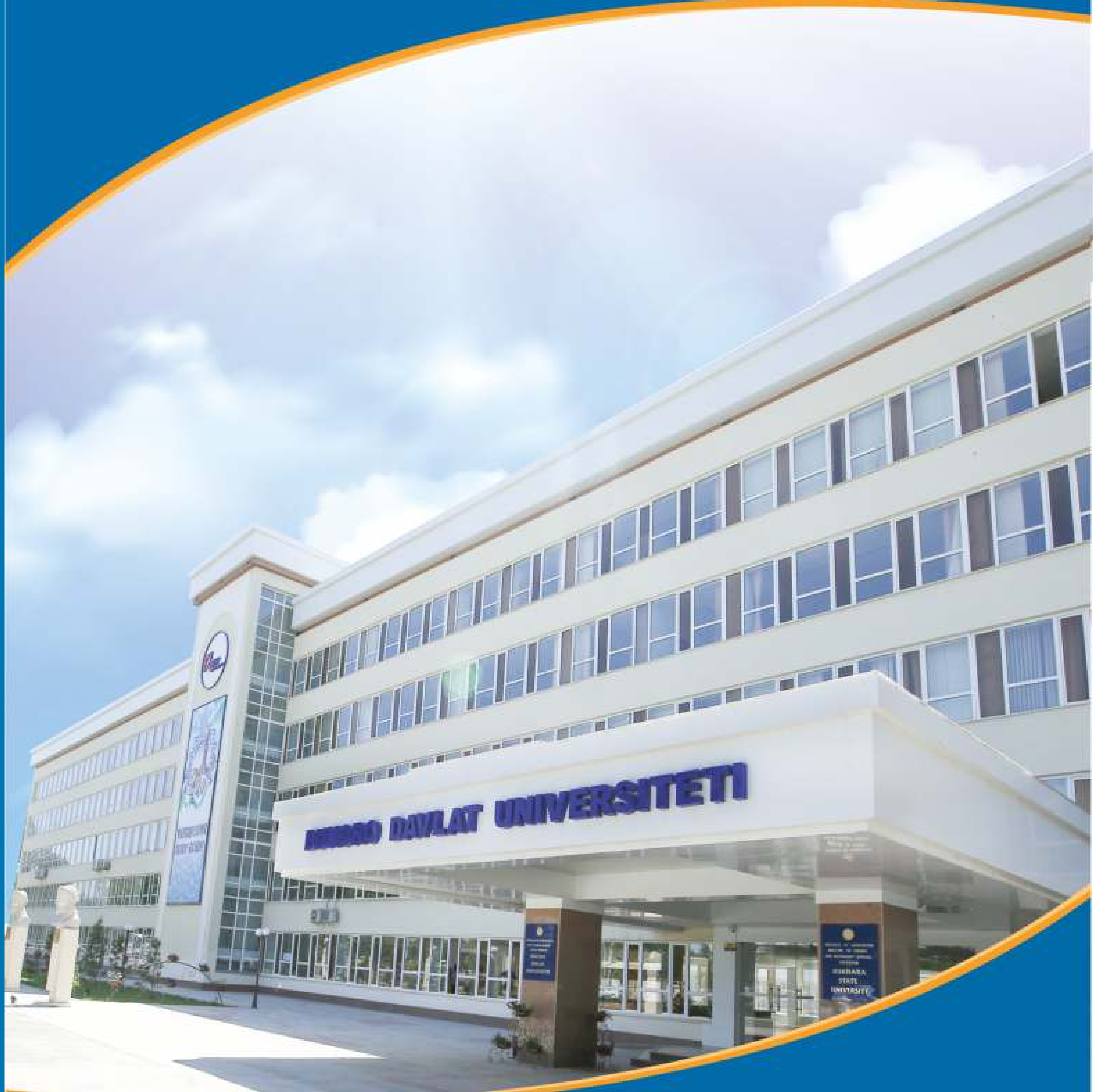
BUXORO DAVLAT UNIVERSITETI 1930

BUXORO DAVLAT UNIVERSITETI
ILMIY AXBOROTI



4/2022

Научный вестник Бухарского государственного университета
Scientific reports of Bukhara State University

4/2022

E-ISSN 2181-1466
9 772181 146004

ISSN 2181-6875
9 772181 687004

@buxdu_uz @buxdu1 @buxdu1 www.buxdu.uz

PUBLISHED SINCE 2000 (Online since 2020)

PUBLISHED SIX TIMES A YEAR

2022/4(92)

SCIENTIFIC REPORTS OF BUKHARA STATE UNIVERSITY

***

BUXORO DAVLAT UNIVERSITETI ILMIY AXBOROTI

***

НАУЧНЫЙ ВЕСТНИК БУХАРСКОГО ГОСУДАРСТВЕННОГО УНИВЕРСИТЕТА

The journal is published in the Bukhara Regional Department of Press and Information of the Press and Information Agency of Uzbekistan on August 24, 2020 With registered certificate № 1103

The journal "Scientific reports of Bukhara state university" is included in the list of scientific publications recommended to publish the main scientific results of doctoral dissertations of the Higher Attestation Commission under the Cabinet of Ministers of the Republic of Uzbekistan on philology and physical and mathematical sciences.

The journal is intended for professors and teachers of higher educational institutions, senior researchers, students, scientific staff of scientific research institutions, teachers of academic lyceums, professional colleges, as well as researchers in general secondary education and various fields.

Founder: BUKHARA STATE UNIVERSITY

CONTENTS

UDC: 517.956.47

THE PROBLEM OF DETERMINING THE ONE-DIMENSIONAL KERNEL OF THE VISCOELASTICITY SYSTEM

Turdiev Khalim Khamroyevich
Bukhara State University, Bukhara, Uzbekistan
hturdiev@mail.ru

Boltaev Asliddin Askar o'g'li
Institute of Mathematics at the Academy of Sciences
of the Republic of Uzbekistan, Bukhara, Uzbekistan
asliddinboltayev@mail.ru

Abstract.

Background. *Many physical processes are described by a system of equations of the hyperbolic type of the first order. For example, systems of an incompressible viscoelastic polymeric fluid, etc. It is well known that second-order equations are derived from them with the help of a number of additional restrictions. The solution of inverse problems leads directly to the solution of these systems. Hyperbolic systems of equations of the first order describe many physical processes associated with the propagation of waves of various nature. For example, we can indicate the systems of equations of acoustics, electromagnetic oscillations, the theory of elasticity. Systematic research in this area was carried out in the 1970s by L.P. Nizhnik, S.P. Belinsky, V.G. Romanov and L.I. Slinyuchev began in the work of scientists.*

Methods. *In this work we will use the differential equations, functional analysis, algebraic methods and also the principle of contraction mappings.*

Results. *For the reduced canonical system of integro-differential equations of viscoelasticity, direct and inverse problems of determining the velocity field of elastic waves and the relaxation matrix are posed. The problems are replaced by a closed system of integral equations of the second kind of Volterra type with respect to the Fourier transform in the variables x_1, x_2, for solving the direct problem and unknowns of the inverse problem. Further, the method of contraction mappings in the space of continuous functions with a weighted norm is applied to this system. Thus, we prove global existence and uniqueness theorems for solutions to the problems posed.*

Discussions. *The study of direct and inverse problems posed to a mixed type equation is one of the advanced critical and rapidly emerging areas of world science. Their numerical implementation provides an applied application for the study of these problems. In this paper, we numerically study the boundary value problem posed to a model equation of mixed type. To do this, you need to know the concept of approximation and stability. The stability of the difference scheme has been proven. The order of approximation is calculated in the work. Further, when the stability and approximation are proved, it is possible to show the approximation of the numerical solution to the exact solution.*

Conclusion. *To sum up, we look the kernel is 9×9 dimensional diagonal matrix that depends on time. To define that function, we put initial-boundary conditions on characteristic lines. Proved the theorem of unique solvability. We get the following results, firstly consider the inverse problem of the determination kernel in hyperbolic system of n number first-order integro–differential equations, which is of the form of 9×9 matrices depending on variable t, next obtain the theorem of exists unique solution, finally proved local theorem in a small interval.*

Keywords: *Hyperbolic system, diagonal and inverse quadratic matrices, vector functions, convolution kernel, integral equations of Volterra type, principle of contraction mappings, viscoelasticity, resolvent, inverse problem, hyperbolic system, Fourier transform.*

Introduction.

Let us denote by σ_{ij} the projection onto the x_i axis of the stress acting on the area with the normal parallel to the x_j axis, and $\overline{u}_i$ are the projection onto the x_i axis of the vector particle displacement. In viscoelastic anisotropic media, the stress tensor has the following representation[1]:

$$\sigma_{ij}(\overline{x},t) = \sum_{k,l=1}^{3} c_{ijkl}\left[S_{kl} + \int_0^t K_{ij}(t-\tau)S_{kl}(x,\tau)\right], \qquad i,j = 1,2,3, \qquad (1)$$

$$S_{ij} = \frac{1}{2}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right), \qquad i,j = 1,2,3,$$

here $c_{ijkl} = c_{ijkl}(x_3)$ are module of elasticity $K_{ij}(t)$ are functions responsible for the viscosity of the medium and $K_{ij} = K_{ji},\ i,j = \overline{1,3}$.

The equations of motion of a viscoelastic body particles in the absence of external forces have the form[2]:

$$\rho \frac{\partial^2 \overline{u}_i}{\partial t^2} = \sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j}, \quad i = \overline{1,3}, \tag{2}$$

where $\rho = \rho(x_3)$ is medium density and $\rho > 0$,

$\overline{u}(\overline{x},t) = (\overline{u}_1(\overline{x},t), \overline{u}_2(\overline{x},t), \overline{u}_3(\overline{x},t))$ is displacement vector.

Note that (1) can be considered as integral Volterra equations of the second kind with respect to the expression $\sum_{k,l=1}^{3} c_{ijkl}S_{kl}$. For each fixed pair (i,j) solving these equations, we get

$$\sigma_{ij}(\overline{x},t) = \sum_{k,l=1}^{3} c_{ijkl}S_{kl} + \int_0^t r_{ij}(t-\tau)\sigma_{ij}(\overline{x},\tau)d\tau, \tag{3}$$

where r_{ij} are the resolvents of the kernels K_{ij} and they are related by the following integral relations [3]:

$$r_{ij}(t) = -K_{ij}(t) - \int_0^t K_{ij}(t-\tau)r_{ij}(\tau)d\tau, \quad i,j = \overline{1,3}. \tag{4}$$

From the condition $K_{ij} = K_{ji}$ implies the $r_{ij} = r_{ji}$.

Differentiating (3) with respect to t and introducing the notation $u_i = \frac{\partial}{\partial t}\overline{u}_i$, we get

$$\frac{\partial}{\partial t}\sigma_{ij}(\overline{x},t) = \sum_{k,l=1}^{3} c_{ijkl}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right) +$$

$$r_{ij}(0)\sigma_{ij}(\overline{x},t) + \int_0^t r'_{ij}(t-\tau)\sigma_{ij}(\overline{x},\tau)d\tau, \qquad i,j = 1,2,3. \tag{5}$$

Let's $c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}$. The symmetry of the stress tensor reduces the number of independent elastic module from 81 to 21. If we assume that $c_{\alpha\beta} = c_{ijkl}$, where $\alpha = (ij)$ and $\beta = (kl)$, in accordance with the notation $(11) \to 1$, $(22) \to 2$, $(33) \to 3$, $(23) = (32) \to 4$, $(13) = (31) \to 5$, $(12) = (21) \to 6$, then the matrix of independent elastic module can be given the form of a 6×6 symmetrical matrix. We will consider anisotropic media with a matrix of independent elastic module of the following form[4]:

$$c_{\alpha\beta} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{c_{11}-c_{12}}{2} \end{pmatrix}.$$

Then the system of equations (1) and (2) for the velocity u_i and strain $\sigma_{ij}\left(\sigma_{ij} = \sigma_{ji}\right)$ in view of (3)-(5) can be written as a system of first-order integro-differential equations.

