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The Problem of Finding the Kernels in the System of Integro-Differential Maxwell's Equations

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Abstract—We pose the direct and inverse problem of finding the electromagnetic field and the diagonal memory matrix for the reduced canonical system of integro-differential Maxwell's equations. The problems are replaced by a closed system of Volterra-type integral equations of the second kind with respect to the Fourier transform in the variables x_1 and x_2 of t he solution to the direct problem and the unknowns of the inverse problem. To this system, we then apply the method of contraction mapping in the space of continuous functions with a weighted norm. Thus, we prove the global existence and uniqueness theorems for solutions to the posed problems.

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INTRODUCTION

The propagation of various waves is described by hyperbolic systems of first-order equations. In media with aftereffect, such a phenomenon in generally depends on the previous state of the process. An example is given by the phenomenon of the propagation of electromagnetic waves in media with dispersion. It turns out that, in these media, a violation occurs of the unique dependence of D and B (the induction of the electric and magnetic fields respectively) on E and H (the intensities of the corresponding fields) at the same time. The more general kind of linear dependence between $D(x,t)$, $B(x,t)$ and the corresponding values of the functions $E(x,t)$, $H(x,t)$ at all previous time can be written in the form of the integral relations (see [1, pp. 357–376]):

$$D(x,t) = \hat{\varepsilon}E + \int_0^t \varphi(t-\tau)E(x,\tau)\,d\tau, \qquad B(x,t) = \hat{\mu}H + \int_0^t \psi(t-\tau)H(x,\tau)\,d\tau. \tag{1}$$

Here

$$E = (E_1, E_2, E_3), \qquad H = (H_1, H_2, H_3), \qquad D = (D_1, D_2, D_3),$$
$$B = (B_1, B_2, B_3), \qquad x = (x_1, x_2, x_3),$$

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$\varphi(t) = \text{diag}(\varphi_1, \varphi_2, \varphi_3)$ and $\psi(t) = \text{diag}(\psi_1, \psi_2, \psi_3)$ are diagonal matrices that represent the memory.

In an anisotropic medium with dispersion, the system of Maxwell's equations has the form

$$\begin{gathered}\nabla \times H = \frac{\partial}{\partial t} D(x,t) + \hat{\sigma} E + J, \qquad \nabla \times E = -\frac{\partial}{\partial t} B(x,t), \\ \text{div } B = 0, \qquad \text{div } D = \rho.\end{gathered} \tag{2}$$

The matrices $\hat{\varepsilon}$ and $\hat{\mu}$ are assumed positive definite, symmetric, and depending only on x_3, and $\hat{\sigma}$ is a constant matrix. In the first equation of (2), $J = J(x,t)$ is a vector-function characterizing the external current density. The third equation in (2) is a consequence of the second equation and the condition

$$\text{div } (\hat{\mu} H)|_{t=0} = 0. \tag{3}$$

Indeed, applying the operation div to the second equation in (2), taking into account the equality $\text{div } (\nabla \times E) = 0$, we have

$$\frac{\partial}{\partial t}(\text{div } B) = 0.$$

Hence, integrating the last equality under condition (3), we obtain the third equality in (2). The fourth equation in (2) determines the electric charge density after the distribution of the induction of the electric field has been found.

Supposing the fulfillment of condition (3), we will consider system (1), (2) as an independent object of study. Write it as a symmetric hyperbolic system:

$$A_0 \frac{\partial}{\partial t} U + \sum_{j=1}^{3} A_j \frac{\partial}{\partial x_j} U + A_4 U = \int_0^t K(t-\tau) U(x,\tau)\, d\tau + \hat{J}(x,t), \tag{4}$$

in which $U = (U_1, \ldots, U_6)^*$ is the column vector with entries $U_k = E_k$, $U_{k+3} = H_k$, $k = \overline{1,3}$; A_j, $j = \overline{0,4}$, are symmetric matrices, where A_0 is positive definite; $K(t) = \text{diag}(\varphi_1, \varphi_2, \varphi_3, \psi_1, \psi_2, \psi_3)$, $J = (\chi_1, \ldots, \chi_6)^*$ is the column-vector with entries $\chi_k = \chi_k(x,t)$, $\chi_{k+3} = 0$, $k = \overline{1,3}$, and χ_k are some given sufficiently smooth functions. The matrices A_j have the cell structure:

$$A_0 = \begin{pmatrix} \hat{\varepsilon} & 0 \\ 0 & \hat{\mu} \end{pmatrix}_{6\times 6}, \qquad A_j = \begin{pmatrix} 0 & p_j \\ p_j^* & 0 \end{pmatrix}_{6\times 6}, \qquad j = 1,2,3,$$

$$\begin{gathered} p_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad p_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad p_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ A_4 = \begin{pmatrix} \varphi(0) + \hat{\sigma} & 0 \\ 0 & \psi(0) \end{pmatrix}_{6\times 6}, \qquad K(t) = \begin{pmatrix} -\varphi'(t) & 0 \\ 0 & -\psi'(t) \end{pmatrix}_{6\times 6}, \\ U = (E, H)^*, \qquad \hat{J} = (-J, 0_{1\times 3})^*, \end{gathered} \tag{5}$$

where $*$ is the transposition symbol; $0_{1\times 3}$ stands for the row vector $(0; 0; 0)$.

Multiplying equation (4) from the left by the inverse matrix A_0^{-1}, we obtain

$$I_6\frac{\partial}{\partial t}U+\sum_{j=1}^{3}B_j\frac{\partial}{\partial x_j}U+B_4U=\int_0^t K_0(x_3,t-\tau)U(x,\tau)d\tau+J_0. \tag{6}$$

Here and in what follows, I_6 stands for the identity matrix of order 6 and $B_j=A_0^{-1}A_j$, $j=\overline{1,4}$.

Introduce the notations

$$\varepsilon=\hat{\varepsilon}^{-1}=(\varepsilon_{ij}),\qquad \mu=\hat{\mu}^{-1}=(\mu_{ij}),\qquad \sigma=\varepsilon\hat{\sigma}=(\sigma_{ij}). \tag{7}$$

In accordance with (4), we have

$$B_j=\begin{pmatrix}0 & \varepsilon p_j\\ \mu p_j^* & 0\end{pmatrix},\quad j=\overline{1,3},\qquad B_4=\begin{pmatrix}\varepsilon\varphi(0)+\sigma & 0\\ 0 & \mu\psi(0)\end{pmatrix},$$

$$J_0=A_0^{-1}\hat{J},\qquad K_0(x_3,t)=A_0^{-1}K(x_3,t)=\begin{pmatrix}-\varepsilon\varphi'(t) & 0\\ 0 & -\mu\psi'(t)\end{pmatrix}.$$

Reduce system (6) to the canonical form. As is known from linear algebra (see [2, p. 149—153]), in the case under consideration, there exists a nondegenerate matrix T such that $T^{-1}B_3T=\Lambda$, where Λ is the diagonal matrix whose diagonal contains the eigenvalues of B_3.

Some matrix T with the above properties was constructed in [3, pp. 5-20]. It looks as

$$T(x_3)=\begin{pmatrix}q_1 & 0 & q_1 & 0 & 0 & 0\\ 0 & q_2 & 0 & q_2 & 0 & 0\\ q_3 & q_4 & q_3 & q_4 & 1 & 0\\ 0 & 1/q_2 & 0 & -1/q_2 & 0 & 0\\ -1/q_1 & 0 & 1/q_1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix}, \tag{8}$$

where

$$q_1=\left(\frac{\varepsilon_{11}}{\mu_{22}}\right)^{1/4},\qquad q_2=\left(\frac{\varepsilon_{22}}{\mu_{11}}\right)^{1/4},\qquad q_3=\frac{\varepsilon_{31}}{\varepsilon_{11}^{3/4}\mu_{22}^{1/4}},\qquad q_4=\frac{\varepsilon_{32}}{\varepsilon_{22}^{3/4}\mu_{11}^{1/4}}. \tag{9}$$

Note that T is defined not uniquely.

The inverse matrix to T is defined by the formula

$$T^{-1}(x_3) = \begin{pmatrix} 1/(2q_1) & 0 & 0 & 0 & -q_1/2 & 0 \\ 0 & 1/(2q_2) & 0 & q_2/2 & 0 & 0 \\ 1/(2q_1) & 0 & 0 & 0 & q_1/2 & 0 \\ 0 & 1/(2q_2) & 0 & -q_2/2 & 0 & 0 \\ -q_3/q_1 & -q_4/q_2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{10}$$

In (6), introduce the new function by the equality

$$U = T\overline{U} \tag{11}$$

and multiply (11) by T^{-1} from the left. Then for $\overline{U}$ we obtain the equation

$$\left(I_6 \frac{\partial}{\partial t} + \Lambda \frac{\partial}{\partial x_3} + \sum_{j=1}^{2} C_j \frac{\partial}{\partial x_j} + C\right)\overline{U} = \int\limits_0^t \overline{K}(x_3, t-\tau)\overline{U}(x,\tau)\, d\tau + F, \tag{12}$$

where

$$\begin{gathered} C = C_0 + C_4, \qquad C_0 = T^{-1}B_3 \frac{\partial}{\partial x_3} T, \qquad C_i = T^{-1}B_i T, \quad i = \overline{1,4}, \\ C_3 = \Lambda = \sqrt{p}\Lambda_0, \qquad p = \varepsilon_{11}\mu_{22} = \varepsilon_{22}\mu_{11}, \qquad \Lambda_0 = \operatorname{diag}(-1,-1,1,1,0,0), \\ \overline{K}(x_3,t) = T^{-1}K_0(x_3,t)T = (\overline{a}_{ij})_{i,j=1}^6 (x_3,t), \qquad F = T^{-1}J_0. \end{gathered} \tag{13}$$

