equation (for H)

$$\max\left(n, \frac{n^2\Psi_0}{\Phi_0}\right)\left[\frac{f_0+2\lambda_0\varphi_0}{\mu} + \frac{HR(1+2n\varphi_0)}{2\mu^2}\right]HR = \rho,$$

then $\|U\psi - \psi^0\| \le \rho$ for $H \le H_1$; i.e., $U\psi \in S(\psi^0, \rho)$.

Now let us take any functions $\psi, \widetilde{\psi} \in S(\psi^0, \rho)$ and estimate the norm of the difference $U\psi - U\widetilde{\psi}$. Using the obvious inequality

$$|\psi_{il}^2\psi_{il}^1 - \widetilde{\psi}_{il}^2\widetilde{\psi}_{il}^1| \le |\psi_{il}^2 - \widetilde{\psi}_{il}^2||\widetilde{\psi}_{il}^1| + |\widetilde{\psi}_{il}^2||\psi_{il}^1 - \widetilde{\psi}_{il}^1| \le 2R\|\psi - \widetilde{\psi}\|$$

and estimates for integrals similar to the ones given above, we obtain

$$
\begin{aligned}
|U_{il}^1\psi - U_{il}^1\widetilde{\psi}| = \Bigg| &\int\limits_{t_i(x)}^{t} \sum_{j=1}^{n} \bigg\{ \big(\psi_{ij}^2(\tau) - \widetilde{\psi}_{ij}^2(\tau)\big) \Big[f_j^l(\xi,0) - 2\lambda_i \frac{d}{d\xi}\varphi_j^l(\xi) \Big] \\
&+ \int\limits_0^{\tau} \psi_{ij}^2(\eta) \bigg\{ \psi_{jl}^1(\xi, \tau-\eta) \\
&+ \sum_{k=1}^{n} \Big[\psi_{jk}^2(\tau-\eta)\varphi_k^l(\xi) - \psi_{jk}^2\big(t_i(x)\big)\varphi_k^l\Big(\xi + \lambda_i\big(t_i(x) - (\tau-\eta)\big)\Big)\Big] \bigg\} d\eta \bigg\}\bigg|_{\xi = x+\lambda_i(\tau-t)} d\tau \\
&- \int\limits_{t_i(x)}^{t} \sum_{j=1}^{n} \bigg\{ \int\limits_0^{\tau} \widetilde{\psi}_{ij}^2(\eta) \bigg\{ \widetilde{\psi}_{jl}^1(\xi, \tau-\eta) \\
&+ \sum_{k=1}^{n} \Big[\widetilde{\psi}_{jk}^2(\tau-\eta)\varphi_k^l(\xi) - \widetilde{\psi}_{jk}^2\big(t_i(x)\big)\varphi_k^l\Big(\xi + \lambda_i\big(t_i(x) - (\tau-\eta)\big)\Big)\Big] \bigg\} d\eta \bigg\}\bigg|_{\xi = x+\lambda_i(\tau-t)} d\tau \, \Bigg| \\
&\le n\|\psi - \widetilde{\psi}\| \int\limits_0^{t} \big[R(f_0 + 2\lambda_0\varphi_0) + 2R(1 + 2nR\varphi_0)\tau \big] d\tau \\
&\le n \bigg[\frac{f_0 + 2\lambda_0\varphi_0}{\mu} + \frac{H(1+2nR\varphi_0)}{\mu^2} \bigg] HR\, \|\psi - \widetilde{\psi}\|,
\end{aligned}
$$