$$\left(A\frac{\partial}{\partial t} + B\frac{\partial}{\partial x_1} + C\frac{\partial}{\partial x_2} + D\frac{\partial}{\partial x_3} + F\right)U(\overline{x},t) = \int_0^t R(t-\tau)U(\overline{x},\tau)d\tau, \quad (6)$$

where $U = (u_1, u_2, u_3, \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{22}, \sigma_{23}, \sigma_{33})^*$, $*$ is the transposition sign,

$$A = \begin{pmatrix} \rho I_{3\times3} & O_{3\times6} \\ O_{6\times3} & I_{6\times6} \end{pmatrix},$$

$$B = \left(\begin{array}{ccc|cccccc} & O_{3\times3} & & -1 & 0 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & -1 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & -1 & 0 \\ -c_{11} & 0 & 0 & & & & & & \\ -c_{12} & 0 & 0 & & & & & & \\ -c_{13} & 0 & 0 & & & & & & \\ 0 & -c_{44} & 0 & & & & O_{6\times6} & & \\ 0 & 0 & -c_{44} & & & & & & \\ 0 & 0 & 0 & & & & & & \end{array}\right),$$

$$C = \left(\begin{array}{ccc|cccccc} & O_{3\times3} & & 0 & 0 & 0 & -1 & 0 & 0 \\ & & & 0 & -1 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -c_{12} & 0 & & & & & & \\ 0 & -c_{11} & 0 & & & & & & \\ 0 & -c_{13} & 0 & & & & & & \\ -c_{44} & 0 & 0 & & & & O_{6\times6} & & \\ 0 & 0 & 0 & & & & & & \\ 0 & 0 & \frac{c_{11}-c_{12}}{2} & & & & & & \end{array}\right),$$

$$D = \left(\begin{array}{ccc|cccccc} & O_{3\times3} & & 0 & 0 & 0 & 0 & -1 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 & -1 \\ & & & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -c_{13} & & & & & & \\ 0 & 0 & -c_{13} & & & & & & \\ 0 & 0 & -c_{33} & & & & & & \\ 0 & 0 & 0 & & & & O_{6\times6} & & \\ -c_{44} & 0 & 0 & & & & & & \\ 0 & \frac{c_{11}-c_{12}}{2} & 0 & & & & & & \end{array}\right),$$

$$F=\begin{pmatrix} 0_{3\times 3} & O_{3\times 6} & & & & & \\ & -r_{11}(0) & 0 & 0 & 0 & 0 & 0 \\ & 0 & -r_{22}(0) & 0 & 0 & 0 & 0 \\ 0_{6\times 3} & 0 & 0 & -r_{33}(0) & 0 & 0 & 0 \\ & 0 & 0 & 0 & -r_{12}(0) & 0 & 0 \\ & 0 & 0 & 0 & 0 & -r_{13}(0) & 0 \\ & 0 & 0 & 0 & 0 & 0 & -r_{23}(0) \end{pmatrix},$$

$$R=R(t)=\begin{pmatrix} O_{3\times 3} & O_{3\times 6} & & & & & \\ & r'_{11} & 0 & 0 & 0 & 0 & 0 \\ & 0 & r'_{22} & 0 & 0 & 0 & 0 \\ O_{6\times 3} & 0 & 0 & r'_{33} & 0 & 0 & 0 \\ & 0 & 0 & 0 & r'_{12} & 0 & 0 \\ & 0 & 0 & 0 & 0 & r'_{13} & 0 \\ & 0 & 0 & 0 & 0 & 0 & r'_{23} \end{pmatrix}. \qquad (7)$$

The system (6) can be reduced to a symmetric hyperbolic system [5],[6].

We reduce the system (6) to canonical form with respect to the variables t and x_3. To do this, multiply (6) on the left by A^{-1} and compose the equation

$$|A^{-1}D-\lambda I|=0, \qquad (8)$$

where I is the identity matrix of dimension 9. The last equation with respect to λ has following solutions:

$$\lambda_1(x_3)=-\sqrt{\frac{c_{33}}{\rho}},\quad \lambda_2(x_3)=-\sqrt{\frac{c_{44}}{\rho}},\quad \lambda_3(x_3)=-\sqrt{\frac{c_{11}-c_{12}}{2\rho}},\quad \lambda_{4,5,6}=0, \quad (9)$$

$$\lambda_7(x_3)=\sqrt{\frac{c_{11}-c_{12}}{2\rho}},\quad \lambda_8(x_3)=\sqrt{\frac{c_{44}}{\rho}},\quad \lambda_9(x_3)=\sqrt{\frac{c_{33}}{\rho}}. \qquad (10)$$

Now we choose a nondegenerate matrix $T(x_3,t)$ so that the equality

$$T^{-1}A^{-1}DT=\Lambda \qquad (11)$$

is hold, where Λ is a diagonal matrix, the diagonal of which contains the eigenvalues (for each fixed x_3) (9)-(10) of the matrix $A^{-1}D$ that is

$$\Lambda = diag\left(-\sqrt{\frac{c_{33}}{\rho}},-\sqrt{\frac{c_{44}}{\rho}},-\sqrt{\frac{c_{11}-c_{12}}{2\rho}},0,0,0,\sqrt{\frac{c_{11}-c_{12}}{2\rho}},\sqrt{\frac{c_{44}}{\rho}},\sqrt{\frac{c_{33}}{\rho}}\right).$$

From the formula (11) implies the equality

$$A^{-1}DT=T\Lambda,$$

which means that the column with the number i of the matrix T is an eigenvector of the matrix $A^{-1}DT$, corresponding to the eigenvalue λ_i. Direct calculations show that the matrix T, satisfying the above conditions, can be chosen as (not uniquely)

$$T(x_3) =$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -c_{13}\sqrt{\frac{\rho}{c_{33}}} & 0 & 0 & 1 & 0 & 1 & 0 & 0 & c_{13}\sqrt{\frac{\rho}{c_{33}}} \\ -c_{13}\sqrt{\frac{\rho}{c_{33}}} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & c_{13}\sqrt{\frac{\rho}{c_{33}}} \\ -\sqrt{c_{33}\rho} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{c_{33}\rho} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\sqrt{c_{44}\rho} & 0 & 0 & 0 & 0 & 0 & \sqrt{c_{44}\rho} & 0 \\ 0 & 0 & -\sqrt{(c_{11}-c_{12})\rho} & 0 & 0 & 0 & \sqrt{(c_{11}-c_{12})\rho} & 0 & 0 \end{pmatrix}.$$

We introduce the vector function U by the equality

$$U = T\vartheta.$$

Making this change in the equation (6) and then multiplying it on the left by $T^{-1}A^{-1}$, then we get

$$\left(I\frac{\partial}{\partial t} + \Lambda\frac{\partial}{\partial x_3} + B_1\frac{\partial}{\partial x_1} + C_1\frac{\partial}{\partial x_2} + F_1\right)\vartheta = \int_0^t R_1(t-\tau, x_3)\vartheta(\overline{x}, \tau)d\tau, \qquad (12)$$

where

$$B_1 = T^{-1}A^{-1}BT = (b_{ij}), \qquad C_1 = T^{-1}A^{-1}CT = (c_{ij}),$$

$$F_1 = T^{-1}A^{-1}D\frac{\partial T}{\partial x_3} + T^{-1}A^{-1}FT = (p_{ij}),$$

$$R_1 = (\tilde{r}_{ij}) = \begin{pmatrix} \tilde{r}_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{r}_{19} \\ 0 & \tilde{r}_{22} & 0 & 0 & 0 & 0 & \tilde{r}_{28} & 0 & 0 \\ 0 & 0 & \tilde{r}_{33} & 0 & 0 & 0 & 0 & \tilde{r}_{37} & 0 \\ \tilde{r}_{41} & 0 & 0 & \tilde{r}_{44} & 0 & \tilde{r}_{46} & 0 & 0 & \tilde{r}_{49} \\ \tilde{r}_{51} & 0 & 0 & 0 & \tilde{r}_{55} & 0 & 0 & 0 & \tilde{r}_{59} \\ 0 & 0 & 0 & 0 & 0 & \tilde{r}_{66} & 0 & 0 & 0 \\ 0 & 0 & \tilde{r}_{73} & 0 & 0 & 0 & 0 & \tilde{r}_{77} & 0 \\ 0 & \tilde{r}_{82} & 0 & 0 & 0 & 0 & \tilde{r}_{88} & 0 & 0 \\ \tilde{r}_{91} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{r}_{99} \end{pmatrix}, \qquad (13)$$

where $\tilde{r}_{11} = \tilde{r}_{99} = -\tilde{r}_{19} = -\tilde{r}_{91} = \frac{r'_{33}}{2}$, $\tilde{r}_{22} = \tilde{r}_{88} = -\tilde{r}_{28} = -\tilde{r}_{82} = \frac{r'_{13}}{2}$, $\tilde{r}_{33} = \tilde{r}_{77} = -\tilde{r}_{37} = -\tilde{r}_{73} = \frac{r'_{23}}{2}$, $\tilde{r}_{41} = -\tilde{r}_{49} = \frac{c_{13}}{\lambda_1}(r'_{11} - r'_{33})$, $\tilde{r}_{44} = r'_{11}$, $\tilde{r}_{46} = r'_{11} - r'_{12}$, $\tilde{r}_{51} = -\tilde{r}_{59} = \frac{c_{13}}{\lambda_1}(r'_{22} - r'_{23})$, $\tilde{r}_{55} = r'_{22}, \tilde{r}_{66} = r'_{12}$.

The purpose of this article is to study the direct and inverse problems for the system (11). Moreover, the direct problem is an initial-boundary value problem for this system in domain $(x_1, x_2) \in \mathbb{R}^2, x_3 \in (0, H), t > 0$, and in the inverse problem, the elements of the matrix R are assumed to be unknown, which are included in the definition of the matrix R_1 (12).

The outline of this paper organized as follows. Section 1 presents the formulations of the direct and inverse problems and investigates the direct problem. In Section 2, the inverse problem is reduced to solving of an equivalent closed system of integral equations. In Section 3, we present the formulation and proof of the main result, which consists in the unique global solvability of the inverse problem. At the end there is a list of literatures used in the article.

1. Set up problems and investigation

Consider the system of equations (12) in the domain

$$D = \{(x_1, x_2, x_3, t): (x_1, x_2) \in \mathbb{R}^2, x_3 \in (0, H), t > 0\}, \quad H = const,$$

with a bounded $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$:

$$\Gamma_0 = \{(x_1, x_2, x_3, t): (x_1, x_2) \in \mathbb{R}^2, 0 \le x_3 \le H, t = 0\},$$

$$\Gamma_1 = \{(x_1, x_2, x_3, t): (x_1, x_2) \in \mathbb{R}^2, x_3 = 0, t > 0\},$$

$$\Gamma_2 = \{(x_1, x_2, x_3, t): (x_1, x_2) \in \mathbb{R}^2, x_3 = H, t > 0\}.$$

For this system **direct problem** we pose as follows: determine the solution of the system of equations (12)

$$\vartheta_i|_{t=0} = \varphi_i(\overline{x}), \quad i = \overline{1,9}, \tag{14}$$

$$\vartheta_i|_{x_3=H} = g_i(x_1, x_2, t), \quad i = \overline{1,3}, \quad \vartheta_i|_{x_3=0} = g_i(x_1, x_2, t), \quad i = \overline{7,9}. \tag{15}$$

Here $\varphi_i(\overline{x})$, $g_i(x_1, x_2, t)$ are given functions. It is known that [6] the problem (12), (14), (15) is posed well.

The inverse problem is to determine the nonzero components of the matrix kernel R, that is $r_{ij}(t)$, $i, j = \overline{1,3}$ (R is included in R_1 according to the formula (12)) in (12) if the following conditions are known:

$$\vartheta_i|_{x_3=0} = h_i(x_1, x_2, t), \quad i = \overline{1,6}, \tag{16}$$

where $h_i(x_1, x_2, t)$, $i = \overline{1,6}$, are the given functions.

In the inverse problem, the numbers $r_{ij}(0)$, $i, j = \overline{1,3}$ are also considered to be given.

Recently, there has been an increased interest in hyperbolic systems of integro-differential equations containing integrals of the convolution type. Such equations describe processes with memory (with aftereffect) or, as they are also called, eridite processes [7, p. 180-189]. Such processes are characterized by the fact that the change in their state at each moment of time depends on the history of the process. Examples of such processes are the deformation of a viscoelastic medium [2, p. 449-453], the processes of propagation of electromagnetic waves in media with dispersion [8, p. 357-392] and dynamics of coexistence and development of animal and plant populations of various species [7, p. 193-195].