Using the consequences stemming from (13) for the entries of the matrices ε and μ

$$\begin{gathered} \varepsilon_{11} = q_1^2\sqrt{p}, \qquad \varepsilon_{12} = 0, \qquad \varepsilon_{13} = q_1q_3\sqrt{p}, \qquad \varepsilon_{22} = q_2^2\sqrt{p}, \qquad \varepsilon_{23} = q_2q_4\sqrt{p}, \\ \mu_{11} = q_2^{-2}\sqrt{p}, \qquad \mu_{22} = q_1^{-2}\sqrt{p}, \end{gathered} \tag{14}$$

and introducing the additional notations by the equalities

$$\varepsilon_{33} = \left(q_3^2 + q_4^2 + \frac{1}{2}q_5\right)\sqrt{p}, \qquad \mu_{33} = \frac{1}{2}\frac{q_6\sqrt{p}}{q_1q_2}, \tag{15}$$

we can write the matrices C_i involved in (12) in the form

$$C_1 = \frac{\sqrt{p}}{2q_1}\begin{pmatrix} 2q_3 & q_4 & 0 & q_4 & 1 & 0 \\ q_4 & 0 & -q_4 & 0 & 0 & q_1q_2 \\ 0 & -q_4 & -2q_3 & -q_4 & -1 & 0 \\ q_4 & 0 & -q_4 & 0 & 0 & q_1q_2 \\ q_5 & 0 & -q_5 & 0 & 0 & 0 \\ 0 & q_6 & 0 & q_6 & 0 & 0 \end{pmatrix}, \quad C_2 = \frac{\sqrt{p}}{2q_2}\begin{pmatrix} 0 & q_3 & 0 & -q_3 & 0 & -q_1q_2 \\ q_3 & 2q_4 & q_3 & 0 & 1 & 0 \\ 0 & q_3 & 0 & -q_3 & 0 & -q_1q_2 \\ -q_3 & 0 & -q_3 & -2q_4 & -1 & 0 \\ 0 & q_5 & 0 & -q_5 & 0 & 0 \\ -q_6 & 0 & -q_6 & 0 & 0 & 0 \end{pmatrix},$$

$$C_0 = \frac{\sqrt{p}}{2q_2}\begin{pmatrix} 0 & 0 & -\dfrac{\partial}{\partial x_3}\ln q_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\dfrac{\partial}{\partial x_3}\ln q_2 & 0 & 0 \\ \dfrac{\partial}{\partial x_3}\ln q_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \dfrac{\partial}{\partial x_3}\ln q_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad C_4 = \Big(\varkappa_{ij} + M_{ij} \Big)_{i,j=1}^{6}.$$

Here we use the following notations:

$$\varkappa_{11} = \varkappa_{33} = \frac{\sqrt{p}}{2}\left(q_1^2\varphi_1(0) + q_3^2\varphi_3(0) + \frac{\psi_2(0)}{q}{}^{2}_{1}\right),$$

$$\varkappa_{13} = \varkappa_{31} = \frac{\sqrt{p}}{2}\left(q_1^2\varphi_1(0) + q_3^2\varphi_3(0) - \frac{\psi_2(0)}{q_1^2}\right),$$

$$\varkappa_{12} = \varkappa_{14} = \varkappa_{21} = \varkappa_{23} = \varkappa_{32} = \varkappa_{34} = \varkappa_{41} = \varkappa_{43} = \frac{\sqrt{p}}{2}\, q_3q_4\varphi_3(0),$$

$$\varkappa_{15} = \varkappa_{35} = \frac{\sqrt{p}}{2}\, q_3\varphi_3(0), \qquad \varkappa_{22} = \varkappa_{44} = \frac{\sqrt{p}}{2}\left(q_2^2\varphi_2(0) + q_4^2\varphi_3(0) + \frac{\psi_1(0)}{q}{}^{2}_{2}\right),$$

$$\varkappa_{24} = \varkappa_{42} = \frac{\sqrt{p}}{2}\left(q_2^2\varphi_2(0) + q_4^2\varphi_3(0) - \frac{\psi_1(0)}{q_2^2}\right), \qquad \varkappa_{25} = \varkappa_{45} = \frac{\sqrt{p}}{2} q_4\varphi_3(0),$$

$$\varkappa_{51} = \varkappa_{53} = \frac{\sqrt{p}}{2} q_3q_5\varphi_3(0), \qquad \varkappa_{52} = \varkappa_{54} = \frac{\sqrt{p}}{2} q_4q_5\varphi_3(0),$$

$$\varkappa_{55} = \frac{\sqrt{p}}{2} q_5\varphi_3(0), \qquad \varkappa_{6j} = \varkappa_{j6} = 0, \quad j = \overline{1,5}, \qquad \varkappa_{66} = \sqrt{p}\frac{q_6\psi_3(0)}{q_1q_2};$$

$$M_{11} = M_{13} = M_{31} = M_{33} = \frac{1}{2q_1}(\sigma_{11}q_1 + \sigma_{13}q_3),$$

$$M_{12} = M_{14} = M_{32} = M_{34} = \frac{1}{2q_1}(\sigma_{12}q_2 + \sigma_{13}q_3), \qquad M_{15} = \frac{1}{2q_1}\sigma_{13},$$

$$M_{21} = M_{23} = M_{41} = M_{43} = \frac{1}{2q_2}(\sigma_{21}q_1 + \sigma_{23}q_3), \qquad M_{25} = \frac{1}{2q_2}\sigma_{23},$$

$$M_{22} = M_{24} = M_{42} = M_{44} = \frac{1}{2q_2}(\sigma_{22}q_2 + \sigma_{23}q_4),$$

$$M_{51} = M_{53} = \left(-\sigma_{11}\frac{q_3}{q_1} - \sigma_{21}\frac{q_4}{q_2} + \sigma_{31}\right)q_1 + \left(-\sigma_{13}\frac{q_3}{q_1} - \sigma_{23}\frac{q_4}{q_2} + \sigma_{33}\right)q_3,$$

$$M_{52} = M_{54} = \left(-\sigma_{12}\frac{q_3}{q_1} - \sigma_{22}\frac{q_4}{q_2} + \sigma_{32}\right)q_2 + \left(-\sigma_{13}\frac{q_3}{q_1} - \sigma_{23}\frac{q_4}{q_2} + \sigma_{33}\right)q_4,$$

$$M_{55} = -\sigma_{13}\frac{q_3}{q_1} - \sigma_{23}\frac{q_4}{q_2} + \sigma_{33}, \qquad M_{6j} = M_{j6} = 0, \quad j = \overline{1,5}, \qquad M_{66} = 0.$$

The entries of $\overline{K}(x_3,t)$ have the form

$$\bar a_{11}(x_3,t)=\bar a_{33}(x_3,t)=\frac{\sqrt{p(x_3)}}{2}\left(q_1^2(x_3)\varphi_1'(t)+q_3^2(x_3)\varphi_3'(t)+\frac{\psi_2'(t)}{q_1^2(x_3)}\right),$$
$$\bar a_{13}(x_3,t)=\bar a_{31}(x_3,t)=\frac{\sqrt{p(x_3)}}{2}\left(q_1^2(x_3)\varphi_1'(t)+q_3^2(x_3)\varphi_3'(t)-\frac{\psi_2'(t)}{q_1^2(x_3)}\right),$$

$$\bar a_{12}(x_3,t)=\bar a_{14}(x_3,t)=\bar a_{21}(x_3,t)=\bar a_{23}(x_3,t)$$
$$=\bar a_{32}(x_3,t)=\bar a_{34}(x_3,t)=\bar a_{41}(x_3,t)=\bar a_{43}(x_3,t)=\frac{\sqrt{p(x_3)}}{2}\,q_3(x_3)q_4(x_3)\varphi_3'(t),$$

$$\bar a_{15}=\bar a_{35}=\frac{\sqrt{p(x_3)}}{2}\,q_3(x_3)\varphi_3'(t),\qquad \bar a_{25}(x_3,t)=\bar a_{45}(x_3,t)=\frac{\sqrt{p(x_3)}}{2}\,q_4(x_3)\varphi_3'(t),$$
$$\bar a_{22}(x_3,t)=\bar a_{44}(x_3,t)=\frac{\sqrt{p(x_3)}}{2}\left(q_2^2(x_3)\varphi_2'(t)+q_4^2(x_3)\varphi_3'(t)+\frac{\psi_1'(t)}{q_2^2(x_3)}\right),$$
$$\bar a_{24}(x_3,t)=\bar a_{42}(x_3,t)=\frac{\sqrt{p(x_3)}}{2}\left(q_2^2(x_3)\varphi_2'(t)+q_4^2(x_3)\varphi_3'(t)-\frac{\psi_1'(t)}{q_2^2(x_3)}\right),$$
$$\bar a_{51}(x_3,t)=\bar a_{53}(x_3,t)=\frac{\sqrt{p(x_3)}}{2}\,q_3(x_3)q_5(x_3)\varphi_3'(t),$$
$$\bar a_{52}(x_3,t)=\bar a_{54}(x_3,t)=\frac{\sqrt{p(x_3)}}{2}\,q_4(x_3)q_5(x_3)\varphi_3'(t),\quad \bar a_{55}(x_3,t)=\frac{\sqrt{p(x_3)}}{2}\,q_5(x_3)\varphi_3'(t),$$
$$\bar a_{k6}(x_3,t)=\bar a_{6k}(x_3,t)=0,\quad k=\overline{1,5},\qquad \bar a_{66}(x_3,t)=\sqrt{p(x_3)}\frac{q_6(x_3)\psi_3'(t)}{q_1(x_3)q_2(x_3)}.$$

Introduce the new variable z as

$$z=\nu(x_3)=\int_0^{x_3}\frac{d\xi}{\sqrt{p(\xi)}}. \tag{16}$$

Denote by $\nu^{-1}(z)$ the function inverse to $\nu(x_3)$ and let

$$V(x_1,x_2,z,t):=\overline{U}(x_1,x_2,\nu^{-1}(z),t),\qquad \widehat{C}_j(z):=C_j(\nu^{-1}(z)),\qquad \widehat{C}(z)=C(\nu^{-1}(z)),$$
$$\widehat{K}(z,t):=\overline{K}(\nu^{-1}(z),t),\qquad \widehat{F}(x_1,x_2,z):=F(x_1,x_2,\nu^{-1}(z)),$$
$$a_{ij}(z,t):=\bar a_{ij}(\nu^{-1}(z),t),\qquad i,j=\overline{1,6}.$$

Then (12) takes the form

$$\left(I_6\frac{\partial}{\partial t}+\Lambda_0\frac{\partial}{\partial z}+\sum_{j=1}^{2}\widetilde{C}_j\frac{\partial}{\partial x_j}+\widetilde{C}\right)V=\int_0^t\widehat{K}(z,t-\tau)V(x_1,x_2,z,\tau)\,d\tau+\widehat{F}(x_1,x_2,z,t). \tag{17}$$

1. STATEMENT OF THE PROBLEM

In the direct problem, given matrices $\widehat{K}$, $\widehat{C}_1$, $\widehat{C}_2$, and $\widehat{C}$ and vector-function $\widehat{F}$, it is required, in the domain

$$D=\{(x_1,x_2,z,t)\mid 0<z<L,\ t>0,\ (x_1,x_2)\in\mathbb{R}^2\},$$

find a vector-function $V(z,t)$ satisfying equation (17) for the following initial and boundary conditions:

$$V_i(x_1,x_2,z,t)|_{t=0} = \phi_i(x_1,x_2,z), \qquad i=\overline{1,6}, \tag{18}$$

$$\begin{aligned} V_i(x_1,x_2,z,t)|_{z=0} &= g_i(x_1,x_2,t), \qquad i=1,2,\\ V_i(x_1,x_2,z,t)|_{z=L} &= g_i(x_1,x_2,t), \qquad i=3,4, \end{aligned} \tag{19}$$

where

$$\phi(x_1,x_2,z) = (\phi_1,\phi_2,\dots,\phi_6)(x_1,x_2,z), \qquad g(x_1,x_2,t) = (g_1,g_2,\dots,g_6)(x_1,x_2,t)$$

are some given functions.