$$
\begin{aligned}
|U_{il}^2\psi - U_{il}^2\widetilde{\psi}| = \Bigg| &-\frac{1}{\det\Phi(\nu_i)} \int\limits_0^{t} \sum_{j=1}^{n}\sum_{k=1}^{n} \bigg[\big(\psi_{ik}^2(\tau) - \widetilde{\psi}_{ik}^2(\tau)\big) \Big(f_k^j(\xi,0) - 2\lambda_i \frac{d}{d\xi}\varphi_k^j(\xi) \Big) \\
&+ \int\limits_0^{\tau} \psi_{ik}^2(\eta) \Big(\psi_{kj}^1(\xi, \tau-\eta) \\
&+ \sum_{p=1}^{n} \Big[\psi_{ip}^2(\tau-\eta)\varphi_p^j(\xi) - \psi_{ip}^2(t_0^i)\varphi_p^j\Big(\xi + \lambda_i\big(t_0^i - (\tau-\eta)\big)\Big)\Big] \Big) d\eta \bigg]\bigg|_{\xi = \overline{t_i}(t)+\lambda_i\tau} d\tau \times \Psi_j^l(\nu_i) \\
&+ \bigg\{ \frac{1}{\det\Phi(\nu_i)} \int\limits_0^{t} \sum_{j=1}^{n}\sum_{k=1}^{n} \bigg[\int\limits_0^{\tau} \widetilde{\psi}_{ik}^2(\eta) \Big(\psi_{kj}^1(\xi, \tau-\eta) \\
&+ \sum_{p=1}^{n} \Big[\widetilde{\psi}_{ip}^2(\tau-\eta)\varphi_p^j(\xi) - \widetilde{\psi}_{ip}^2(t_0^i)\varphi_p^j\Big(\xi + \lambda_i\big(t_0^i - (\tau-\eta)\big)\Big)\Big] \Big) d\eta \bigg]\bigg|_{\xi = \overline{t_i}(t)+\lambda_i\tau} d\tau \, \Psi_j^l(\nu_i) \bigg\} \Bigg| \\
&\le \frac{n^2\Psi_0}{\Phi_0} \int\limits_0^{t} \big[R(f_0 + 2\lambda_0\varphi_0) + 2R(1 + 2nR\varphi_0)\tau \big] d\tau \, \|\psi - \widetilde{\psi}\| \\
&\le \frac{n^2\Psi_0}{\Phi_0} \bigg[\frac{f_0 + 2\lambda_0\varphi_0}{\mu} + \frac{H(1+2nR\varphi_0)}{\mu^2} \bigg] HR\|\psi - \widetilde{\psi}\|.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|U\psi - U\widetilde{\psi}\| &= \max\bigg\{ \max_{1\le i,l\le n} \sup_{(x,t)\in D(\mu)} |U_{il}^1\psi - U_{il}^1\widetilde{\psi}|, \ \max_{1\le i,l\le n} \sup_{t\in[0,H/\mu]} |U_{il}^2\psi - \psi_{il}^{20}| \bigg\} \\
&\le \max\Big(n, \frac{n^2\Psi_0}{\Phi_0} \Big) \bigg[\frac{f_0 + 2\lambda_0\varphi_0}{\mu} + \frac{H(1+2nR\varphi_0)}{\mu^2} \bigg] HR\|\psi - \widetilde{\psi}\|.
\end{aligned}
$$

Therefore, if H_2 is the positive root of the equation

$$\max\left(n, \frac{n^2\Psi_0}{\Phi_0}\right)\left[\frac{f_0 + 2\lambda_0\varphi_0)}{\mu} + \frac{H(1 + 2nR\varphi_0)}{\mu^2}\right]HR = 1$$

(for H), then for $H \le H_2$ the operator U contracts the distance between the elements ψ and $\widetilde{\psi}$. Under the inequality $H < H^* = \min\{H_1, H_2\}$, the operator U is a contraction operator on the set $S(\psi^0, \rho)$. Consequently, by the Banach principle [19, pp. 87–97], Eq. (17) has a unique solution on this set. The proof of the theorem is complete.

Remark 2. We point out that the above results can be transferred to the case of a variable matrix $A(x, t)$ provided that the family of characteristics generated by this matrix is regular in a certain rigorously defined sense.

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