Inverse problems for hyperbolic integro-differential equations have been studied by many authors, we note the works [9]–[21] closest to the topic of this article. In works [9]–[11], one-dimensional problems of finding the convolution kernel were studied: from the general wave equation [9] and from the viscoelasticity equations [11]–[19]. The main results of these works are theorems on the unique solvability of the problems posed. In [20], [21], results were obtained on the existence and uniqueness of solutions to some multidimensional inverse problems of determining the kernel from second-order hyperbolic integro-differential equations.

It seems completely natural to study inverse problems on the determination of the kernels of integral terms of a system of integro-differential equations directly in terms of the system itself. This article is a natural continuation of this circle of problems and to a certain extent generalizes the results of [22] to the case of a three-dimensional system of viscoelasticity equations (1), (2).

Let functions $F_1(\overline{x},t)$, $\varphi(\overline{x})$, $g_i(x_1,x_2,t)$ included in the right-hand side of (12) and the data (14), (15) are compact support in x_1, x_2 for each fixed x_3,t. From the existence for the system (12) of a compact support domain of dependence and compact support with respect to $x_1,\ x_2$ of the right-hand side (12) and data (14), (15) implies the compact support in $x_1,\ x_2$ solutions to the problem (12) - (15).

Let us study the property of solution to this problem. More precisely, we restrict ourselves to studying the Fourier transform in the variables $x_1,\ x_2$ of the solution. In what follows, for convenience, we put $x_3 = z$ and introduce the notation

$$\hat{\vartheta}(\eta_1,\eta_2,z,t) = \int\limits_{\mathbb{R}^2} \vartheta(x_1,x_2,z,t)e^{i[\eta_1 x_1+\eta_2 x_2]}dx_1 dx_2,$$

$$\hat{F}(\eta_1,\eta_2,z,t) = \int\limits_{\mathbb{R}^2} F_1(x_1,x_2,z,t)e^{i[\eta_1 x_1+\eta_2 x_2]}dx_1 dx_2,$$

where η_1,η_2 are transformation parameters.

In terms of the function $\hat{\vartheta}$ we write the problem (12)-(15) as

$$\left(\frac{\partial}{\partial t}+\nu_j\frac{\partial}{\partial z}\right)\hat{\vartheta}_j(z,t) = \sum_{k=1}^{9}\hat{p}_{jk}(z)\hat{\vartheta}_k(z,t) + \\ +\int\limits_0^t\sum_{k=1}^{9}\tilde{r}_{jk}(z,\tau)\hat{\vartheta}_k(\eta_1,\eta_2,z,t-\tau)d\tau, \quad j=\overline{1,9}, \tag{17}$$

where $\hat{p}_{jk} = -i\eta_1 b_{jk} - i\eta_2 c_{jk} - p_{jk}$.

We fix η_1, η_2 and for convenience, we introduce the notation $\hat{\vartheta}(\eta_1,\eta_2,z,t) = \hat{\vartheta}(z,t)$. We will use a similar notations for the Fourier images of functions included in the initial, boundary and additional conditions(14)-(16):

$$\hat{\vartheta}_i|_{t=0} = \hat{\varphi}_i(z), \quad i=\overline{1,9}, \tag{18}$$

$$\hat{\vartheta}_i|_{z=H} = \hat{g}_i(t), \quad i=\overline{1,3}, \quad \hat{\vartheta}_i|_{z=0} = \hat{g}_i(t), \quad i=\overline{7,9}, \tag{19}$$

$$\hat{\vartheta}_i|_{z=0} = \hat{h}_i(t), \quad i = \overline{1,6}. \tag{20}$$

Where $\hat{\varphi}_i(z),\ i = \overline{1,9},\ \hat{\psi}_i(t),\ i = 1,2,3,7,8,9$ are the Fourier images of the corresponding functions from (14), (15) for $\eta = 0$. We also denote by D_H the projection of D onto the plane z, t. In what follows, we will consider the system of equations (17) in the domain $D_H \cup \tilde{\Gamma}$ under the conditions (18) and (19). Where $\tilde{\Gamma}_0 = \{(z,t):\ 0 \le z \le H, t = 0\}$, $\tilde{\Gamma}_1 = \{z,t): z = 0, t > 0\}$, $\tilde{\Gamma}_2 = \{(z,t): z = H, t > 0\}$.

Let us pass from the equalities (17)-(20) to the integral relations for the components of the vector $\hat{\vartheta}$ with integration flux along the corresponding characteristics of the equations of the system (17). We denote

$$\mu_9(z) = -\mu_1(z) = \int_0^z \frac{\sqrt{\rho(\beta)}}{\sqrt{c_{33}(\beta)}} d\beta, \qquad \mu_8(z) = -\mu_2(z) = \int_0^z \frac{\sqrt{\rho(\beta)}}{\sqrt{c_{44}(\beta)}} d\beta,$$

$$\mu_7(z) = -\mu_3(z) = \int_0^z \frac{\sqrt{\rho(\beta)}}{\sqrt{(c_{11} - c_{12})(\beta)}} d\beta, \quad \mu_4(z) = \mu_5(z) = \mu_6(z) = 0.$$

Inverse functions to $\mu_i(z),\ i = \overline{1,9}$, will be denoted by $z = \mu_i^{-1}(t),\ i = \overline{1,9}$. Using the introduced functions, the equations of characteristics passing through the points (z,t) on the plane of variables ξ, τ can be written in the form

$$\tau = t + \mu_i(\xi) - \mu_i(z), \quad i = \overline{1,9}. \tag{21}$$

Consider an arbitrary point $(z,t) \in D_H$ on the plane of variables ξ, τ and draw through it the characteristic of the i th of the system (17) equation tell to intersection in the domain $\tau \le t$. The intersection point is denoted by (z_0^i, t_0^i). Integrating the equations of the system (17) along the corresponding characteristics from the point (z_0^i, t_0^i) to the point (z,t) we find

$$\hat{\vartheta}_i(z,t) = \hat{\vartheta}_i(z_0^i, t_0^i) + \int_{t_0^i}^{t} \sum_{k=1}^{9} \hat{p}_{jk}(\xi)\hat{\vartheta}_k(\xi,\tau)|_{\xi=\mu_i^{-1}[\tau - t + \mu_i(z)]} d\tau +$$

$$+ \int_{t_0^i}^{t} \int_0^{\tau} \sum_{k=1}^{9} \tilde{r}_{jk}(\xi, \tau - \alpha)\hat{\vartheta}_k(\xi,\alpha) d\alpha|_{\xi=\mu_i^{-1}[\tau - t + \mu_i(z)]} d\tau, \quad i = \overline{1,9}. \tag{22}$$

We define in (22) t_0^i. It depends on the coordinates of the point (z,t). It is not difficult to see that $t_0^i(z,t)$ has the form

$$t_0^i(z,t) = \begin{cases} \begin{cases} t - \mu_i(z) + \mu_i(H), & t \ge \mu_i(z) - \mu_i(H) \\ 0, & 0 < t < \mu_i(z) - \mu_i(H), \end{cases} & i = 1,2,3, \\ 0, & i = 4,5,6, \\ \begin{cases} t - \mu_i(z), & t \ge \mu_i(z), \\ 0, & 0 < t < \mu_i(z), \end{cases} & i = 7,8,9. \end{cases}$$

Then, from the condition that the pair (z_0^i, t_0^i) satisfies the equation (22) it follows

$$z_0^i(z,t)=\begin{cases}H, & t\ge \mu_i(z)-\mu_i(H),\\ \mu_i^{-1}(\mu_i(z)-t), & 0<t<\mu_i(z)-\mu_i(H),\end{cases} i=1,2,3,$$

$$z_0^i(z,t)=z,\ \ i=4,5,6,$$

$$z_0^i(z,t)=\begin{cases}0, & t\ge \mu_i(z),\\ \mu_i^{-1}(\mu_i(z)-t), & 0<t<\mu_i(z),\end{cases} i=7,8,9.$$

The free terms of the integral equations (22) are defined through the initial and boundary conditions (18) and (19) as follows:

$$\hat{\vartheta}_0^i(z_0^i,t_0^i)=\begin{cases}\hat{g}_i(t-\mu_i(z)+\mu_i(H)), & t\ge \mu_i(z)-\mu_i(H),\\ \hat{\varphi}_i\left(\mu_i^{-1}(\mu_i(z)-t)\right), & 0<t<\mu_i(z)-\mu_i(H),\end{cases}\ i=1,2,3,$$

$$\hat{\vartheta}_0^i(z_0^i,t_0^i)=\hat{\varphi}_i(z),\ \ \ i=4,5,6,$$

$$\hat{\vartheta}_0^i(z_0^i,t_0^i)=\begin{cases}\hat{g}_i(t-\mu_i(z)), & t\ge \mu_i(z),\\ \hat{\varphi}_i\left(\mu_i^{-1}(\mu_i(z)-t)\right), & 0<t<\mu_i(z),\end{cases}\ i=7,8,9.$$

We require that the functions $\hat{\vartheta}_i(z_0^i,t_0^i)$ be continuous in the domain D. Note that for the fulfillment of these conditions, the given functions $\hat{\varphi}_i$ and $\hat{g}_i$ must satisfy the matching conditions at the corner points of the domain D, that is

$$\hat{\varphi}_i(0)=\hat{g}_i(H),\ \ i=1,2,3,\qquad \hat{\varphi}_i(0)=\hat{g}_i(0),\ \ i=7,8,9. \tag{23}$$

Here and below, the values of the functions $\hat{g}_i,\ i=1,2,3,7,8,9$ at $t=0$ and functions $\hat{\varphi}_i,\ i=1,2,3,7,8,9$ at $z=0$ and $z=H$ are understood as the limit at these points when the argument tends from the side of the point where these functions are defined.

Suppose that all given functions included in (22) are continuous functions of their arguments in D_H. Then this system of equations is a closed system of integral equations of the Volterra type of the second kind with continuous kernels and free terms. As usual, such a system has a unique solution in the bounded subdomain $D_{HT}=\{(z,t): 0<z<H,\ 0<t<T\},\ T>0$ are some fixed number, domain D_H.

Thus, the following statement is hold:

Theorem 1. *Assume functions $F_1(x,t),\ \varphi(x),\ x\in\mathbb{R}^3,\ g(x_1,x_2,t)$ are compact support in x_1, x_2 for each fixed $x_3,\ t$. Let be $\rho(z), c_{33}(z), c_{44}(z), c_{11}(z), c_{12}(z), \hat{\varphi}(z)\in C^1[0,\infty),\ \hat{g}(t)\in C^1[0,\infty),\ \rho(z)>0,\ c_{33}(z)>0,\ c_{44}(z)>0,\ c_{11}(z)>0,\ c_{12}(z)>0,\ r_{ij}(t)\in C^1[0,\infty),\ i,j=1,2,$ and conditions (23) are satisfied. Then there is a unique solution to the problem (17)-(19) in the domain D_{HT}.*

The problem (17)-(19) in the domain D_{HT} is equivalent to a linear integral equation of the second kind of Volterra type with respect to $\hat{\vartheta}$. As follows from the theory of linear integral equations, it has a unique solutions [23]. So we drop it.