Remark 1. For given initial data, equality (3) takes the form

$$\operatorname{div}\left(\hat{\mu}(z)\begin{pmatrix} \dfrac{1}{q_2}\left(\phi_2(x_1,x_2,z)-\phi_4(x_1,x_2,z)\right)\\ \dfrac{1}{q_2}\left(\phi_3(x_1,x_2,z)-\phi_1(x_1,x_2,z)\right)\\ \phi_6(x_1,x_2,z)\end{pmatrix}\right) = 0. \tag{20}$$

Pose the inverse problem as follows:

Find the functions $\varphi_i(t)$ and $\psi_i(t)$ for $t>0$ and $i=1,2,3$ that are involved in the matrix $\widetilde{K}$ if the extra conditions

$$\begin{aligned} V_i(x_1,x_2,z,t)|_{z=L} &= h_i(x_1,x_2,t), \qquad i=1,2,\\ V_i(x_1,x_2,z,t)|_{z=0} &= h_i(x_1,x_2,t), \qquad i=\overline{3,6} \end{aligned} \tag{21}$$

are given for a solution to problem (17)–(19). *Moreover, we assume that $\varphi_i(0)$ and $\psi_i(0)$ are given as well.*

Remark 2. As follows from (8), (11), and (16), the vector-function V is expressed through $(E,H)^*$ by the formula

$$V(x_1,x_2,z,t) = \begin{pmatrix} \dfrac{1}{2q_1}E_1 - \dfrac{q_1}{2}H_2\\ \dfrac{1}{2q_2}E_2 + \dfrac{q_2}{2}H_1\\ \dfrac{1}{2q_1}E_1 + \dfrac{q_1}{2}H_2\\ \dfrac{1}{2q_2}E_2 - \dfrac{q_2}{2}H_1\\ -\dfrac{q_3}{q_1}E_1 - \dfrac{q_4}{q_2}E_2 + E_3\\ H_6 \end{pmatrix}(x_1,x_2,\nu^{-1}(z),t).$$

In terms of the vector-function $(E, H)^*$, the boundary and extra conditions (19) and (21) take the form

$$(E_i, H_i)^*(x_1, x_2, \nu^{-1}(z), t)|_{z=0} = T(0) \times (g_1, g_2, h_3, h_4, h_5, h_6)^*(x_1, x_2, t);$$
$$(E_i, H_i)^*(x_1, x_2, \nu^{-1}(z), t)|_{z=L} = T(\nu^{-1}(L)) \times (h_1, h_2, g_3, g_4, g_5, g_6)^*(x_1, x_2, t), \qquad i = \overline{1,3}.$$

By now, the problems of finding the kernels from one second-order integro-differential equation have been widely studied (see [4–23]. The numerical solution of direct and inverse problems for such equations were under study in [24–38]. As a rule, the second-order equations are derived from systems of first-order partial differential equations under some additional assumptions.

The inverse problem of finding the kernels of the integral terms from a system of first-order integro-differential equations of general form with two independent variables was studied in [39]. Some theorem of local existence and global uniqueness was obtained.

It seems quite natural to carry out the study of inverse problems of finding the kernels of the integral terms of a system of integro-differential equations directly in terms of the system itself. The present article is a natural continuation of this circle of problems and, to a certain extent, generalizes the results of [39] to the case of the system of Maxwell's equations with memory (1), (2).

Suppose that functions $\widetilde{F}(x_1, x_2, z, t)$, $\phi_i(x_1, x_2, z)$, and $g_i(x_1, x_2, t)$ occurring on the right-hand side of (17) and the data (18), (19) have some compact support with respect to x_1, x_2 for every fixed z and t. The existence for (17) of a finite dependence domain and the property of having compact support with respect to x_1, x_2 of the right-hand side of (17) and the data (18), (19) imply that solutions to problem (17)–(19) have the compact support with respect to x_1, x_2.

We investigate the properties of solutions to this problem. More exactly, we will confine ourselves to the study of the Fourier transform of a solution with respect to x_1 and x_2. Put

$$\begin{aligned} \widetilde{V}(\eta_1, \eta_2, z, t) &= \int\limits_{\mathbb{R}^2} V(x_1, x_2, z, t) e^{i[\eta_1 x_1 + \eta_2 x_2]}\, dx_1 dx_2, \\ \widetilde{F}(\eta_1, \eta_2, z, t) &= \int\limits_{\mathbb{R}^2} \widehat{F}(x_1, x_2, z, t) e^{i[\eta_1 x_1 + \eta_2 x_2]}\, dx_1 dx_2, \end{aligned} \tag{22}$$

where η_1 and η_2 are the parameters of the transform.

In terms of the function $\widetilde{V}$, we write problem (17)–(19) as follows:

$$\begin{aligned} \frac{\partial \widetilde{V}_i}{\partial t} + \gamma_i \frac{\partial \widetilde{V}_i}{\partial z} = &-\sum_{j=1}^{6} b_{ij}(\eta_1, \eta_2, z) \widetilde{V}_j(\eta_1, \eta_2, z, t) \\ &+ \int_0^t \sum_{j=1}^{6} a_{ij}(z, \tau) \widetilde{V}_j(\eta_1, \eta_2, z, t-\tau)\, d\tau + F_i(\eta_1, \eta_2, z, t), \qquad i = 1, 2, 3, 4, \end{aligned} \tag{23}$$

$$\frac{\partial \widetilde{V}_i}{\partial t} = -\sum_{j=1}^{6} b_{ij}(\eta_1, \eta_2, z) \widetilde{V}_j(\eta_1, \eta_2, z, t) + \int_0^t \sum_{j=1}^{6} a_{ij}(z, \tau) \widetilde{V}_j(\eta_1, \eta_2, z, t-\tau)\, d\tau, \qquad i = 5, 6. \tag{24}$$

Here γ_i takes real values

$$\gamma_i = \begin{cases} -1, & i = 1, 2 \\ 1, & i = 3, 4, \end{cases}$$

and the coefficients b_{ij} are defined as follows:

$$b_{11} = \varkappa_{11} + M_{11} - i\eta_1 \frac{q_3\sqrt{p}}{q_1}, \qquad b_{12} = \varkappa_{12} + M_{12} - i\eta_1 \frac{q_4\sqrt{p}}{2q_1} - i\eta_2 \frac{q_3\sqrt{p}}{2q_2},$$

$$b_{13} = \varkappa_{13} + M_{13} - \frac{\sqrt{p}}{2q_2}\frac{q_1^{'}}{q_1}, \qquad b_{14} = \varkappa_{14} + M_{14} - i\eta_1 \frac{q_4\sqrt{p}}{2q_1} + i\eta_2 \frac{q_3\sqrt{p}}{2q_2},$$

$$b_{15} = \varkappa_{15} + M_{15} - i\eta_1 \frac{\sqrt{p}}{2q_1}, \qquad b_{16} = \varkappa_{16} + M_{16} - \frac{i\eta_2}{2} q_1\sqrt{p},$$

$$b_{21} = \varkappa_{21} + M_{21} - i\eta_1 \frac{q_4\sqrt{p}}{2q_1} - i\eta_2 \frac{q_3\sqrt{p}}{2q_2}, \qquad b_{22} = \varkappa_{22} + M_{22} - i\eta_2 \frac{q_4\sqrt{p}}{q_2},$$

$$b_{23} = \varkappa_{23} + M_{23} + i\eta_1 \frac{q_4\sqrt{p}}{2q_1} - i\eta_2 \frac{q_3\sqrt{p}}{2q_2}, \qquad b_{24} = \varkappa_{24} + M_{24} - \frac{q_1^{'}\sqrt{p}}{2q_2^2},$$

$$b_{25} = \varkappa_{25} + M_{25} - i\eta_1 \frac{\sqrt{p}}{2q_2}, \qquad b_{26} = \varkappa_{26} + M_{26} - \frac{i\eta_1}{2} q_2\sqrt{p},$$

$$b_{31} = \varkappa_{31} + M_{31} - \frac{\sqrt{p}}{2q_2}\frac{q_1^{'}}{q_1}, \qquad b_{32} = \varkappa_{32} + M_{32} + i\eta_1 \frac{q_4\sqrt{p}}{2q_1} - i\eta_2 \frac{q_3\sqrt{p}}{2q_2},$$

$$b_{33} = \varkappa_{33} + M_{33} + i\eta_1 \frac{q_3\sqrt{p}}{q_1}, \qquad b_{34} = \varkappa_{34} + M_{34} + i\eta_1 \frac{q_4\sqrt{p}}{2q_1} + i\eta_2 \frac{q_3\sqrt{p}}{2q_2},$$

$$b_{35} = \varkappa_{35} + M_{35} + i\eta_1 \frac{\sqrt{p}}{2q_1}, \qquad b_{36} = \varkappa_{36} + M_{36} - \frac{i\eta_1}{2} q_1\sqrt{p},$$

$$b_{41} = \varkappa_{41} + M_{41} - i\eta_1 \frac{q_4\sqrt{p}}{2q_1} + i\eta_2 \frac{q_3\sqrt{p}}{2q_2}, \qquad b_{42} = \varkappa_{42} + M_{42} - \frac{q_1^{'}\sqrt{p}}{2q_2^2},$$

$$b_{43} = \varkappa_{43} + M_{43} + i\eta_1 \frac{q_4\sqrt{p}}{2q_1} + i\eta_2 \frac{q_3\sqrt{p}}{2q_2}, \qquad b_{44} = \varkappa_{44} + M_{44} + i\eta_2 \frac{q_4\sqrt{p}}{q_2},$$

$$b_{45} = \varkappa_{45} + M_{45} + i\eta_2 \frac{\sqrt{p}}{2q_2}, \qquad b_{46} = \varkappa_{46} + M_{46} - \frac{i\eta_1}{2} q_2\sqrt{p},$$

$$b_{51} = \varkappa_{51} + M_{51} - i\eta_1 \frac{q_5\sqrt{p}}{2q_1}, \qquad b_{52} = \varkappa_{52} + M_{52} - i\eta_2 \frac{q_5\sqrt{p}}{2q_2},$$

$$b_{53} = \varkappa_{51} + M_{51} + i\eta_1 \frac{q_5\sqrt{p}}{2q_1}, \qquad b_{54} = \varkappa_{52} + M_{52} + i\eta_2 \frac{q_5\sqrt{p}}{2q_2}, \qquad b_{55} = \varkappa_{55} + M_{55},$$

$$b_{56} = 0, \qquad b_{61} = \varkappa_{61} + M_{61} + i\eta_2 \frac{q_6\sqrt{p}}{2q_2}, \qquad b_{62} = \varkappa_{62} + M_{62} - i\eta_1 \frac{q_6\sqrt{p}}{2q_1},$$

$$b_{63} = \varkappa_{63} + M_{63} + i\eta_2 \frac{q_6\sqrt{p}}{2q_2}, \qquad b_{64} = \varkappa_{64} + M_{64} - i\eta_1 \frac{q_6\sqrt{p}}{2q_1}, \qquad b_{65} = 0, \qquad b_{66} = \varkappa_{66}.$$

Fix η_1 and η_2 and for convenience introduce the notation $\widetilde{V}(\eta_1, \eta_2, z, t) = \widetilde{V}(z, t)$. We will adopt these notations for the Fourier transforms of the functions occurring in the initial, boundary, and additional conditions (18), (19), and (21):

$$\widetilde{V}_i\big|_{t=0} \equiv \tilde{\phi}_i(z), \qquad i = 1, 2, \dots, 6, \tag{25}$$

$$\widetilde{V}_i|_{z=0} = \tilde{g}_i(t), \quad i = 1, 2, \qquad \widetilde{V}_i|_{z=L} = \tilde{g}_i(t), \quad i = 3, 4, \tag{26}$$

$$\widetilde{V}_i|_{z=L} = \tilde{h}_i(t), \quad i = 1, 2, \qquad \widetilde{V}_i|_{z=0} = \tilde{h}_i(t), \quad i = 3, 4, 5, 6. \tag{27}$$

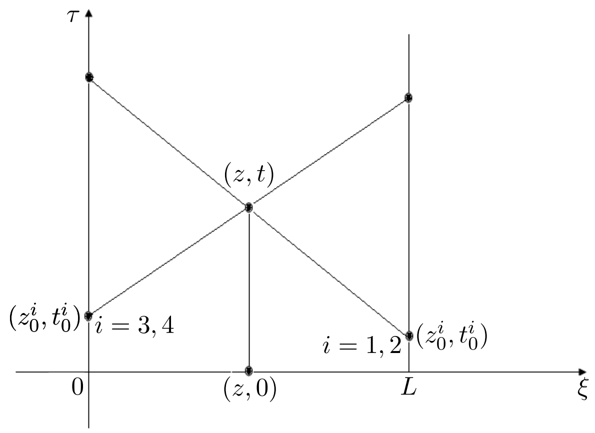

Fig. 1. Characteristic lines

2. EXAMINATION OF THE DIRECT PROBLEM

Let $\Pi = \{(z,t) \mid 0 < z < L,\ t > 0\}$ be the projection of the domain D to the plane of the variables z and t. Consider an arbitrary point $(z,t) \in \Pi$ on the plane of the variables ξ and τ and draw a characteristic of the ith equation of system (23), (24) through (z,t) till the intersection with the boundary of Π in the domain $\tau < t$. The equation looks as

$$\xi = z + \gamma_i(\tau - t). \tag{28}$$

For $\gamma_i = 1$ (i.e., $i = 3, 4$), this point lies either on the interval $[0, L]$ of the axis $t = 0$ or on the straight line $z = 0$, and for $\gamma_i = -1$ (i.e., $i = 1, 2$), either on the interval $[0, L]$ or on the straight line $z = L$ (see Fig. 1).

Integrating the ith component of equations (23), (24) over characteristic (28) from $\left(z_0^i, t_0^i\right)$ to (z,t), we find

$$\begin{aligned}\widetilde{V}_i(z,t) = \widetilde{V}_i\left(z_0^i, t_0^i\right) &+ \int\limits_{t_0^i}^{t} \left[F_i(\xi,\tau) - \sum_{j=1}^{6} b_{ij}(\xi)\widetilde{V}_j(\xi,\tau)\right]\Bigg|_{\xi = z+\gamma_i(\tau-t)} d\tau \\ &+ \int\limits_{t_0^i}^{t} \int\limits_{0}^{\tau} \sum_{j=1}^{6} a_{ij}(\xi,\alpha)\widetilde{V}_j(\xi,\tau-\alpha)\, d\alpha\Bigg|_{\xi = z+\gamma_i(\tau-t)} d\tau, \qquad i = 1,2,3,4, \end{aligned} \tag{29}$$

$$\widetilde{V}_i(z,t) = \widetilde{V}_i(z,0) - \int\limits_0^t \sum_{j=1}^{6} b_{ij}(z)\widetilde{V}_j(z,\tau)\, d\tau + \int\limits_0^t \int\limits_0^\tau \sum_{j=1}^{6} a_{ij}(z,\alpha)\widetilde{V}_j(z,\tau-\alpha)\, d\alpha d\tau, \quad i = 5,6. \tag{30}$$

Find t_0^i in (29) and (30). It depends on the coordinates of (z,t). It is easy to observe that $t_0^i(z,t)$ has the form

$$t_0^i(z,t) = \begin{cases} t + \dfrac{L-z}{\gamma_i}, & t \geq \dfrac{L-z}{\gamma_i}, \\ 0, & 0 < t < \dfrac{L-z}{\gamma_i}, \end{cases} \qquad i = 1,2,$$

$$t_0^i(z,t) = \begin{cases} t - \dfrac{z}{\gamma_i}, & t \geq z/\gamma_i, \\ 0, & 0 < t < z/\gamma_i, \end{cases} \qquad i = 3,4, \qquad t_0^i(z,t) = 0, \quad i = 5,6.$$

Then the condition that the pair (z_0^i, t_0^i) enjoys (28) implies

$$z_0^i(z,t) = \begin{cases} L, & t \geq \dfrac{L-z}{\gamma_i}, \\ z - \gamma_i t, & 0 < t < \dfrac{L-z}{\gamma_i}, \end{cases} \qquad i = 1,2,$$

$$z_0^i(z,t) = \begin{cases} 0, & t \geq z/\gamma_i, \\ z - \gamma_i t, & 0 < t < z/\gamma_i, \end{cases} \qquad i = 3,4, \qquad z_0^i(z,t) = z, \quad i = 5,6.$$

The free terms of the integral equations (28) are defined through the initial and boundary conditions (25) and (26) as follows:

$$\widetilde{V}_i\big(z_0^i, t_0^i\big) = \begin{cases} \tilde{g}_i\left(t + \dfrac{L-z}{\gamma_i}\right), & t \geq \dfrac{L-z}{\gamma_i}, \\ \tilde{\phi}_i(z - \gamma_i t), & 0 \leq t < \dfrac{L-z}{\gamma_i}, \end{cases} \qquad i = 1,2,$$

$$\widetilde{V}_i(z_0^i, t_0^i) = \begin{cases} \tilde{g}_i(t - z/\gamma_i), & t \geq z/\gamma_i, \\ \tilde{\phi}_i(z - \gamma_i t), & 0 \leq t < z/\gamma_i, \end{cases} \qquad i = 3,4.$$

It is required that $\widetilde{V}_i\big(z_0^i, t_0^i\big)$ be continuous in Π. Note that, for these conditions to be fulfilled, the given functions $\tilde{\phi}_i$ and $\tilde{g}_i$ must satisfy the fitting conditions at the angular points of Π:

$$\tilde{\phi}_i(0) = \tilde{g}_i(0), \quad i = 1,2; \qquad \tilde{\phi}_i(L) = \tilde{g}_i(0), \quad i = 3,4. \tag{31}$$

Here and below, the values of $\tilde{g}_i$ for $t = 0$ and $\tilde{\phi}_i$ for $z = 0$ and $z = L$ are understood as the limit values at these points as the argument tends from the side of the point where these functions are defined.

Suppose that all given functions in (29) and (30) are continuous functions of their arguments in Π. Then we have a closed system of Volterra-type integral equations with continuous kernels and free terms. As usual, such a system has a unique solution in the bounded subdomain

$$\Pi_T = \{(z,t) \mid 0 \leq z \leq L,\ 0 \leq t \leq T\}$$

of Π, where $T > 0$ is some fixed number.

Introduce the vector-function

$$w(z,t) = \frac{\partial}{\partial t}\widetilde{V}(z,t).$$

For obtaining a problem for $w(z,t)$ similar to (23)–(26), we differentiate (23), (24) and the boundary conditions (29) with respect to t and find the condition for $t = 0$ by means of (23), (24) and the initial conditions (25). We infer

$$\frac{\partial w_i}{\partial t}+\gamma_i\frac{\partial w_i}{\partial z}=-\sum_{j=1}^{6}b_{ij}(z)w_j(z,t)+\sum_{j=1}^{6}a_{ij}(z,t)\tilde{\phi}_j(z)$$
$$+\int\limits_0^t\sum_{j=1}^{6}a_{ij}(z,\tau)w_j(z,t-\tau)\,d\tau+\frac{\partial}{\partial t}F_i(z,t),\qquad i=1,2,3,4,\tag{32}$$

$$\frac{\partial w_i}{\partial t}=-\sum_{j=1}^{6}b_{ij}(z)w_j(z,t)+\sum_{j=1}^{6}a_{ij}(z,t)\tilde{\phi}_j(z)+\int\limits_0^t\sum_{j=1}^{6}a_{ij}(z,\tau)w_j(z,t-\tau)\,d\tau,\qquad i=5,6,\tag{33}$$

$$w_i(z,t)|_{t=0}=\Phi_i(z),\qquad i=\overline{1,6},\tag{34}$$

$$w_i(z,t)|_{z=0}=\frac{d}{dt}\tilde{g}_i(t),\quad i=1,2;\qquad w_i(z,t)|_{z=L}=\frac{d}{dt}\tilde{g}_i(t),\quad i=3,4,\tag{35}$$

where

$$\begin{gathered}\Phi_i(z)=F_i(z,0)-\gamma_i\frac{\partial}{\partial z}\tilde{\phi}_i(z)-\sum_{j=1}^{6}b_{ij}(z)\tilde{\phi}_i(z),\qquad i=\overline{1,4};\\ \Phi_i(z)=-\sum_{j=1}^{6}b_{ij}(z)\tilde{\phi}_i(z),\qquad i=5,6.\end{gathered}\tag{36}$$