Let us introduce the vector function $\omega(z,t)=\frac{\partial\hat{\vartheta}}{\partial t}(z,t)$. To obtain a problem for a function $\omega(z,t)$ similar to (17) – (20) differentiate the equations (17) and the boundary conditions (22) with respect to the variable t, and the condition for $t=0$ is found using the equations (17) and the initial conditions (18). In this case, we get

$$\left(\frac{\partial}{\partial t}+\lambda_i\frac{\partial}{\partial z}\right)\omega_i(z,t)=\sum_{k=1}^{9}\hat{p}_{ik}(z)\omega_k(z,t)+$$

$$+\sum_{k=1}^{9}\tilde{r}_{ik}(z,t)\hat{\varphi}_i(z)+\int_0^t\sum_{k=1}^{9}\tilde{r}_{ik}(z,\tau)\omega_k(z,t-\tau)d\tau,\quad i=\overline{1,9},\qquad(24)$$

$$\omega_i|_{t=0}=\Phi_i(z),\quad i=\overline{1,9},\qquad(25)$$

$$\omega_i|_{z=H}=\frac{d}{dt}\hat{g}_i(t),\quad i=\overline{1,3},\quad \omega_i|_{z=0}=\frac{d}{dt}\hat{g}_i(t),\quad i=\overline{7,9},\qquad(26)$$

where $\Phi_i(z)=-\lambda_i\frac{\partial\hat{\varphi}_i(z)}{\partial z}+\sum_{j=1}^{9}\hat{p}_{ji}(z)\hat{\varphi}_i(z),\ i=\overline{1,9}.$

For functions ω_i additional conditions (20) gets

$$\omega_i|_{z=0}=\frac{d}{dt}\hat{h}_i(t),\quad i=\overline{1,6}.\qquad(27)$$

Integration along the corresponding characteristics again brings the problem (24)—(26) integral equations

$$\omega_i(z,t)=\omega_i(z_0^i,t_0^i)+$$
$$+\int_{t_0^i}^{t}\sum_{k=1}^{9}\hat{p}_{ik}(\xi)\omega_k(\xi,\tau)|_{\xi=\mu_i^{-1}[\tau-t+\mu_i(z)]}d\tau+\int_{t_0^i}^{t}\Bigg[\sum_{k=1}^{9}\tilde{r}_{ik}(\xi,\tau)\hat{\varphi}_i(\xi)+$$
$$+\int_0^{\tau}\sum_{k=1}^{9}\tilde{r}_{ik}(\xi,\tau-\alpha)\omega_k(\xi,\alpha)d\alpha\Bigg]_{\xi=\mu_i^{-1}[\tau-t+\mu_i(z)]}d\tau,\ i=\overline{1,9}.\qquad(28)$$

In equations (28), $\omega_i(z_0^i,t_0^i)$ are defined as follows:

$$\omega_i(z_0^i,t_0^i)=\begin{cases}\dfrac{\partial}{\partial t}\hat{g}_i(t-\mu_i(z)+\mu_i(H)), & t\ge\mu_i(z)-\mu_i(H),\\ \Phi_i\left(\mu_i^{-1}(\mu_i(z)-t)\right), & 0<t<\mu_i(z)-\mu_i(H),\end{cases}\quad i=1,2,3,$$

$$\hat{\vartheta}_0^i(z_0^i,t_0^i)=\Phi_i(z),\quad i=4,5,6,$$

$$\omega_i(z_0^i,t_0^i)=\begin{cases}\dfrac{\partial}{\partial t}\hat{g}_i(t-\mu_i(z)), & t\ge\mu_i(z),\\ \Phi_i\left(\mu_i^{-1}(\mu_i(z)-t)\right), & 0<t<\mu_i(z),\end{cases}\quad i=7,8,9.$$

Let the following conditions hold

$$\left[\frac{\partial\hat{g}_i(t)}{\partial t}\right]_{t=0}=-\lambda_j\left[\frac{\partial\hat{\varphi}_i(z)}{\partial z}\right]_{z=H}+\sum_{j=1}^{9}p_{ij}(H)\hat{\varphi}_j(H),\quad i=\overline{1,3},\qquad(29)$$

$$\left[\frac{\partial \hat{g}_i(t)}{\partial t}\right]_{t=0} = -\lambda_j \left[\frac{\partial \hat{\varphi}_i(z)}{\partial z}\right]_{z=0} + \sum_{j=1}^{9} p_{ij}(0)\hat{\varphi}_j(0), \quad i = \overline{7,9}. \quad (30)$$

It is easy to see that the conditions for matching the initial and boundary data (25), (26) in corner points of the domain D_H coincide with the relations (29) and (30). Hence it is clear that for the fulfillment of the same equalities (29) and (30) equations (28) will have unique continuous solutions $\omega_i(z,t)$, or the same $\frac{\partial}{\partial t}\hat{\vartheta}_i(z,t)$.

2. Study of the inverse problem

Consider an arbitrary point $(z,0) \in D_{HT}$ and draw through it the characteristics (21) for $i = 1,2,3$, up to the intersection with the left lateral boundary of the domain D_H. Integrating the first three components of the equation (24), we obtain

$$\omega_i(z,0) = \omega_i\left(\gamma_i H, t_1^i\right) -$$

$$-\int_0^{t_1^i} \sum_{k=1}^{9} \hat{p}_{ik}(\xi)\omega_k(\xi,\tau)|_{\xi=\mu_i^{-1}[\tau+\mu_i(z)]} d\tau - \int_0^{t_1^i} \Big[\tilde{r}_{ik}(\xi,\tau)\hat{\varphi}_i(\xi) +$$

$$+\sum_{k=1}^{9}\int_0^{\tau}\sum_{k=1}^{9} \tilde{r}_{ik}(\xi,\tau-\alpha)\omega_k(\xi,\alpha)d\alpha\Big]_{\xi=\mu_i^{-1}[\tau+\mu_i(z)]} d\tau, \quad i = \overline{1,9}. \quad (31)$$

where $t_1^i = \begin{cases} \gamma_i\mu_i(H) - \mu_i(z), & i = 1,2,3,7,8,9, \\ 0 & i = 4,5,6, \end{cases}$ $\gamma_i = \begin{cases} 0, & i = 1,2,3,4,5,6, \\ 1, & i = 7,8,9. \end{cases}$

After many simple observation, we taken into account the initial conditions (27), those characteristics that are perpendicular to the OZ axis are differentiable with respect to the variable t, and other with respect to the variable z. Used equality (13), (26), (27) and (31), we get following the integral equation:

$$r'_{11}(t) = M_1 Q_1 - M_1 \int_0^t (r'_{11} - r'_{12})(\tau)\frac{d}{dt}\hat{h}_6(t-\tau)d\tau +$$

$$+\int_0^t r'_{12}(\tau)\frac{d}{dt}\hat{h}_6(t-\tau)d\tau - M_1\int_0^t \Big[\frac{c_{13}}{\lambda_1}(r'_{11} - r'_{33})(\tau)\frac{d}{dt}\left(\hat{h}_1 - \hat{g}_9\right)(t-\tau) +$$

$$+r'_{11}(\tau)\frac{d}{dt}\hat{h}_4(t-\tau)\Big]d\tau + 2c_{13}\int_0^t \frac{\partial}{\partial z}\sum_{j=1}^{9} \hat{p}_{1j}(\xi)\omega_j(\xi,\tau)|_{\xi=\mu_1^{-1}[t-\tau]}d\tau +$$

$$+c_{13}\int_0^t r'_{33}(\tau)\frac{\partial}{\partial z}(\hat{\varphi}_1 - \hat{\varphi}_9)(\xi)|_{\xi=\mu_1^{-1}[t-\tau]}d\tau$$

$$-\frac{c_{13}}{\lambda_1}\int_0^t r'_{33}(\tau)\frac{d}{dt}\left(\hat{h}_1 - \hat{g}_9\right)(t-\tau)d\tau +$$

$$+c_{13}\int_0^t\int_0^\tau r'_{33}(\alpha)\frac{\partial}{\partial z}(\omega_1-\omega_9)(\xi,\tau-\alpha)d\alpha|_{\xi=\mu_1^{-1}[t-\tau]}d\tau, \quad (32)$$

$$r'_{12}(t)=Q_2-[\hat{\varphi}_6(0)]^{-1}\int_0^t r'_{12}(\tau)\frac{d}{dt}\hat{h}_6(t-\tau)d\tau, \quad (33)$$

$$r'_{13}(t)=-2M_2Q_3(z)+2M_2\int_0^t\frac{\partial}{\partial z}\sum_{j=1}^{9}\hat{p}_{2j}(\xi)\omega_j(\xi,\tau)|_{\xi=\mu_2^{-1}[t-\tau]}d\tau+$$

$$+M_2\int_0^t r'_{13}(\tau)\frac{\partial}{\partial z}(\hat{\varphi}_2-\hat{\varphi}_8)(\xi)|_{\xi=\mu_2^{-1}[t-\tau]}d\tau-$$

$$-M_3\int_0^t r'_{13}(\tau)\frac{d}{dt}\left(\hat{h}_2-\hat{g}_8\right)(t-\tau)d\tau+$$

$$+M_2\int_0^t\int_0^\tau r'_{13}(\alpha)\frac{\partial}{\partial z}(\omega_2-\omega_8)(\xi,\tau-\alpha)d\alpha|_{\xi=\mu_2^{-1}[t-\tau]}d\tau, \quad (34)$$

$$r'_{22}(t)=M_4Q_4+M_5\lambda_1M_6\int_0^t r'_{23}(\tau)\frac{\partial}{\partial z}(\hat{\varphi}_3-\hat{\varphi}_7)(\xi)|_{\xi=\mu_3^{-1}[t-\tau]}d\tau-$$

$$-M_5\int_0^t r'_{23}(\tau)\frac{d}{dt}\left(\hat{h}_3-\hat{g}_7\right)(t-\tau)d\tau+$$

$$+2M_6\int_0^t\frac{\partial}{\partial z}\sum_{j=1}^{9}\hat{p}_{3j}(\xi)\omega_j(\xi,\tau)|_{\xi=\mu_3^{-1}[t-\tau]}d\tau+$$

$$+M_5\lambda_1\int_0^t\int_0^\tau r'_{23}(\alpha)\frac{\partial}{\partial z}(\omega_3-\omega_7)(\xi,\tau-\alpha)d\alpha|_{\xi=\mu_3^{-1}[t-\tau]}d\tau+$$

$$+M_4\int_0^t r'_{23}(\tau)\left[\frac{c_{13}}{\lambda_1}\frac{d}{dt}\left(\hat{h}_1-\hat{g}_9\right)(t-\tau)\right]d\tau+$$

$$+M_4\int_0^t r'_{33}(\tau)\frac{c_{13}}{\lambda_1}\frac{d}{dt}\left(\hat{h}_1-\hat{g}_9\right)(t-\tau)d\tau+$$

$$+M_5\int_0^t r'_{33}(\tau)\frac{\partial}{\partial z}(\hat{\varphi}_1-\hat{\varphi}_9)(\xi)|_{\xi=\mu_1^{-1}[t-\tau]}d\tau+$$

$$+2M_5\int_0^t \frac{\partial}{\partial z}\sum_{j=1}^{9} \hat{p}_{1j}(\xi)\omega_j(\xi,\tau)|_{\xi=\mu_1^{-1}[t-\tau]}d\tau +$$

$$+M_5\int_0^t\int_0^\tau r'_{33}(\alpha)\frac{\partial}{\partial z}(\omega_1-\omega_9)(\xi,\tau-\alpha)d\alpha|_{\xi=\mu_1^{-1}[t-\tau]}d\tau, \qquad (35)$$

$$r'_{23}(t) = -2M_6Q_5(z) + M_6\int_0^t r'_{23}(\tau)\frac{\partial}{\partial z}(\hat{\varphi}_3-\hat{\varphi}_7)(\xi)|_{\xi=\mu_3^{-1}[t-\tau]}d\tau -$$

$$-M_7\int_0^t r'_{23}(\tau)\frac{d}{dt}\left(\hat{h}_3-\hat{g}_7\right)(t-\tau)d\tau +$$

$$+2M_6\int_0^t \frac{\partial}{\partial z}\sum_{j=1}^{9} \hat{p}_{3j}(\xi)\omega_j(\xi,\tau)|_{\xi=\mu_3^{-1}[t-\tau]}d\tau +$$

$$+M_6\int_0^t\int_0^\tau r'_{23}(\alpha)\frac{\partial}{\partial z}(\omega_3-\omega_7)(\xi,\tau-\alpha)d\alpha|_{\xi=\mu_3^{-1}[t-\tau]}d\tau, \quad (36)$$

$$r'_{33}(t) = -2M_8Q_6(z) + 2M_8\int_0^t \frac{\partial}{\partial z}\sum_{j=1}^{9} \hat{p}_{1j}(\xi)\omega_j(\xi,\tau)|_{\xi=\mu_1^{-1}[t-\tau]}d\tau +$$

$$+M_8\int_0^t r'_{33}(\tau)\frac{\partial}{\partial z}(\hat{\varphi}_1-\hat{\varphi}_9)(\xi)|_{\xi=\mu_1^{-1}[t-\tau]}d\tau - M_9\int_0^t r'_{33}(\tau)\frac{d}{dt}\left(\hat{h}_1-\hat{g}_9\right)(t-\tau)d\tau$$

$$+M_8\int_0^t\int_0^\tau r'_{33}(\alpha)\frac{\partial}{\partial z}(\omega_1-\omega_9)(\xi,\tau-\alpha)d\alpha|_{\xi=\mu_1^{-1}[t-\tau]}d\tau, \quad (37)$$

where, $M_1 = \left[\frac{c_{13}}{\lambda_1}(\hat{\varphi}_1(0)-\hat{\varphi}_9(0)) + \hat{\varphi}_4(0)+\hat{\varphi}_6(0)\right]^{-1}$, $M_2 = \lambda_2[\hat{\varphi}_2(0)-\hat{\varphi}_8(0)]^{-1}$, $M_3 = [\hat{\varphi}_2(0)-\hat{\varphi}_8(0)]^{-1}$, $M_4 = \left[\frac{c_{13}}{\lambda_1}(\hat{\varphi}_9(0)-\hat{\varphi}_1(0)) + \hat{\varphi}_5(0)\right]^{-1}$,