Once again, integration along the corresponding characteristics reduces (32)–(35) to the integral equations

$$w_i(z,t)=w_i(z_0^i,t_0^i)$$
$$+\int\limits_{t_0^i}^t\left[\frac{\partial}{\partial t}F_i(\xi,\tau)-\sum_{j=1}^{6}b_{ij}(\xi)w_j(\xi,\tau)+\sum_{j=1}^{6}a_{ij}(\xi,\tau)\tilde{\phi}_j(\xi)\right]\Bigg|_{\xi=z+\gamma_i(\tau-t)}d\tau$$
$$+\int\limits_{t_0^i}^t\int\limits_0^\tau\sum_{j=1}^{6}a_{ij}(\xi,\alpha)w_j(\xi,\tau-\alpha)\,d\alpha\Bigg|_{\xi=z+\gamma_i(\tau-t)}d\tau,\qquad i=\overline{1,4},\tag{37}$$

$$w_i(z,t)=w_i(z,0)+\int\limits_0^t\left[-\sum_{j=1}^{6}b_{ij}(z)w_j(z,\tau)+\sum_{j=1}^{6}a_{ij}(z,\tau)\tilde{\phi}_j(z)\right]d\tau$$
$$+\int\limits_0^t\int\limits_0^\tau\sum_{j=1}^{6}a_{ij}(z,\alpha)w_j(z,\tau-\alpha)\,d\alpha d\tau,\qquad i=5,6.\tag{38}$$

For the functions w_i, the additional conditions (27) look as

$$w_i(0,t)=\frac{d}{dt}\tilde{h}_i(t),\quad i=\overline{3,6},\qquad w_i(L,t)=\frac{d}{dt}\tilde{h}_i(t),\quad i=1,2.\tag{39}$$

In equations (37), the functions $w_i\big(z_0^i,t_0^i\big)$ are defined as follows:

$$w_i(z_0^i,t_0^i)=\begin{cases}\dfrac{d}{dt}\tilde{g}_i\left(t+\dfrac{L-z}{\gamma_i}\right), & t\geq\dfrac{L-z}{\gamma_i},\\ \Phi_i(z-\gamma_i t), & 0\leq t<\dfrac{L-z}{\gamma_i},\end{cases}\qquad i=1,2,$$

$$w_i(z_0^i, t_0^i) = \begin{cases} \dfrac{d}{dt}\tilde{g}_i(t - z/\gamma_i), & t \geq z/\gamma_i, \\ \Phi_i(z - \gamma_i t), & 0 \leq t < z/\gamma_i, \end{cases} \qquad i = 3, 4.$$

Suppose the fulfillment of the conditions

$$F_i(0,0) - \gamma_i \left[\frac{\partial}{\partial z}\tilde{\phi}_i(z)\right]_{z=0} - \sum_{j=1}^{6} b_{ij}(0)\tilde{\phi}_i(0) = \left[\frac{d}{dt}\tilde{g}_i(t)\right]_{t=0}, \qquad i = 1, 2, \tag{40}$$

$$F_i(L,0) - \gamma_i \left[\frac{\partial}{\partial z}\tilde{\phi}_i(z)\right]_{z=L} - \sum_{j=1}^{6} b_{ij}(L)\tilde{\phi}_i(L) = \left[\frac{d}{dt}\tilde{g}_i(t)\right]_{t=0}, \qquad i = 3, 4. \tag{41}$$

It is not hard to see that the fitting conditions for the initial data (34) and the boundary data (35) coincide with (40) and (41) at the angular points of Π. It is clear that if the same equalities (40) and (41) are fulfilled then (37) and (38) have unique continuous solutions $w_i(z,t)$ and the same $\dfrac{\partial}{\partial t}\widetilde{V}_i(z,t)$.

Thus, we have proved the following

Theorem 1. *Suppose that*

$$\hat{\varepsilon}(x_3) \in C^1[0,\infty), \qquad \hat{\mu}(x_3) \in C^1[0,\infty), \qquad \tilde{\phi}(x_3) \in C^1[0,\infty],$$

$$\tilde{g}(t) \in C^1[0,\infty), \qquad K(t) \in C^1[0,\infty), \qquad \widetilde{F}(x_3,t) \in C^1(\Pi)$$

and conditions (20), (31), (40), *and* (41) *are fulfilled. Then there is a unique solution to problem* (23)–(26) *in* Π.

3. EXAMINATION OF THE INVERSE PROBLEM. DEDUCTION OF AN EQUIVALENT SYSTEM OF INTEGRAL EQUATIONS.

Consider an arbitrary point $(z,0) \in \Pi$ and draw the characteristic (8) through $(z,0)$ till the intersection with the lateral boundaries of Π. Iterating the ith component of equation (32), using the data (39), we infer

$$w_i(z,0) = \frac{d}{dt}\tilde{h}_i(t_i(z)) + \int\limits_0^{t_i(z)} \left[\frac{\partial}{\partial t}F_i(\xi,\tau) - \sum_{j=1}^{6} b_{ij}(\xi)w_j(\xi,\tau) + \sum_{j=1}^{6} a_{ij}(\xi,\tau)\tilde{\phi}_j(\xi)\right]\Bigg|_{\xi=z+\gamma_i\tau} d\tau$$
$$+ \int\limits_0^{t_i(z)} \int\limits_0^{\tau} \sum_{j=1}^{6} a_{ij}(\xi,\alpha)w_j(\xi,\tau-\alpha)d\alpha\Bigg|_{\xi=z+\gamma_i\tau} d\tau, \qquad i = \overline{1,4}, \tag{42}$$

where

$$t_i(z) = \frac{1}{\gamma_i}\begin{cases} z, & i = 1,2, \\ L - z, & i = 3,4. \end{cases}$$

Integrating (33) leads to the integral equations

$$w_i(z,t) = \Phi_i(z) + \int\limits_0^{t} \left[-\sum_{j=1}^{6} b_{ij}(z)w_j(z,\tau) + \sum_{j=1}^{6} a_{ij}(z,\tau)\tilde{\phi}_j(z)\right]d\tau$$
$$+ \int\limits_0^{t}\int\limits_0^{\tau} \sum_{j=1}^{6} a_{ij}(z,\alpha)w_j(z,\tau-\alpha)\,d\alpha d\tau, \qquad i = 5,6. \tag{43}$$

Reckoning with the initial data (34), rewrite (42) and (43) as

$$\int_0^{t_i(z)} \sum_{j=1}^{6} a_{ij}(z+\gamma_i\tau,\tau)\tilde{\phi}_j(z+\gamma_i\tau)\,d\tau + \int_0^{t_i(z)}\int_0^{\tau} \sum_{j=1}^{6} a_{ij}(z+\gamma_i\tau,\alpha)w_j(z+\gamma_i\tau,\tau-\alpha)\,d\alpha d\tau$$

$$= \Phi_i(z) - \frac{d}{dt}\tilde{h}_i(t_i(z)) - \int_0^{t_i(z)} \left[\frac{\partial}{\partial t}F_i(z+\gamma_i\tau,\tau) - \sum_{j=1}^{6} b_{ij}(z+\gamma_i\tau)w_j(z+\gamma_i\tau,\tau)\right]d\tau, \qquad i=\overline{1,4},$$

$$\int_0^{t} \sum_{j=1}^{6} a_{ij}(0,\tau)\tilde{\phi}_j(0)\,d\tau + \int_0^{t}\int_0^{\tau} \sum_{j=1}^{6} a_{ij}(0,\alpha)w_j(0,\tau-\alpha)\,d\alpha d\tau$$

$$= \frac{d}{dt}\tilde{h}_i(t) - \Phi_i(0) + \int_0^{t}\sum_{j=1}^{6} b_{ij}(0)w_j(0,\tau)d\tau, \qquad i=5,6.$$

Differentiate the first equations with respect to z, and the second, with respect to t. Then

$$\sum_{j=1}^{6} a_{ij}(z+\gamma_i t_i(z),t_i(z))\tilde{\phi}_j(z+\gamma_i t_i(z)) - \gamma_i \int_0^{t_i(z)} \sum_{j=1}^{6} \frac{\partial}{\partial z}(a_{ij}(z+\gamma_i\tau,\tau)\tilde{\phi}_j(z+\gamma_i\tau))\,d\tau$$

$$+ \int_0^{t_i(z)} \sum_{j=1}^{6} a_{ij}(z+\gamma_i t_i(z),\tau)w_j(z+\gamma_i t_i(z),t_i(z)-\tau)\,d\tau$$

$$-\gamma_i \int_0^{t_i(z)}\int_0^{\tau} \sum_{j=1}^{6} \frac{\partial}{\partial z}(a_{ij}(z+\gamma_i\tau,\alpha)w_j(z+\gamma_i\tau,\tau-\alpha))\,d\alpha d\tau = -\gamma_i\frac{d}{dz}\Phi_i(z) - \frac{d^2}{dt^2}\tilde{h}_i(t_i(z))$$

$$-\left[\frac{\partial}{\partial t}F_i(z+\gamma_i t_i(z),t_i(z)) - \sum_{j=1}^{6} b_{ij}(z+\gamma_i t_i(z))w_j(z+\gamma_i t_i(z),t_i(z))\right]$$

$$+\gamma_i \int_0^{t_i(z)} \left[\frac{\partial^2}{\partial t\partial z}F_i(z+\gamma_i\tau,\tau) - \sum_{j=1}^{6}\frac{\partial}{\partial z}\big(b_{ij}(z+\gamma_i\tau)w_j(z+\gamma_i\tau,\tau)\big)\right]d\tau, \quad i=\overline{1,4}, \quad (44)$$

$$\sum_{j=1}^{6} a_{ij}(0,t)\tilde{\phi}_j(0) + \int_0^{t}\sum_{j=1}^{6} a_{ij}(0,\tau)w_j(0,t-\tau)\,d\tau = \frac{d^2}{dt^2}\tilde{h}_i(t) + \sum_{j=1}^{6} b_{ij}(0)w_j(0,t), \quad i=5,6. \quad (45)$$

Now, replace $t_i(z)$ by t in (43). We infer

$$\sum_{j=1}^{6} a_{ij}(0,t)\tilde{\phi}_j(0) = P_i(-\gamma_i t) + \int_0^{t}\sum_{j=1}^{6}\frac{\partial}{\partial z}(b_{ij}(-\gamma_i(t-\tau))w_j(-\gamma_i(t-\tau,\tau)))\,d\tau$$

$$-\gamma_i\int_0^{t}\sum_{j=1}^{6}\frac{\partial}{\partial z}\big(a_{ij}(-\gamma_i(t-\tau),\tau)\tilde{\phi}_j(-\gamma_i(t-\tau))\big)\,d\tau - \int_0^{t}\sum_{j=1}^{6} a_{ij}(0,\tau)\frac{d}{dt}\tilde{h}_j(-\gamma_i(t-\tau))\,d\tau$$

$$-\int_0^{t}\int_0^{\tau}\sum_{j=1}^{6}\frac{\partial}{\partial z}\big(a_{ij}(-\gamma_i(t-\tau),\alpha)w_j(-\gamma_i(t-\tau),\tau-\alpha)\big)\,d\alpha d\tau, \qquad i=1,2, \quad (46)$$

$$\sum_{j=1}^{6} a_{ij}(L,t)\tilde{\phi}_j(L) = P_i(L-\gamma_i t) + \int_0^t \sum_{j=1}^{6} \frac{\partial}{\partial z}\big(a_{ij}(L-\gamma_i(t-\tau),\tau)\tilde{\phi}_j(L-\gamma_i(t-\tau))\big)\, d\tau$$
$$- \int_0^t \sum_{j=1}^{6} a_{ij}(L,\tau)\frac{d}{dt}\tilde{h}_j(-\gamma_i t-\tau)\, d\tau - \int_0^t \sum_{j=1}^{6} \frac{\partial}{\partial z}\big(b_{ij}(H-\gamma_i(t-\tau))w_j(L-\gamma_i(t-\tau),\tau)\big)\, d\tau$$
$$+ \int_0^t \int_0^\tau \sum_{j=1}^{6} \frac{\partial}{\partial z}\big(a_{ij}(L-\gamma_i(t-\tau),\alpha)w_j(L-\gamma_i(t-\tau),\tau-\alpha)\big)\, d\alpha d\tau, \quad i=3,4, \quad (47)$$

where $P_i(z)$ are defined by the formulas

$$P_i(z) = -\gamma_i \frac{d}{dz}\Phi_i(z) - \frac{d^2}{dt^2}\tilde{h}_i(t_i(z)) - \frac{\partial}{\partial t}F_i(z+\gamma_i t_i(z), t_i(z))$$
$$+ \sum_{j=1}^{6} b_{ij}(z+\gamma_i t_i(z))w_j(z+\gamma_i t_i(z), t_i(z)) + \gamma_i \int_0^{t_i(z)} \frac{\partial^2}{\partial t\partial z}F_i(z+\gamma_i\tau,\tau)\, d\tau, \qquad i=\overline{1,4},$$

and the notations

$$P_i(t) = \frac{d^2}{dt^2}\tilde{h}_i(t) + \sum_{j=1}^{6} b_{ij}(0)\frac{d}{dt}\tilde{h}_j(t), \qquad i=5,6,$$

are introduced.

Put

$$Q(z;\tilde{\phi}) := \begin{pmatrix} c_{11}(z) & 0 & c_{13}(z) & 0 & c_{15}(z) & 0 \\ 0 & c_{22}(z) & c_{23}(z) & c_{24}(z) & 0 & 0 \\ c_{31}(z) & 0 & c_{33}(z) & 0 & c_{35}(z) & 0 \\ 0 & c_{42}(z) & c_{43}(z) & c_{44}(z) & 0 & 0 \\ 0 & 0 & c_{53}(z) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66}(z) \end{pmatrix}, \qquad (48)$$

where

$$c_{11}(z) = c_{31}(z) = \frac{\sqrt{p}}{2}\big(q_1^2(z)\tilde{\phi}_1(z) + q_1^2(z)\tilde{\phi}_3(z)\big), \qquad c_{12}(z)=0,$$
$$c_{13}(z) = c_{33}(z) = \frac{\sqrt{p}}{2}\big(q_3^3(z\tilde{\phi}_1(z) + q_3(z)q_4(z)\tilde{\phi}_2(z) + q_3^2(z)\tilde{\phi}_3(z) + q_3(z)q_4(z)\tilde{\phi}_4(z) + q_3(z)\tilde{\phi}_5(z)\big),$$
$$c_{14}(z)=0, \qquad c_{15}(z) = -c_{35}(z) = \frac{\sqrt{p}}{2}\frac{\tilde{\phi}_1(z)-\tilde{\phi}_3(z)}{q_1^2(z)}, \qquad c_{16}(z)=0, \qquad c_{21}(z)=0,$$
$$c_{22}(z) = c_{42}(z) = \frac{\sqrt{p}}{2}\big(q_2^2(z)\tilde{\phi}_2(z) + q_2^2(z)\tilde{\phi}_4(z)\big),$$
$$c_{23}(z) = c_{43}(z) = \frac{\sqrt{p}}{2}\big(q_3(z)q_4(z)\tilde{\phi}_1(z) + q_4^2(z)\tilde{\phi}_2(z) + q_3(z)q_4(z)\tilde{\phi}_3(z) + q_4^2(z)\tilde{\phi}_4(z) + q_4(z)\tilde{\phi}_5(z)\big),$$
$$c_{24}(z) = -c_{44}(z) = \frac{\sqrt{p}}{2}\frac{\tilde{\phi}_2(z)-\tilde{\phi}_4(z)}{q_2^2(z)}, \qquad c_{25}(z)=0, \qquad c_{26}(z)=0,$$

$$c_{31}(z)=\frac{\sqrt{p}}{2}\big(q_1^2\tilde{\phi}_1(z)+q_1^2\tilde{\phi}_3(z)\big),\qquad c_{32}(z)=0,\qquad c_{34}(z)=0,\qquad c_{36}(z)=0,$$
$$c_{41}(z)=0,\qquad c_{45}(z)=0,\qquad c_{46}(z)=0,\qquad c_{51}(z)=0,\qquad c_{52}(z)=0,$$
$$c_{53}(z)=\frac{\sqrt{p}}{2}\big(q_3q_5\tilde{\phi}_1(z)+q_4q_5\tilde{\phi}_2(z)+q_3q_5\tilde{\phi}_3(z)+q_4q_5\tilde{\phi}_4(z)+q_5\tilde{\phi}_5(z)\big),$$
$$c_{54}(z)=0,\qquad c_{55}(z)=0,\qquad c_{56}(z)=0,\qquad c_{66}(z)=\sqrt{p}\frac{q_6}{q_1q_2}\tilde{\phi}_6(z),$$
$$c_{6i}(z)=0,\qquad i=\overline{1,5}.$$

Reckoning with (48), rewrite (37) and (38) as follows:

$$w_i(z,t)=w_i(z_0^i,t_0^i)+\int\limits_{t_0^i}^{t}\left[\frac{\partial}{\partial t}F_i(\xi,\tau)-\sum_{j=1}^{6}b_{ij}(\xi)w_j(\xi,\tau)+\sum_{j=1}^{6}Q_{ij}(\xi;\tilde{\phi})\Psi_j(\tau)\right]\bigg|_{\xi=z+\gamma_i(\tau-t)}d\tau$$
$$+\int\limits_{t_0^i}^{t}\int\limits_{0}^{\tau}\sum_{j=1}^{6}Q_{ij}(\xi;w(\xi,\tau-\alpha))\Psi_j(\alpha)\,d\alpha\bigg|_{\xi=z+\gamma_i(\tau-t)}d\tau,\qquad i=\overline{1,6}. \quad (49)$$

Using (48), we can also rewrite (44) and (45) so that

$$\sum_{j=1}^{6}Q_{ij}(\nu_i;\tilde{\phi}(\nu_i))\Psi_j(t)=P_i(\bar{t}_i(t))$$
$$+\beta_i\int\limits_0^t\sum_{j=1}^{6}\left[\frac{\partial}{\partial z}b_{ij}(-\gamma_i(t-\tau))w_j(-\gamma_i(t-\tau,\tau))+b_{ij}(-\gamma_i(t-\tau))\frac{\partial}{\partial z}w_j(-\gamma_i(t-\tau,\tau))\right]d\tau$$
$$-\int\limits_0^t\sum_{j=1}^{6}\left[\gamma_i\beta_i\frac{\partial}{\partial z}Q_{ij}\big(-\gamma_i(t-\tau);\tilde{\phi}(-\gamma_i(t-\tau))\big)+Q_{ij}\big(-\gamma_i(t-\tau);\frac{d}{dt}\tilde{h}(-\gamma_i(t-\tau))\big)\right]\Psi_j(\tau)\,d\tau$$
$$-\beta_i\int\limits_0^t\int\limits_0^\tau\sum_{j=1}^{6}\frac{\partial}{\partial z}Q_{ij}\big(-\gamma_i(t-\tau);w_j(-\gamma_i(t-\tau),\tau-\alpha)\big)\Psi_j(\alpha)d\alpha d\tau,\quad i=\overline{1,6}. \quad (50)$$

Here

$$\beta_i=\begin{cases}1, & i=\overline{1,4},\\ 0, & i=5,6,\end{cases}\qquad \nu_i=\begin{cases}L, & i=3,4,\\ 0, & i=1,2,5,6,\end{cases}\qquad \bar{t}_i(t)=\begin{cases}-\lambda_i t, & i=1,2,\\ L-\lambda_i t, & i=3,4,\\ t, & i=5,6.\end{cases}$$

Let $\Psi(t)=\big(\varphi_1',\varphi_2',\varphi_3',\psi_1',\psi_2',\psi_3'\big)^*$ be the vector-function composed of the derivatives of the unknown functions of the inverse problem, where $\Psi_i(t)$ are the entries of this vector-function.

In what follows, we assume the fulfillment of the condition

$$\det Q(\nu_i;\tilde{\phi})\neq 0, \quad (51)$$

which is equivalent to the inequalities

$$c_{11}\neq 0,\qquad c_{15}\neq 0,\qquad c_{22}\neq 0,\qquad c_{24}\neq 0,\qquad c_{53}\neq 0,\qquad c_{66}\neq 0.$$

Now, solving (50) with respect to $\Psi_i(t)$, we obtain

$$\Psi_i(t) = \frac{1}{\det Q(\nu_i;\tilde{\phi})}\sum_{j=1}^{6}\left[P_j(\bar{t}_j(t)) + \beta_j\int\limits_0^t\sum_{k=1}^{6}\frac{\partial}{\partial z}b_{jk}(-\gamma_j(t-\tau))w_k(-\gamma_j(t-\tau,\tau))\,d\tau\right]\mathcal{Q}_{ji}(\nu_i;\tilde{\phi})$$
$$+\frac{1}{\det Q(\nu_i;\tilde{\phi})}\sum_{j=1}^{6}\left[\beta_j\int\limits_0^t\sum_{k=1}^{6}b_{jk}(-\gamma_j(t-\tau))\frac{\partial}{\partial z}w_k(-\gamma_j(t-\tau,\tau))\,d\tau\right]\mathcal{Q}_{ji}(\nu_i;\tilde{\phi})$$
$$-\frac{1}{\det Q(\nu_i;\tilde{\phi})}\sum_{j=1}^{6}\left[\gamma_j\beta_j\int\limits_0^t\sum_{k=1}^{6}\frac{\partial}{\partial z}Q_{jk}(-\gamma_j(t-\tau);\tilde{\phi}(-\gamma_j(t-\tau)))\Psi_k(\tau)\,d\tau\right]\mathcal{Q}_{ji}(\nu_i;\tilde{\phi})$$
$$-\frac{1}{\det Q(\nu_i;\tilde{\phi})}\sum_{j=1}^{6}\left[\int\limits_0^t\sum_{k=1}^{6}Q_{jk}(-\gamma_j(t-\tau);\frac{d}{dt}\tilde{h}(-\gamma_j(t-\tau)))\Psi_k(\tau)d\tau\right]\mathcal{Q}_{ji}(\nu_i;\tilde{\phi})$$
$$-\frac{1}{\det Q(\nu_i;\tilde{\phi})}\sum_{j=1}^{6}\left[\beta_j\int\limits_0^t\int\limits_0^\tau\sum_{k=1}^{6}\frac{\partial}{\partial z}Q_{jk}(-\gamma_j(t-\tau);w_k(-\gamma_j(t-\tau),\tau-\alpha))\Psi_k(\alpha)\,d\alpha d\tau\right]\mathcal{Q}_{ji}(\nu_i;\tilde{\phi}), \tag{52}$$

where $\mathcal{Q}_{ji}$ are the algebraic complements to the entries c_{ji} of Q, $i=\overline{1,6}$.