$M_5 = (1-\hat{\varphi}_5(0)M_4)[\hat{\varphi}_3(0)-\hat{\varphi}_7(0)]^{-1}$, $M_6 = \lambda_2[\hat{\varphi}_3(0)-\hat{\varphi}_7(0)]^{-1}$, $M_7 = [\hat{\varphi}_3(0)-\hat{\varphi}_7(0)]^{-1}$, $M_8 = \lambda_1[\hat{\varphi}_1(0)-\hat{\varphi}_9(0)]^{-1}$, $M_9 = [\hat{\varphi}_1(0)-\hat{\varphi}_9(0)]^{-1}$,

$$Q_1 = M_1\left[\frac{d^2}{dt^2}\hat{h}_5(t) - \sum_{j=1}^{9}\hat{p}_{5j}(0)\omega_j(0,t)\right] - 2c_{13}\left[\frac{\partial}{\partial z}\left[\frac{d}{dt}\hat{h}_1(t_1^3)\right]\frac{\partial}{\partial z}\Phi_1(z) - \frac{1}{\lambda_1}\sum_{i=1}^{9}\hat{p}_{1i}(0)\omega_i(0,t)\right],\ Q_2 = \frac{d^2}{dt^2}\hat{h}_6(t) - \sum_{j=1}^{9}\hat{p}_{6j}(0)\omega_j(0,t),$$

$$Q_3 = \frac{\partial}{\partial z}\left[\frac{d}{dt}\hat{h}_2(t_1^2)\right]\frac{\partial}{\partial z}\Phi_2(z) - \frac{1}{\lambda_2}\sum_{i=1}^{9}\hat{p}_{2i}(0)\omega_i(0,t),$$

$$Q_4 = M_4\left[\frac{d^2}{dt^2}\hat{h}_4(t) - \sum_{j=1}^{9}\hat{p}_{4j}(0)\omega_j(0,t)\right] - 2M_5\lambda_1\left[\frac{\partial}{\partial z}\left[\frac{d}{dt}\hat{h}_3(t_1^3)\right]\frac{\partial}{\partial z}\Phi_3(z) - \frac{1}{\lambda_3}\sum_{i=1}^{9}\hat{p}_{3i}(0)\omega_i(0,t)\right],$$

$$Q_5 = -2M_6\left[\frac{\partial}{\partial z}\left[\frac{d}{dt}\hat{h}_3(t_1^3)\right]\frac{\partial}{\partial z}\Phi_3(z) - \frac{1}{\lambda_3}\sum_{i=1}^{9}\hat{p}_{3i}(0)\omega_i(0,t)\right],$$

$$Q_6 = -2M_8\left[\frac{\partial}{\partial z}\left[\frac{d}{dt}\hat{h}_1(t_1^3)\right]\frac{\partial}{\partial z}\Phi_1(z) - \frac{1}{\lambda_1}\sum_{i=1}^{9}\hat{p}_{1i}(0)\omega_i(0,t)\right],$$

$$\frac{c_{13}}{\lambda_1}(\hat{\varphi}_1(0) - \hat{\varphi}_9(0)) + \hat{\varphi}_4(0) + \hat{\varphi}_6(0) \neq 0,\ \hat{\varphi}_2(0) - \hat{\varphi}_8(0) \neq 0, \tag{38}$$

$$\frac{c_{13}}{\lambda_1}(\hat{\varphi}_9(0) - \hat{\varphi}_1(0)) + \hat{\varphi}_5(0) \neq 0,\ \hat{\varphi}_3(0) - \hat{\varphi}_7(0) \neq 0,\ \hat{\varphi}_1(0) - \hat{\varphi}_9(0) \neq 0. \tag{39}$$

The equation $(32) - (3237)$ contains unknown functions $\frac{\partial \omega_j}{\partial z}$, $j = \overline{1,9}$. For them we will receive integral equations from (28) by differentiating them with respect to the variable z. Moreover, we have

$$\frac{\partial}{\partial z}\omega_i(z,t) = \frac{\partial}{\partial z}\omega_i(z_0^i, t_0^i) - \frac{\partial}{\partial z}t_0^i\left[\sum_{k=1}^{9}\hat{p}_{ik}(z_0^i)\omega_k(z_0^i, t_0^i) + \sum_{k=1}^{9}\tilde{r}_{ik}(z_0^i, t_0^i)\hat{\varphi}_i(z_0^i)\right]$$

$$+\int_{t_0^i}^{t}\frac{\partial}{\partial z}\left[\sum_{k=1}^{9}\hat{p}_{ik}(\xi)\omega_k(\xi,\tau) + \sum_{k=1}^{9}\tilde{r}_{ik}(\xi,\tau)\hat{\varphi}_i(\xi)\right]|_{\xi=\mu_i^{-1}[\tau-t+\mu_i(z)]}d\tau +$$

$$+\frac{\partial}{\partial z}t_0^i\int_{0}^{t_0^i}\sum_{k=1}^{9}\tilde{r}_{ik}(\xi, t_0^i - \tau)\omega_k(\xi,\tau)|_{\xi=\mu_i^{-1}[t_0^i-t+\mu_i(z)]}d\tau +$$

$$+\int_{t_0^i}^{t}\int_{0}^{\tau}\frac{\partial}{\partial z}\sum_{k=1}^{9}\tilde{r}_{ik}(\xi,\tau-\alpha)\omega_k(\xi,\alpha)d\alpha|_{\xi=\mu_i^{-1}[\tau-t+\mu_i(z)]}d\tau, \quad i = \overline{1,9}. \tag{40}$$

We require the fulfillment of the matching conditions

$$[-\lambda_j\frac{\partial\hat{\varphi}_i(z)}{\partial z}]_{z=0} + \sum_{j=1}^{9}p_{ij}(0)\hat{\varphi}_j(0) = [\frac{d}{dt}\hat{h}_i]_{t=0}, \qquad i = \overline{1,6}. \tag{41}$$

3. Main result and its proof

The main result of this work is the following theorem:

Theorem 2: *Let the conditions of Theorem 1 are satisfaed, besides* $\varphi_i(z) \in C^2[0,H]$, $i = \overline{1,9}$, $\psi_i(t) \in C^2[0,\ \infty)$, $i = 1,2,3,7,8,9$, $h_i(t) \in C^2(0,\infty)$, $i = \overline{1,6}$, *equality (38),(39) and matching conditions (23), (29), (30), (41) hold. Then for any* $H > 0$ *on the segment* $[0,H]$ *there is a unique solution to the inverse problems* (12), (14), (15) *and* (16) *from class* $R_1(z,t) \in C^1[0,H]$.

Proof. Equations (28), (32)–(37) and (40) supplemented by the initial and boundary conditions from the equalities (24) forms a closed system of equations for the unknowns $\omega_i(z,t)$, $i = \overline{1,9}$, $r'_{ij}(t)$, $i,j = \overline{1,3}$, $\frac{\partial}{\partial z}\omega_i(z,t)$, $i = \overline{1,9}$. Consider now a square

$$D_0 := \{(z,t): 0 \le z \le H, \qquad 0 \le t \le \mu^0\},$$

where $\mu^0 = \max_{1\le i\le 3, 7\le i\le 9}\{\mu_{m(i)}(H)\}$.

Equations (28), (32)–(37) and (40) show that the values of the functions $\omega_i(z,t)$, $i=\overline{1,9}$, $r'_{ij}(t)$, $i,j=\overline{1,3}$, $\frac{\partial}{\partial z}\omega_i(z,t)$, $i=\overline{1,9}$ at $(z,t)\in D_0$ are expressed through integrals of some combinations of the same functions over segments lying in D_0.

Let us write down the equations (28), (32)–(37) and (40) in the form of a closed system of integral equations of Volterra type of the second kind. To do this, we introduce into consideration the vector function $\upsilon(z,t)=(\upsilon_i^1,\ \upsilon_j^2,\ \upsilon_i^3)$, $i=\overline{1,9}$, $j=\overline{1,6}$, setting their components by equalities

$$\upsilon_i^1(z,t)=\omega_i(z,t),\quad i=\overline{1,9}, \tag{42}$$

$$\upsilon_1^2(t)=r'_{11}(t),\quad \upsilon_2^2(t)=r'_{12}(t),\quad \upsilon_3^2(t)=r'_{13}(t),\quad \upsilon_4^2(t)=r'_{22}(t),$$

$$\upsilon_5^2(t)=r'_{23}(t),\quad \upsilon_6^2(t)=r'_{33}(t), \tag{43}$$

$$\upsilon_i^3(z,t)=\frac{\partial}{\partial z}\omega_i(z,t)-\frac{r'_{33}(t_0^i)}{2}\left(\tilde{\varphi}_1(z_0^i)-\tilde{\varphi}_9(z_0^i)\right)\frac{\partial}{\partial z}t_0^i,\quad i=1,9, \tag{44}$$

$$\upsilon_i^3(z,t)=\frac{\partial}{\partial z}\omega_i(z,t)-\frac{r'_{23}(t_0^i)}{2}\left(\tilde{\varphi}_2(z_0^i)-\tilde{\varphi}_8(z_0^i)\right)\frac{\partial}{\partial z}t_0^i,\quad i=2,7, \tag{45}$$

$$\upsilon_i^3(z,t)=\frac{\partial}{\partial z}\omega_i(z,t)-\frac{r'_{13}(t_0^i)}{2}\left(\tilde{\varphi}_3(z_0^i)-\tilde{\varphi}_7(z_0^i)\right)\frac{\partial}{\partial z}t_0^i,\quad i=3,8, \tag{46}$$

$$\upsilon_i^3(z,t)=\frac{\partial}{\partial z}\omega_i(z,t),\quad i=\overline{4,6}. \tag{47}$$

Then the system of equations (28), (32)–(37) and (40) takes operator-vector form

$$\upsilon=A\upsilon, \tag{48}$$

where the operator $A=(A_i^1,\ A_j^2,\ A_i^3)$, $i=\overline{1,9}$, $j=\overline{1,6}$, in accordance with the right-hand sides follow equations (28), (32)–(37) and (40). Taking into the equations (28) and the expression of (42), defined as follows

$$A_i^1\upsilon=\upsilon_i^{01}(z,t)+\int_{t_0^i}^{t}\left[\sum_{j=1}^{9}\hat{p}_{ij}(\xi)\upsilon_j^1(\xi,\tau)-\frac{\upsilon_6^2(\tau)}{2}(\hat{\varphi}_1-\hat{\varphi}_9)(\xi)\right]\Big|_{\xi=\mu_1^{-1}[\tau-t+\mu_1(z)]}d\tau$$

$$-\int_{t_0^i}^{t}\int_0^{\tau}\frac{\upsilon_6^2(\alpha)}{2}(\upsilon_1^1-\upsilon_9^1)(\xi,\tau-\alpha)d\alpha\Big|_{\xi=\mu_1^{-1}[\tau-t+\mu_1(z)]}d\tau,\quad i=1,9, \tag{49}$$

$$A_i^1\upsilon=\upsilon_i^{01}(z,t)+\int_{t_0^i}^{t}\left[\sum_{j=1}^{9}\hat{p}_{ij}(\xi)\upsilon_j^1(\xi,\tau)-\frac{\upsilon_3^2(\tau)}{2}(\hat{\varphi}_2-\hat{\varphi}_8)(\xi)\right]\Big|_{\xi=\mu_2^{-1}[\tau-t+\mu_2(z)]}d\tau$$

$$-\int_{t_0^i}^{t}\int_0^{\tau}\frac{\upsilon_3^2(\alpha)}{2}(\upsilon_2^1-\upsilon_8^1)(\xi,\tau-\alpha)d\alpha\Big|_{\xi=\mu_2^{-1}[\tau-t+\mu_2(z)]}d\tau,\quad i=2,8, \tag{50}$$

$$A_i^1 \upsilon = \upsilon_i^{01}(z,t) + \int_{t_0^i}^{t} \left[\sum_{j=1}^{9} \hat{p}_{ij}(\xi) \upsilon_j^1(\xi,\tau) - \frac{\upsilon_5^2(\tau)}{2} (\hat{\varphi}_3 - \hat{\varphi}_7)(\xi) \right] \Big|_{\xi=\mu_2^{-1}[\tau - t + \mu_3(z)]} d\tau$$