Equations (52) contain the unknown functions $\dfrac{\partial w_j}{\partial z}$, $j=\overline{1,6}$. For them we obtain integral equations from (49) by differentiating in z. Moreover,

$$\frac{\partial}{\partial z}w_i(z,t) = \frac{\partial}{\partial z}w_i(z_0^i,t_0^i) - \frac{\partial}{\partial z}t_0^i\left[\frac{\partial}{\partial t}F_i(z_0^i,t_0^i) - \sum_{j=1}^{6}b_{ij}(z_0^i)w_j(z_0^i,t_0^i) + \sum_{j=1}^{6}Q_{ij}(z_0^i;\tilde{\phi})\Psi_j(t_0^i)\right]$$
$$+\int\limits_{t_0^i}^{t}\left[\frac{\partial}{\partial t\partial z}F_i(\xi,\tau) - \sum_{j=1}^{6}\frac{d}{dz}b_{ij}(\xi)w_j(\xi,\tau) - \sum_{j=1}^{6}b_{ij}(\xi)\frac{\partial}{\partial z}w_j(\xi,\tau)\right.$$
$$\left.+\sum_{j=1}^{6}\frac{\partial}{\partial z}Q_{ij}(\xi;\tilde{\phi})\Psi_j(\tau)\right]\Bigg|_{\xi=z+\gamma_i(\tau-t)}d\tau - \frac{\partial}{\partial z}t_0^i\int\limits_0^{t_0^i}\sum_{j=1}^{6}Q_{ij}(z_0^i;G_j(z_0^i,t_0^i-\tau))\Psi_j(\tau)\,d\tau$$
$$+\int\limits_{t_0^i}^{t}\int\limits_0^\tau\sum_{j=1}^{6}\frac{\partial}{\partial z}Q_{ij}(\xi;w_j(\xi,\tau-\alpha))\Psi_j(\alpha)\,d\alpha\Bigg|_{\xi=z+\gamma_i(\tau-t)}d\tau, \quad i=\overline{1,6}, \tag{53}$$

where

$$G_j\big(z_0^i,t_0^i-\tau\big) = \begin{cases}\dfrac{d}{dt}h_j\left(\dfrac{L-z}{\gamma_i}-\tau\right), & j=\overline{3,6},\\ \dfrac{d}{dt}g_j\left(\dfrac{L-z}{\gamma_i}-\tau\right), & j=1,2.\end{cases}$$

The fulfillment of the following fitting conditions is required:

$$\begin{aligned} &\frac{d}{dt}\,\tilde{g}_i(0) = F_i(0,0) - \gamma_i \frac{\partial}{\partial z}\,\tilde{\phi}_i(z)\Big|_{z=0} - \sum_{j=1}^{6} b_{ij}(0)\tilde{\phi}_i(0), \qquad i = 1,2, \\ &\frac{d}{dt}\,\tilde{g}_i(0) = F_i(L,0) - \gamma_i \frac{\partial}{\partial z}\,\tilde{\phi}_i(z)\Big|_{z=L} - \sum_{j=1}^{6} b_{ij}(L)\tilde{\phi}_i(L), \qquad i = 3,4, \end{aligned} \tag{54}$$

$$\frac{d}{dt}\,\tilde{h}_i(t)\Big|_{t=0} = -\sum_{j=1}^{6} b_{ij}(0)\tilde{\phi}_i(0), \qquad i = 5,6. \tag{55}$$

4. THE MAIN RESULT AND THE PROOF

The main result of the present article is as follows:

Theorem 2. *Suppose the fulfillment of the conditions of Theorem* 1 *and also the conditions*

$$\tilde{\phi}(z) \in C^2[0,L], \qquad \tilde{g}(t) \in C^2[0,\infty), \qquad \tilde{h}(t) \in C^2(0,\infty), \qquad F(z,t) \in C^2(\Pi)$$

condition (51)*, and the fitting conditions* (31), (40), (41), (54)*, and* (55)*. Then, for every* $L > 0$*, on the interval* $[0,L]$*, there exists a unique solution to the inverse problem* (32)–(35) *of the class* $\Psi(t) \in C^1[0,L]$*, and each component* $\varphi_i \in C^1[0,L]$ *is determined by defining* $h_i(t)$ *for* $t \in [0,L]$*,* $i = 1,2,3$*; and each* $\psi_i \in C^1[0,L]$*, by defining* $h_i(t)$ *for* $t \in [0,L]$*,* $i = 4,5,6$.

Proof. Equations (49), (52), and (53) supplemented with the initial and boundary value conditions from (32) and (33) constitute the closed system of equations on the unknown $w_i(z,t)$, $\Psi_j(t)$, and $\dfrac{\partial}{\partial z} w_i(z,t)$ for $i = \overline{1,6}$. Now, consider the square

$$\Pi_0 := \{(z,t) \mid 0 \le z \le L,\ 0 \le t \le L\}.$$

Equation (49), (52), and (53) show that the values of $w_i(z,t)$, $\Psi_j(t)$, and $\dfrac{\partial}{\partial z} w_i(z,t)$ for $(z,t) \in \Pi_0$ are expressed in terms of the integrals of some combinations of these functions over segments lying in Π_0.

Write (49), (52), and (53) as a closed system of Volterra-type integral equations. For this introduce the vector-functions $v(z,t) = \left(v_i^1, v_i^2, v_i^3\right)$, $i = \overline{1,6}$, by defining their components by the equalities

$$v_i^1(z,t) = w_i(z,t), \qquad v_i^2(z,t) = \Psi_i(t), \qquad v_i^3(z,t) = \frac{\partial}{\partial z} w_i(z,t) + \sum_{j=1}^{6} Q_{ij}(z_0^i;\tilde{\phi})\Psi_j(t_0^i)\frac{\partial}{\partial z} t_0^i.$$

Then the system (49), (52), and (53) takes the operator form

$$v = \mathcal{A}v, \tag{56}$$

where $\mathcal{A}$ is the operator $\mathcal{A} = \big(\mathcal{A}_i^1, \mathcal{A}_i^2, \mathcal{A}_i^3\big)$, $i = \overline{1,6}$, that is defined in accordance with the right-hand sides of (49), (52), and (53) by the equalities

$$\begin{aligned}\mathcal{A}_i^1 \upsilon &= \upsilon_i^{01}(z,t) \\ &+ \int\limits_{t_0^i}^{t} \Big[\sum_{j=1}^{6} Q_{ij}(z+\gamma_i(\tau-t);\tilde{\phi})\upsilon_j^2(\tau) - \sum_{j=1}^{6} b_{ij}(z+\gamma_i(\tau-t))\upsilon_j^1(z+\gamma_i(\tau-t),\tau)\Big]\,d\tau \\ &+ \int\limits_{t_0^i}^{t}\int\limits_{0}^{\tau} \sum_{j=1}^{6} Q_{ij}(z+\gamma_i(\tau-t);\upsilon_j^1(z+\gamma_i(\tau-t),\tau-\alpha))\upsilon_j^2(\alpha)\,d\alpha\,d\tau, \end{aligned} \tag{57}$$

$$\begin{aligned}\mathcal{A}_i^2 \upsilon = \upsilon_i^{02}(z,t) &+ \frac{1}{\det Q(\nu_i;\tilde{\phi})} \int\limits_0^t \sum_{j=1}^{6}\sum_{k=1}^{6} \beta_j \frac{\partial}{\partial z} b_{jk}(-\gamma_j(t-\tau))\upsilon_k^1(-\gamma_j(t-\tau,\tau))\,d\tau\,\mathcal{Q}_{ji}(\nu_i;\tilde{\phi}) \\ &+ \frac{1}{\det Q(\nu_i;\tilde{\phi})} \int\limits_0^t \sum_{j=1}^{6}\sum_{k=1}^{6} \beta_j b_{jk}(-\gamma_j(t-\tau))\Big[\upsilon_k^3(-\gamma_j(t-\tau),\tau) \\ &\qquad - \sum_{p=1}^{6} Q_{kp}\big(z_0^k;\tilde{\phi}\big)\upsilon_p^2\big(t_0^k\big)\frac{\partial}{\partial z}t_0^j\Big]\,d\tau\,\mathcal{Q}_{ji}(\nu_i;\tilde{\phi}) \\ &- \frac{1}{\det Q(\nu_i;\tilde{\phi})} \int\limits_0^t \sum_{k=1}^{6}\sum_{j=1}^{6} \gamma_j\beta_j \frac{\partial}{\partial z} Q_{jk}(-\gamma_j(t-\tau);\tilde{\phi}(-\gamma_j(t-\tau)))\upsilon_k^2(\tau)\,d\tau\,\mathcal{Q}_{ji}(\nu_i;\tilde{\phi}) \\ &- \frac{1}{\det Q(\nu_i;\tilde{\phi})} \int\limits_0^t \sum_{j=1}^{6}\sum_{k=1}^{6} Q_{jk}\Big(-\gamma_j(t-\tau);\frac{d}{dt}\tilde{h}(-\gamma_j(t-\tau))\Big)\upsilon_k^2(\tau)\,d\tau\,\mathcal{Q}_{ji}(\nu_i;\tilde{\phi}) \\ &- \frac{1}{\det Q(\nu_i;\tilde{\phi})} \int\limits_0^t\int\limits_0^\tau \sum_{j=1}^{6}\sum_{k=1}^{6} \beta_j \frac{\partial}{\partial z} Q_{jk}\big(-\gamma_j(t-\tau);\upsilon_k^1(-\gamma_j(t-\tau),\tau-\alpha)\big)\upsilon_k^2(\alpha)\,d\alpha d\tau\,\mathcal{Q}_{ji}(\nu_i;\tilde{\phi}), \end{aligned} \tag{58}$$

$$\begin{aligned}\mathcal{A}_i^3 \upsilon = \upsilon_i^{03}(z,t) &- \int\limits_{t_0^i}^{t} \Big[\sum_{j=1}^{6} \frac{d}{dz} b_{ij}(\xi)\upsilon_j^1(\xi,\tau) + \sum_{j=1}^{6} b_{ij}(\xi)\Big(\upsilon_j^3(\xi,\tau) - \sum_{k=1}^{6} Q_{jk}\big(z_0^j;\tilde{\phi}\big)\upsilon_k^2\big(t_0^j\big)\frac{\partial}{\partial z}t_0^j\Big) \\ &- \sum_{j=1}^{6} \frac{\partial}{\partial z} Q_{ij}(\xi;\tilde{\phi})\upsilon_j^2(\tau)\Big]\Big|_{\xi=z+\gamma_i(\tau-t)} d\tau - \frac{\partial}{\partial z}t_0^i \int\limits_0^{t_0^i} \sum_{j=1}^{6} Q_{ij}\big(z_0^i G_j z_0^i, t_0^i-\tau\big)\upsilon_j^2(\tau)\,d\tau \\ &+ \int\limits_{t_0^i}^{t}\int\limits_0^\tau \sum_{j=1}^{6} \frac{\partial}{\partial z} Q_{ij}\big(\xi;\upsilon_j^1(\xi,\tau-\alpha)\big)\upsilon_j^2(\alpha)\,d\alpha\Big|_{\xi=z+\gamma_i(\tau-t)} d\tau, \end{aligned} \tag{59}$$

where $i = \overline{1,6}$.