$$- \int_{t_0^3}^{t} \int_0^{\tau} \frac{\upsilon_5^2(\alpha)}{2} (\upsilon_3^1 - \upsilon_7^1)(\xi, \tau - \alpha) d\alpha \Big|_{\xi=\mu_2^{-1}[\tau - t + \mu_3(z)]} d\tau, \quad i = 3,7, \quad (51)$$

$$A_4^1 \upsilon = \int_0^t \sum_{j=1}^{9} \hat{p}_{4j}(z) \upsilon_j^1(z,\tau) d\tau + \int_0^t [\upsilon_1^2(\tau) \hat{\varphi}_4(z) + (\upsilon_1^2 - \upsilon_2^2)(\tau) \hat{\varphi}_6(z)] d\tau +$$

$$+ \int_0^t \frac{c_{13}}{\lambda_1} (\upsilon_1^2 - \upsilon_6^2)(\tau) (\hat{\varphi}_1 - \hat{\varphi}_9)(z) d\tau +$$

$$+ \int_0^t \int_0^{\tau} \frac{c_{13}}{\lambda_1} (\upsilon_1^2 - \upsilon_6^2)(\tau) (\upsilon_1^1 - \upsilon_9^1)(z,\alpha) d\alpha d\tau +$$

$$+ \int_0^t \int_0^{\tau} [\upsilon_1^2(\alpha) \upsilon_4^1(z, \tau - \alpha) + (\upsilon_1^2(\alpha) - \upsilon_2^2(\alpha)) \upsilon_6^1(z, \tau - \alpha)] d\alpha d\tau, \quad (52)$$

$$A_5^1 \upsilon = \int_0^t \sum_{j=1}^{9} \hat{p}_{5j}(z) \upsilon_j^1(z,\tau) d\tau + \int_0^t \upsilon_1^2(\tau) \hat{\varphi}_4(z) d\tau + \int_0^t \int_0^{\tau} \upsilon_1^2(\alpha) \upsilon_4^1(z, \tau - \alpha) d\alpha d\tau$$

$$+ \int_0^t \frac{c_{13}}{\lambda_1} \left(\upsilon_4^2 - \upsilon_5^2\right)(\tau) (\hat{\varphi}_1 - \hat{\varphi}_9)(z) d\tau +$$

$$+ \int_0^t \int_0^{\tau} \frac{c_{13}}{\lambda_1} (\upsilon_1^2 - \upsilon_6^2)(\tau) (\upsilon_1^1 - \upsilon_9^1)(z,\alpha) d\alpha d\tau, \quad (53)$$

$$A_6^1 \upsilon = \int_0^t \sum_{j=1}^{9} \hat{p}_{6j}(z) \upsilon_j^1(z,\tau) d\tau +$$

$$+ \int_0^t \int_0^{\tau} \upsilon_2^2(\alpha) \upsilon_6^1(z, \tau - \alpha) d\alpha d\tau + \int_0^t \upsilon_2^2(\tau) \hat{\varphi}_6(z) d\tau. \quad (54)$$

So taking into the equations (32)–(37) and the expression of (43)–(??), defined as follows

$$A_1^2 \upsilon = \upsilon_1^{02}(z,t) - M_1 \int_0^t (\upsilon_1^2 - \upsilon_2^2)(\tau) \frac{d}{dt} \hat{h}_6(t - \tau) d\tau + \int_0^t \upsilon_2^2(\tau) \frac{d}{dt} \hat{h}_6(t - \tau) d\tau -$$

$$-M_1 \int_0^t \left[\frac{c_{13}}{\lambda_1} (\upsilon_1^1 - \upsilon_6^2)(\tau) \frac{d}{dt} \left(\hat{h}_1 - \hat{g}_9\right)(t - \tau) + \upsilon_1^2(\tau) \frac{d}{dt} \hat{h}_4(t - \tau) \right] d\tau +$$

$$+2c_{13}\int_0^t \frac{\partial}{\partial z}\sum_{j=1}^{9} \hat{p}_{1j}(\xi)\upsilon_j^1(\xi,\tau)|_{\xi=\mu_1^{-1}[t-\tau]}d\tau +$$

$$+c_{13}\int_0^t \upsilon_6^2(\tau)\frac{\partial}{\partial z}(\hat{\varphi}_1-\hat{\varphi}_9)(\xi)|_{\xi=\mu_1^{-1}[t-\tau]}d\tau - \frac{c_{13}}{\lambda_1}\int_0^t \upsilon_6^2(\tau)\frac{d}{dt}\left(\hat{h}_1-\hat{g}_9\right)(t-\tau)d\tau +$$

$$+c_{13}\int_0^t\int_0^\tau \upsilon_6^2(\alpha)\frac{\partial}{\partial z}(\upsilon_1^1-\upsilon_9^1)(\xi,\tau-\alpha)d\alpha|_{\xi=\mu_1^{-1}[t-\tau]}d\tau, \qquad (55)$$

$$A_2^2\upsilon = \upsilon_2^{02}(z,t) - [\hat{\varphi}_6(0)]^{-1}\int_0^t \upsilon_2^2(\tau)\frac{d}{dt}\hat{h}_6(t-\tau)d\tau, \qquad (56)$$

$$A_3^2\upsilon = \upsilon_3^{02}(z,t) + 2M_2\int_0^t \frac{\partial}{\partial z}\sum_{j=1}^{9} \hat{p}_{2j}(\xi)\upsilon_j^1(\xi,\tau)|_{\xi=\mu_2^{-1}[t-\tau]}d\tau +$$

$$+M_2\int_0^t \upsilon_3^2(\tau)\frac{\partial}{\partial z}(\hat{\varphi}_2-\hat{\varphi}_8)(\xi)|_{\xi=\mu_2^{-1}[t-\tau]}d\tau - M_3\int_0^t \upsilon_3^2(\tau)\frac{d}{dt}\left(\hat{h}_2-\hat{g}_8\right)(t-\tau)d\tau +$$

$$+M_2\int_0^t\int_0^\tau \upsilon_3^2(\alpha)\frac{\partial}{\partial z}(\upsilon_2^1-\upsilon_8^1)(\xi,\tau-\alpha)d\alpha|_{\xi=\mu_2^{-1}[t-\tau]}d\tau, \qquad (57)$$

$$A_4^2\upsilon = \upsilon_4^{02}(z,t) + M_5\lambda_1 M_6\int_0^t \upsilon_5^2(\tau)\frac{\partial}{\partial z}(\hat{\varphi}_3-\hat{\varphi}_7)(\xi)|_{\xi=\mu_3^{-1}[t-\tau]}d\tau -$$

$$M_5\int_0^t \upsilon_5^2(\tau)\frac{d}{dt}\left(\hat{h}_3-\hat{g}_7\right)(t-\tau)d\tau + 2M_6\int_0^t \frac{\partial}{\partial z}\sum_{j=1}^{9} \hat{p}_{3j}(\xi)\upsilon_j^1(\xi,\tau)|_{\xi=\mu_3^{-1}[t-\tau]}d\tau +$$

$$+M_5\lambda_1\int_0^t\int_0^\tau \upsilon_5^2(\alpha)\frac{\partial}{\partial z}(\upsilon_3^1-\upsilon_7^1)(\xi,\tau-\alpha)d\alpha|_{\xi=\mu_3^{-1}[t-\tau]}d\tau +$$

$$+M_4\int_0^t \upsilon_5^2(\tau)\frac{c_{13}}{\lambda_1}\frac{d}{dt}\left(\hat{h}_1-\hat{g}_9\right)(t-\tau)d\tau + M_4\int_0^t \upsilon_6^2(\tau)\frac{c_{13}}{\lambda_1}\frac{d}{dt}\left(\hat{h}_1-\hat{g}_9\right)(t-\tau)d\tau +$$

$$+M_5\int_0^t \upsilon_6^2(\tau)\frac{\partial}{\partial z}(\hat{\varphi}_1-\hat{\varphi}_9)(\xi)|_{\xi=\mu_1^{-1}[t-\tau]}d\tau +$$

$$+2M_5\int_0^t \frac{\partial}{\partial z}\sum_{j=1}^{9} \hat{p}_{1j}(\xi)\upsilon_j^1(\xi,\tau)|_{\xi=\mu_1^{-1}[t-\tau]}d\tau +$$

$$+M_5\int_0^t\int_0^\tau \upsilon_6^2(\alpha)\frac{\partial}{\partial z}(\upsilon_1^1-\upsilon_9^1)(\xi,\tau-\alpha)d\alpha|_{\xi=\mu_1^{-1}[t-\tau]}d\tau, \qquad (58)$$

$$A_5^2 \upsilon = \upsilon_5^{02}(z,t) + M_6 \int_0^t \upsilon_5^2(\tau) \frac{\partial}{\partial z} (\hat{\varphi}_3 - \hat{\varphi}_7)(\xi)|_{\xi=\mu_3^{-1}[t-\tau]} d\tau -$$

$$-M_7 \int_0^t \upsilon_5^2(\tau) \frac{d}{dt} \big(\hat{h}_3 - \hat{g}_7\big)(t-\tau) d\tau + 2M_6 \int_0^t \frac{\partial}{\partial z} \sum_{j=1}^9 \hat{p}_{3j}(\xi) \upsilon_j^1(\xi,\tau)|_{\xi=\mu_3^{-1}[t-\tau]} d\tau +$$

$$+M_6 \int_0^t \int_0^\tau \upsilon_5^2(\alpha) \frac{\partial}{\partial z} (\upsilon_3^1 - \upsilon_7^1)(\xi, \tau-\alpha) d\alpha|_{\xi=\mu_3^{-1}[t-\tau]} d\tau, \qquad (59)$$

$$A_6^2 \upsilon = \upsilon_6^{02}(z,t) + 2M_8 \int_0^t \frac{\partial}{\partial z} \sum_{j=1}^9 \hat{p}_{1j}(\xi) \upsilon_j^1(\xi,\tau)|_{\xi=\mu_1^{-1}[t-\tau]} d\tau +$$

$$+M_8 \int_0^t \upsilon_6^2(\tau) \frac{\partial}{\partial z} (\hat{\varphi}_1 - \hat{\varphi}_9)(\xi)|_{\xi=\mu_1^{-1}[t-\tau]} d\tau - M_9 \int_0^t \upsilon_6^2(\tau) \frac{d}{dt} \big(\hat{h}_1 - \hat{g}_9\big)(t-\tau) d\tau +$$

$$+M_8 \int_0^t \int_0^\tau \upsilon_6^2(\alpha) \frac{\partial}{\partial z} (\upsilon_1^1 - \upsilon_9^1)(\xi, \tau-\alpha) d\alpha|_{\xi=\mu_1^{-1}[t-\tau]} d\tau. \qquad (60)$$

So taking into the equations (40) and the expression of (46)–(47), defined as follows

$$A_i^3 \upsilon = \upsilon_i^{03}(z,t) +$$

$$+ \int_{t_0^i}^t \frac{\partial}{\partial z} \left[\sum_{k=1}^9 \hat{p}_{ik}(\xi) \upsilon_k^1(\xi,\tau) + \frac{1}{2} \upsilon_6^2(\xi,\tau) (\hat{\varphi}_1 - \hat{\varphi}_9)(\xi) \right] |_{\xi=\mu_i^{-1}[\tau-t+\mu_i(z)]} d\tau +$$

$$+ \frac{\partial}{\partial z} t_0^i \int_0^{t_0^i} \upsilon_6^2(\xi, t_0^i - \tau)(\upsilon_1^1 - \upsilon_9^1)(\xi,\tau)|_{\xi=\mu_i^{-1}[t_0^i - t + \mu_i(z)]} d\tau +$$

$$+ \int_{t_0^i}^t \int_0^\tau \frac{\partial}{\partial z} [\upsilon_6^2(\xi, \tau-\alpha)(\upsilon_1^1 - \upsilon_9^1)(\xi,\alpha)] d\alpha|_{\xi=\mu_i^{-1}[\tau-t+\mu_i(z)]} d\tau, \quad i = 1,9, \quad (61)$$

$$A_i^3 \upsilon = \upsilon_i^{03}(z,t) +$$

$$+ \int_{t_0^i}^t \frac{\partial}{\partial z} \left[\sum_{k=1}^9 \hat{p}_{ik}(\xi) \upsilon_k^1(\xi,\tau) + \frac{1}{2} \upsilon_3^2(\xi,\tau) (\hat{\varphi}_2 - \hat{\varphi}_8)(\xi) \right] |_{\xi=\mu_2^{-1}[\tau-t+\mu_2(z)]} d\tau +$$

$$+ \frac{\partial}{\partial z} t_0^i \int_0^{t_0^i} \upsilon_3^2(\xi, t_0^i - \tau)(\upsilon_2^1 - \upsilon_8^1)(\xi,\tau)|_{\xi=\mu_2^{-1}[t_0^i - t + \mu_2(z)]} d\tau +$$

$$+ \int_{t_0^i}^t \int_0^\tau \frac{\partial}{\partial z} [\upsilon_3^2(\xi, \tau-\alpha)(\upsilon_2^1 - \upsilon_8^1)(\xi,\alpha)] d\alpha|_{\xi=\mu_2^{-1}[\tau-t+\mu_2(z)]} d\tau, \quad i = 2,8, \qquad (62)$$

$$A_i^3 \upsilon = \upsilon_i^{03}(z,t) +$$

$$+\int_{t_0^i}^{t} \frac{\partial}{\partial z}\left[\sum_{k=1}^{9} \hat{p}_{ik}(\xi)\upsilon_k^1(\xi,\tau) + \frac{1}{2}\upsilon_5^2(\xi,\tau)(\hat{\varphi}_3 - \hat{\varphi}_7)(\xi)\right]\Big|_{\xi=\mu_3^{-1}[\tau-t+\mu_3(z)]} d\tau +$$

$$+\frac{\partial}{\partial z} t_0^i \int_0^{t_0^i} \upsilon_5^2(\xi, t_0^i - \tau)(\upsilon_3^1 - \upsilon_7^1)(\xi,\tau)\Big|_{\xi=\mu_3^{-1}[t_0^i - t+\mu_3(z)]} d\tau +$$

$$+\int_{t_0^i}^{t}\int_0^{\tau} \frac{\partial}{\partial z}[\upsilon_5^2(\xi,\tau-\alpha)(\upsilon_3^1 - \upsilon_7^1)(\xi,\alpha)]d\alpha\Big|_{\xi=\mu_3^{-1}[\tau-t+\mu_3(z)]} d\tau, \quad i = 3,7, \quad (63)$$

$$A_4^3 \upsilon = \int_0^t \frac{\partial}{\partial z}\left[\sum_{j=1}^{9} \hat{p}_{4j}(z)\upsilon_j^1(z,\tau)\right] d\tau + \int_0^t \frac{\partial}{\partial z}[\upsilon_1^2(\tau)\hat{\varphi}_4(z) + (\upsilon_1^2 - \upsilon_2^2)(\tau)\hat{\varphi}_6(z)]d\tau +$$

$$+\int_0^t \frac{\partial}{\partial z}\left[\frac{c_{13}}{\lambda_1}(\upsilon_1^2 - \upsilon_6^2)(\tau)(\hat{\varphi}_1 - \hat{\varphi}_9)(z)\right] d\tau +$$

$$+\int_0^t \int_0^{\tau} \frac{\partial}{\partial z}\left[\frac{c_{13}}{\lambda_1}(\upsilon_1^2 - \upsilon_6^2)(\tau)(\upsilon_1^1 - \upsilon_9^1)(z,\alpha)\right] d\alpha d\tau +$$

$$+\int_0^t \int_0^{\tau} \frac{\partial}{\partial z}[\upsilon_1^2(\alpha)\upsilon_4^1(z,\tau-\alpha) + (\upsilon_1^2 - \upsilon_2^2)(\alpha)\upsilon_6^1(z,\tau-\alpha)]d\alpha d\tau, \quad (64)$$

$$A_5^3 \upsilon = \int_0^t \frac{\partial}{\partial z}\left[\sum_{j=1}^{9} \hat{p}_{5j}(z)\upsilon_j^1(z,\tau)\right] d\tau +$$

$$+\int_0^t \frac{\partial}{\partial z}\upsilon_1^2(\tau)\hat{\varphi}_4(z)d\tau + \int_0^t \int_0^{\tau} \frac{\partial}{\partial z}\upsilon_1^2(\alpha)\upsilon_4^1(z,\tau-\alpha)d\alpha d\tau +$$

$$+\int_0^t \frac{\partial}{\partial z}\left[\frac{c_{13}}{\lambda_1}\left(\upsilon_4^2 - \upsilon_5^2\right)(\tau)(\hat{\varphi}_1 - \hat{\varphi}_9)(z)\right] d\tau +$$

$$+\int_0^t \int_0^{\tau} \frac{\partial}{\partial z}\left[\frac{c_{13}}{\lambda_1}(\upsilon_1^2 - \upsilon_6^2)(\tau)(\upsilon_1^1 - \upsilon_9^1)(z,\alpha)\right] d\alpha d\tau, \quad (65)$$

$$A_6^3 \upsilon = \int_0^t \frac{\partial}{\partial z}\left[\sum_{j=1}^{9} \hat{p}_{6j}(z)\upsilon_j^1(z,\tau)\right] d\tau +$$

$$+\int_0^t\int_0^\tau \frac{\partial}{\partial z}\upsilon_2^2(\alpha)\upsilon_6^1(z,\tau-\alpha)d\alpha d\tau+\int_0^t \frac{\partial}{\partial z}\upsilon_2^2(\tau)\hat{\varphi}_6(z)d\tau, \quad (66)$$

where

$$\upsilon_i^{01}(z,t)=\omega_i(z_0^i,t_0^i),\quad i=1,2,3,7,8,9,$$

$$\upsilon_1^{02}(z,t)=M_1(z)Q_1(z,t),\quad \upsilon_2^{02}(z,t)=Q_2(z,t),\quad \upsilon_3^{02}(z)=-2M_2(z)Q_3(z),$$

$$\upsilon_4^{02}(z,t)=M_4(z)Q_4(z,t),\quad \upsilon_5^{02}(z)=-2M_6(z)Q_5(z)\quad \upsilon_3^{02}(z)=-2M_8(z)Q_6(z),$$

$$\upsilon_i^{03}(z,t)=\frac{\partial}{\partial z}\omega_i(z_0^i,t_0^i)-\frac{\partial}{\partial z}t_0^i\left[\sum_{j=1}^{9}\hat{p}_{ij}(z_0^i)\omega_j(z_0^i,t_0^i)\right],\quad i=1,2,3,7,8,9.$$

Let $C_s(D_0)$, $(s\le 0)$ be the Banach space of continuous functions with the ordinary norm, denoted by $\|\cdot\|_s$,

$$\|\upsilon\|_s=\max\{\max_{1\le i\le 9,(z,t)\in D_0}\left|\upsilon_i^1(z,t)\right|,\max_{1\le i\le 6,t\in[0,T]}\left|\upsilon_i^2(t)\right|,\max_{1\le i\le 9,(z,t)\in D_0}\left|\upsilon_i^3(z,t)\right|\}.$$

Obviously, C_s with $s=0$ is the usual space of continuous functions with the ordinary norm, denoted by $\|\cdot\|$ in what follows, because

$$e^{-sH}\|\upsilon\|\le\|\upsilon\|_s\le\|\upsilon\|.$$

The norms $\|\upsilon\|_s$ and $\|\upsilon\|$ are equivalent for any $T\in(0,+\infty)$, $H\in(0,\infty)$, where $s\in(0,1)$ and we choose that number later.

Next, consider the set of functions $S(\upsilon^0,r)\subset C_s(D_0)$, satisfying the inequality

$$\|\upsilon-\upsilon^0\|_s\le r, \quad (67)$$

where r is a known number, the vector function $\upsilon^0(z,t)=(\upsilon_i^{01}(z,t),\ i=\overline{1,9},\ \ \upsilon_i^{02}(t),\ i=\overline{1,6},\ \ \upsilon_i^{03}(z,t),\ i=\overline{1,9})$, defined by the free terms of the operator equation (48). It is easy to see that for $\upsilon\in S(\upsilon^0,r)$ the estimate $\|\upsilon\|_s\le\|\upsilon^0\|_s+r\le\|\upsilon^0\|+r:=r_0$. Thus, r_0 is known.

Let us introduce the following notation:

$$\varphi_0:=\max_{i=\overline{1,9}}\|\hat{\varphi}_i\|_{C^2[0,H]},\quad g_0:=\max_{i=1,2,3,7,8,9}\|\hat{g}_i\|_{C^2[0,\infty)},\quad h_0:=\max_{i=\overline{1,6}}\|h_i\|_{C^2[0,\infty)},$$

$$M^1=\max_{i,j=\overline{1,9}}\|M_i(z)\|_{C[0,H]},M^2=\max_{i,j=\overline{1,9}}\|\hat{p}_{ij}(z)\|_{C[0,H]},$$

$$M^3=\max_{i=1,3}\left\|\frac{c_{13}(z)}{\lambda_i(z)}\right\|_{C^1[0,H]},\qquad M^4=\max\|c_{13}(z)\|_{C^1[0,H]},$$

$$M^5=\max_{i=1,3}\|\lambda_i(z)\|_{C^1[0,H]},M^6=\max_{i=\overline{1,3,7,9}}\left\|\frac{\partial t_1^i}{\partial z}\right\|_{C[0,H]},$$

$$M^0=\max_{i=\overline{1,6}}\|M^i,[\hat{\varphi}_6(0)]^{-1}\|.$$

Note that the operator A maps the space $C_s(D_0)$ into itself. Let us show that for a suitable choice of s (recall that $H>0$ is an arbitrary fixed number) it is on the set $S(\upsilon^0,r)$ a contraction operator. First, let us make sure that the operator A takes the set $S(\upsilon^0, r)$ into itself, that is, it follows from the condition $\upsilon(z,t) \in S(\upsilon^0, r)$ that $A\upsilon \in S(\upsilon^0, r)$, if s satisfies some constraints. In fact, for any $(z,t) \in D_0$ and $\upsilon \in S(\upsilon^0, r)$ the following inequalities hold:

$$\left\|A_1^1\upsilon - \upsilon_1^{01}\right\| = \max\left|(A_1^1\upsilon - \upsilon_1^{01})e^{-st}\right| \leq$$

$$\leq \left|\int_{t_0^1}^{t}\left[\sum_{j=1}^{9}\hat{p}_{1j}(\xi)\upsilon_j^1(\xi,\tau)e^{-s\tau} + \frac{\upsilon_6^2(\tau)e^{-s\tau}}{2}(\hat{\varphi}_1(\xi) - \hat{\varphi}_9(\xi))\right]e^{-s(t-\tau)}|_{\xi=\mu_1^{-1}[t-\tau+\mu_1(z)]}d\tau + \right.$$

$$\left. + \int_{t_0^1}^{t} e^{-s(t-\tau)}\int_0^{\tau}\frac{\upsilon_6^2(\alpha)e^{-s\alpha}}{2}(\upsilon_1^1(\xi,\tau-\alpha) - \upsilon_9^1(\xi,\tau-\alpha))e^{-s(\tau-\alpha)}d\alpha|_{\xi=\mu_1^{-1}[t-\tau+\mu_1(z)]}d\tau\right| \leq$$

$$\leq (9M^0 \parallel \upsilon \parallel_s + \varphi_0 \parallel \upsilon \parallel_s + \parallel \upsilon \parallel_s \parallel \upsilon \parallel_s)\int_{t_0^1}^{t} e^{-s(t-\tau)}d\tau \leq$$

$$\leq \frac{r_0}{s}(9M^0 + \varphi_0 + r_0) := \frac{\gamma_{11}}{s}.$$

Using similar calculations, for the remaining equations, we obtain

$$\|A\upsilon - \upsilon^0\| \leq \frac{1}{s}\max\left\{\max_{j=\overline{1,9}}\{\gamma_{1j}\}, \max_{j=\overline{1,6}}\{\gamma_{2j}\}, \max_{j=\overline{1,9}}\{\gamma_{3j}\}\right\} := \frac{1}{s}\gamma^0,$$

where

$$\gamma_{1j} := r_0(9M^0 + \varphi_0 + r_0), \quad j = 2,3,7,8,9,$$

$$\gamma_{14} := r_0(9M^0 + 4M^0\varphi_0 + 4M^0 r_0 + 3r_0 + 3\varphi_0),$$

$$\gamma_{15} := r_0(9M^0 + \varphi_0 + r_0 + 4M^0\varphi_0 + 4M^0 r_0),$$

$$\gamma_{16} := r_0(9M^0 + r_0 + \varphi_0),$$

$$\gamma_{21} := r_0(6M^0 h_0 + 3M^0 g_0 + 2M^0\varphi_0 + h_0 + 36M^0 + 2M^0 r_0),$$

$$\gamma_{22} := r_0 M^0 h_0, \qquad \gamma_{2i} := r_0 M^0(18 + 2\varphi_0 + +h_0 + g_0 + 2r_0), \quad i = 3,5,6$$

$$\gamma_{24} := 2M^0 r_0((M^0)^2\varphi_0 + g_0 + h_0 + \varphi_0 + 18 + M^0 r_0 + r_0 + 2M^0 + M_0 h_0 + M^0 g_0),$$

$$\gamma_{3j} := 2r_0(9M^0 + \varphi_0 + r_0 + M^0 r_0 + 2r_0), \quad j = 1,2,3,7,8,9,$$

$$\gamma_{34} := r_0(9M^0 + 12M^0 r_0 + 3\varphi_0(12M^0 + 3) + 6r_0),$$

$$\gamma_{35} := r_0(18M^0 + r_0 + \varphi_0 + 12M^0\varphi_0 + 12M^0 r_0),$$

$$\gamma_{36} := r_0(18M^0 + r_0 + \varphi_0).$$

Choosing $s > (1/r)\gamma^0$, we get that the operator A maps the set $S(\upsilon^0, r)$ into itself.

Now, let υ and $\tilde{\upsilon}$ be two arbitrary elements in $S(\upsilon^0, r)$. Using the obvious inequality

$$\left|\upsilon_i^k\upsilon_i^l - \tilde{\upsilon}_i^k\tilde{\upsilon}_i^l\right|e^{-st} \le \left|\upsilon_i^k - \tilde{\upsilon}_i^k\right|\left|\upsilon_i^l\right|e^{-st} + \left|\tilde{\upsilon}_i^k\right|\left|\upsilon_i^l - \tilde{\upsilon}_i^l\right|e^{-st} \le 2r_0 \parallel \upsilon - \tilde{\upsilon} \parallel_s, \quad (z,t) \in D_T,$$

after some easy estimations, we find that for $(z,t) \in D_T$,

$$\left\|A_1^1\upsilon - A_1^1\tilde{\upsilon}\right\| = \max\left|(A_1^1\upsilon - A_1^1\tilde{\upsilon})e^{-st}\right| =$$

$$\int_{t_0^1}^{t}\left[\sum_{j=1}^{9}\hat{p}_{1j}(\xi)\left(\upsilon_j^1(\xi,\tau) - \widetilde{\upsilon_j^1}(\xi,\tau)\right) + \frac{\upsilon_6^2(\tau) - \widetilde{\upsilon_6^2}(\tau)}{2}(\hat{\varphi}_1(\xi) - \hat{\varphi}_9(\xi))\right]|_{\xi=\mu_1^{-1}[t-\tau+\mu_1(z)]}d\tau +$$

$$+\int_{t_0^1}^{t}\int_{0}^{\tau}\frac{1}{2}\left(\upsilon_6^2(\alpha)\upsilon_1^1(\xi,\tau-\alpha) - \widetilde{\upsilon_6^2}(\alpha)\widetilde{\upsilon_1^1}(\xi,\tau-\alpha) - \upsilon_6^2(\alpha)\upsilon_9^1(\xi,\tau-\alpha)\right)d\alpha|_{\xi=\mu_1^{-1}[t-\tau+\mu_1(z)]}d\tau +$$

$$+\int_{t_0^1}^{t}\int_0^{\tau}\frac{1}{2}\widetilde{\upsilon_6^2}(\alpha)\widetilde{\upsilon_9^1}(\xi,\tau-\alpha)d\alpha|_{\xi=\mu_1^{-1}[t-\tau+\mu_1(z)]}d\tau \le$$

$$\le \frac{1}{s}[\|\upsilon - \tilde{\upsilon}\|_s(9M^0 + \varphi_0)r_0 + 4r_0\|\upsilon - \tilde{\upsilon}\|_s] \le \frac{\|\upsilon - \tilde{\upsilon}\|_s}{s}[9M^0 + \varphi_0 + 4r_0] := \frac{1}{s}\gamma_{41}$$

and hence we have

$$\|A\upsilon - A\tilde{\upsilon}\| \le \frac{1}{s}\max\left\{\max_{j=\overline{1,9}}\{\gamma_{4j}\}, \max_{j=\overline{1,6}}\{\gamma_{5j}\}, \max_{j=\overline{1,9}}\{\gamma_{6j}\}\right\} := \frac{1}{s}\gamma^1,$$

where

$$\gamma_{4j} := \|\upsilon - \tilde{\upsilon}\|_s(9M^0 + \varphi_0 + r_0), \quad j = 2,3,7,8,9,$$

$$\gamma_{44} := \|\upsilon - \tilde{\upsilon}\|_s(9M^0 + 4M^0\varphi_0 + 4M^0r_0 + 3r_0 + 3\varphi_0),$$

$$\gamma_{45} := \|\upsilon - \tilde{\upsilon}\|_s(9M^0 + \varphi_0 + r_0 + 4M^0\varphi_0 + 4M^0r_0),$$

$$\gamma_{46} := \|\upsilon - \tilde{\upsilon}\|_s(9M^0 + r_0 + \varphi_0),$$

$$\gamma_{51} := \|\upsilon - \tilde{\upsilon}\|_s(6M^0h_0 + 3M^0g_0 + 2M^0\varphi_0 + h_0 + 36M^0 + 2M^0r_0),$$

$$\gamma_{52} := \|\upsilon - \tilde{\upsilon}\|_s M^0h_0,$$

$$\gamma_{54} := \|\upsilon - \tilde{\upsilon}\|_s 2M^0((M^0)^2\varphi_0 + g_0 + h_0 + \varphi_0 + 18 + M^0r_0 + r_0 + 2M^0 + M_0h_0 + M^0g_0),$$

$$\gamma_{5j} := \|\upsilon - \tilde{\upsilon}\|_s M^0(18 + 2\varphi_0 + +h_0 + g_0 + 2r_0), \quad j = 3,5,6$$

$$\gamma_{6j} := 2\|\upsilon - \tilde{\upsilon}\|_s(9M^0 + \varphi_0 + r_0 + M^0r_0 + 2r_0), \quad j = 1,2,3,7,8,9,$$

$$\gamma_{64} := \|\upsilon - \tilde{\upsilon}\|_s(9M^0 + 12M^0r_0 + 3\varphi_0(12M^0 + 3) + 6r_0),$$

$$\gamma_{65} := \|\upsilon - \tilde{\upsilon}\|_s(18M^0 + r_0 + \varphi_0 + 12M^0\varphi_0 + 12M^0r_0),$$

$$\gamma_{66}:=\|\upsilon-\tilde{\upsilon}\|_s(18M^0+r_0+\varphi_0).$$

Choosing now $s>\gamma^1$, we get, that the operator A compresses the distance between the elements $\upsilon,\tilde{\upsilon}$ to $S(\upsilon^0,r)$.

As follows from the performed estimates, if the number s is chosen from conditions $s>s^*:=max\{\gamma^0,\gamma^1\}$, then the operator A is contracting on $S(\upsilon^0,r)$. In this case, according to the principle Banach the equation [24, page 87-97] (48) has the only solution in $S(\upsilon^0,r)$ for any fixed $H>0$. Theorem 2 is proved.

By the found functions $r'_{ij}(t),\ i,j=\overline{1,3}$ the functions $r_{ij}(t),\ i,j=\overline{1,3}$ are found by the formulas

$$r_{ij}(t)=r_{ij}(0)+\int_0^t r'_{ij}(\tau)d\tau,\quad i,j=\overline{1,3}.$$

Note that by the functions $r_{ij}(t),\ i,j=\overline{1,3}$, the functions $K_{ij}(t),\ i,j=\overline{1,3}$, are defined as solutions of integral equations (4).

References:

1. *Mura T. Micromechanics of defects in solids, second, revised edition. USA, IL, Evanston, Northwestern University, 1987.*
2. *Galin L.A. Contact problems of the theory of elasticity and viscoelasticity. Moscow: Nauka, 1980(In Russian).*
3. *Durdiev D.K. Some multidimensional inverse problems of memory determination in hyperbolic equations. Zh. Mat. Fiz., Anal., Geom. 2007. Vol.3. Issue 4. pp. 411-423.*
4. *Delesan E., Ruaye D. Elastic waves in solids. Moscow: Nauka, 1982(In Russian).*
5. *Godunov S.K. Equations of Mathematical Physics. Moscow: Nauka, Ch. ed. physical-mat. lit. 1979 (In Russian).*
6. *Romanov V.G. Inverse problems of mathematical physics. Utrecht, The Netherlands, 1987.*
7. *V. Volterra, Theory of functionals, integral and integro-differential equations, Moscow: Nauka, 1982(In Russian).*
8. *L. D. Landau, E. M. Lifshits, Electrodynamics of continuous media, Moscow: Nauka, 1982(In Russian).*
9. *Janno J., Von Wolfersdorf L. Inverse problems for identification of memory kernels in viscoelasticity. Math. Methods Appl. Sci. 1997. Vol. 20. Issue 4. pp. 291-314.*
10. *Z. D. Totieva, D. K. Durdiev, The Problem of Finding the One-Dimensional Kernel of the Thermoviscoelasticity Equation, Mathematical Methods in the Applied Sciences, 103:1-2, (2018), 118-132.*
11. *Durdiev D.K., Totieva Z.D. The problem of determining the one-dimensional matrix kernel of the system of viscoelasticity equations. Mathematical Methods in the Applied Sciences. 2018. V. 41. Issue 17. pp. 8019-8032.*
12. *Romanov V.G. On the determination of the coefficients in the viscoelasticity equations. Sib. Zh. Ind. Mat. 2014. Vol. 55. Issue 3. pp. 503-510.*
13. *Durdiev D.K., Rakhmonov A. A. Inverse problem for the system integro-differential equation SH waves in a visco-elastic porous medium: global solvability. Theoretical and Mathematical Physics. 2018. Vol. 195. Issue 3. pp. 925-940.*
14. *Safarov J.SH., Durdiev D.K. Inverse Problem for an integro-differential Equation of acoustics. Differential Equations. 2018. Vol. 54. Issue 1. pp. 134-142.*
15. *Romanov V.G. Estimates for the stability of the solution in the problem of determining the kernel of the viscoelasticity equation. Sib. Zh. Ind. Mat. 2012. Vol. 15. Issue. 1. pp. 86-98.*
16. *Totieva Zh.D., Durdiev D. Q. The problem of determining the multidimensional kernel of viscoelasticity equation. Vladikavkaz Mathematical Journal 2015. Vol. 17. Issue 4. pp. 18-43.*
17. *Durdiev D.K., Totieva Z.D. The problem of determining the one-dimensional kernel of viscoelasticity equation with a source of explosive type. Journal of Inverse and Ill-Posed Problems. 2020. Vol. 28. Issue 1. pp. 43-52.*

18. Durdiev D.K., Rakhmonov A. A. The problem of determining two-dimensional kernel in a system of integro-differential equations of a viscoelastic porous medium. Sib. Zh. Ind. Mat. 2020. Vol. 23. Issue 2. pp. 63-80.

19. Durdiev U.D. An inverse problem for a system of equations of viscoelasticity in homogeneous anisotropic medium. Sib. Zh. Ind. Mat. 2019, Vol 22, Issue 4 (80), pp. 26-32.

20. Durdiev D. K., Bozorov Z. R. A problem of determining the kernel of integrodifferential wave equation with weak horizontal properties. Far Eastern Mathematical Journal. 2013. V. 13. No 2. P. 209-221.

21. Durdiev D.K., Turdiev Kh.Kh. Inverse Problem for a First-Order Hyperbolic System with Memory. Differential equations. 2020. Ò. 56. Issue 12. pp.666-1675.

22. Durdiev D.K., Turdiev Kh.Kh. The problem of finding the kernels in the system of integro-differential Maxwell's equations. Sib. Zh. Ind. Mat. 2021, Vol. 24:2, pp.38-61.

23. Kilbas A.A. Integral Equations: Course of Lectures. Minsr: BSU, 2005. (In Russian)

24. Kolmogorov A.N., Fomin S.V. Elements of function theory and functional analysis. Nauka, M., 1989 (in Russian).