In these formulas, we used the notations

$$\upsilon_i^{01}(z,t) = w_i\big(z_0^i, t_0^i\big) + \int\limits_{t_0^i}^{t} \frac{\partial}{\partial t} F_i(z+\gamma_i(\tau-t),\tau)\,d\tau,$$

$$\upsilon_i^{02}(z,t) = \frac{1}{\det Q(\nu_i;\tilde{\phi})} \sum_{j=1}^{6} P_j(\bar{t}_j(t))\mathcal{Q}_{ji}(\nu_i;\tilde{\phi}),$$

$$\upsilon_i^{03}(z,t) = \frac{\partial}{\partial z} w_i\big(z_0^i, t_0^i\big) - \frac{\partial}{\partial z} t_0^i \frac{\partial}{\partial t} F_i\big(z_0^i, t_0^i\big) + \frac{\partial}{\partial z} t_0^i \sum_{j=1}^{6} b_{ij}\big(z_0^i\big) w_j\big(z_0^i, t_0^i\big) + \int\limits_{t_0^i}^{t} \frac{\partial}{\partial t \partial z} F_i(z+\gamma_i(\tau-t),\tau)\,d\tau.$$

Endow the set of continuous functions $C_s(\Pi_0)$ with the norm

$$\|\upsilon\|_s = \max_{1\le i\le 6,\ 1\le l\le 3}\ \sup_{(z,t)\in\Pi_0} \big|\upsilon_i^l(z,t)e^{-st}\big|,$$

where $s \ge 0$ is a number to be chosen below. Obviously, for $s = 0$ this space coincides with the set of continuous functions with the norm $\|\upsilon\|_s$. By the inequality,

$$e^{-sL}\|\upsilon\|_s \le \|\upsilon\|_s \le \|\upsilon\|$$

the norms $\|\upsilon\|_s$ and $\|\upsilon\|$ are equivalent for ant fixed $L \in (0,\infty)$.

Further, consider the set of functions $S(\upsilon^0, r) \subset C_s(\Pi_0)$ satisfying the inequality

$$\|\upsilon - \upsilon^0\|_s \le r, \tag{60}$$

where the vector-function

$$\upsilon^0(z,t) = \big(\upsilon_i^{01}(z,t), \upsilon_i^{02}(t), \upsilon_i^{03}(z,t)\big), \qquad i = \overline{1,6},$$

is defined by the free terms of the operator equation (56). It is not hard to observe that the following estimate holds for $\upsilon \in S(\upsilon^0, r)$:

$$\|\upsilon\|_s \le \|\upsilon^0\|_s + r \le \|\upsilon^0\| + r := r_0.$$

Thus, r_0 is known.

Introduce the notations

$$\tilde{\phi}_0 := \max_{1\le i\le n} \|\tilde{\phi}_i\|_{C^2[0,L]}, \qquad g_0 := \max_{1\le i\le n} \|g_i\|_{C^2[0,L]}, \qquad F_0 := \max_{1\le i\le n} \|F_i\|_{C^2[\Pi_0]},$$

$$h_0 := \max_{1\le i\le n} \|h_i\|_{C^2[0,L]}, \qquad \Gamma_0 := \max\{g_0,\ f_0\}, \qquad P_0 := \min\{|\mathcal{Q}(0)|, |\mathcal{Q}(L)|\},$$

$$\Upsilon_0\tilde{\phi}_0 = \max_{1\le i\le n} \|Q_{ij}(z+\gamma_i(\tau-t);\tilde{\phi})\|_{C^1[0,L]}, \qquad Q_0 := \max\big\{ \max_{1\le i\le n} |Q_i(0)|,\ \max_{1\le i\le n} |Q_i(L)|\big\}.$$

The operator $\mathcal{A}$ takes $C_s(\Pi_0)$ into itself. Show that for a suitable choice of s (note that $L > 0$ is an arbitrary fixed number) it is a contraction operator on $S(\upsilon^0, r)$. Let us first verify that $\mathcal{A}$ takes the set $S(\upsilon^0, r)$ into itself; i.e., the condition $\upsilon(z,t) \in S(\upsilon^0, r)$ implies that $\mathcal{A}\upsilon \in S(\upsilon^0, r)$ if s satisfies some constraints. Indeed, given $(z,t) \in \Pi_0$ and $\upsilon \in S(\upsilon^0, r)$, we have

$$\begin{aligned}\big|(\mathcal{A}_i^1\upsilon - \upsilon_i^{01})e^{-st}\big| = \Big|\int\limits_{t_0^i}^{t}\Big[\sum_{j=1}^{6} Q_{ij}(z+\gamma_i(\tau-t);\tilde{\phi})e^{-s(t-\tau)}\upsilon_j^2(\tau)e^{-s\tau} \\ - \sum_{j=1}^{6} b_{ij}(z+\gamma_i(\tau-t))e^{-s(t-\tau)}\upsilon_j^1(z+\gamma_i(\tau-t),\tau)e^{-s\tau}\Big]d\tau \\ + \int\limits_{t_0^i}^{t}\int\limits_{0}^{\tau}\sum_{j=1}^{6} Q_{ij}\big(z+\gamma_i(\tau-t);\upsilon_j^1(z+\gamma_i(\tau-t),\tau-\alpha)\big)e^{-s(\tau-\alpha)}\upsilon_j^2(\alpha)e^{-s\alpha}\,d\alpha d\tau\Big| \\ \le 6\big[(\Upsilon_0\tilde{\phi}_0 + b_0)\|\upsilon\|_s + \Upsilon_0\|\upsilon\|_s^2\tau\big]\int_0^t e^{-s(t-\tau)}\,d\tau \le \frac{6}{s}\big((\Upsilon_0\tilde{\phi}_0+b_0)+\Upsilon_0 L r_0\big)r_0 := \frac{1}{s}\alpha_1.\end{aligned}$$

Likewise, we obtain

$$\big|(\mathcal{A}_i^2\upsilon - \upsilon_i^{02})e^{-st}\big| \le \frac{36P_0}{sQ_0}\big(b_0(2+6\Upsilon_0\tilde{\phi}_0) + \Upsilon_0\tilde{\phi}_0 + \Upsilon_0\Gamma_0 + \Upsilon_0 L r_0\big)r_0 := \frac{1}{s}\alpha_2,$$

$$\big|(\mathcal{A}_i^3\upsilon - \upsilon_i^{03})e^{-st}\big| \le \frac{6}{s}\big(b_0(2+6\Upsilon_0\tilde{\phi}_0) + \Upsilon_0\tilde{\phi}_0 + \Upsilon_0\Gamma_0 + \Upsilon_0 L r_0\big)r_0 := \frac{1}{s}\alpha_3.$$

These together with (56) and (57)–(59) imply the estimates

$$\begin{aligned}\|\mathcal{A}\upsilon - \upsilon^0\|_s = \max\Big\{\max_{1\le i\le 6}\sup_{(z,t)\in\Pi_0}\big|(\mathcal{A}_i^1\upsilon - \upsilon_i^{01})e^{-st}\big|, \\ \max_{1\le i\le 6}\sup_{t\in[0,L]}\big|(\mathcal{A}_i^2\upsilon - \upsilon_i^{02})e^{-st}\big|,\ \max_{1\le i\le 6}\sup_{t\in[0,L]}\big|(\mathcal{A}_i^3\upsilon - \upsilon_i^{03})e^{-st}\big|\Big\} \le \frac{1}{s}\alpha_0,\end{aligned}$$

where $\alpha_0 := \max(\alpha_1, \alpha_2, \alpha_3)$. Choosing $s > (1/r)\alpha_0$, we obtain that $\mathcal{A}$ takes $S(\upsilon^0, \rho)$ into itself.

Now, take $\upsilon, \tilde{\upsilon} \in S(\upsilon^0, r)$ and estimate the norm of the difference $U\upsilon - U\tilde{\upsilon}$. Using the obvious inequality

$$\big|\upsilon_i^k\upsilon_i^l - \tilde{\upsilon}_i^k\tilde{\upsilon}_i^l\big|e^{-st} \le \big|\upsilon_i^k - \tilde{\upsilon}_i^k\big|\cdot\big|\upsilon_i^l\big|e^{-st} + \big|\tilde{\upsilon}_i^k\big|\cdot\big|\upsilon_i^l - \tilde{\upsilon}_i^l\big|e^{-st} \le 2r_0\|\upsilon - \tilde{\upsilon}\|_s$$

and estimates for the integrals analogous to those above, we arrive at

$$\big|(\mathcal{A}_i^1 \upsilon - \mathcal{A}_i^1 \tilde{\upsilon})e^{-st}\big| = \Bigg| \int\limits_{t_0^i}^{t} \Bigg[\sum_{j=1}^{6} Q_{ij}(z+\gamma_i(\tau-t);\tilde{\phi})e^{-s(\tau-\alpha)}\big(\upsilon_j^2(\tau)-\tilde{\upsilon}_j^2(\tau)\big)e^{-s\tau}$$

$$-\sum_{j=1}^{6} b_{ij}(z+\gamma_i(\tau-t))e^{-s(\tau-\alpha)}\big(\upsilon_j^1(z+\gamma_i(\tau-t),\tau)-\tilde{\upsilon}_j^1(z+\gamma_i(\tau-t),\tau)\big)e^{-s\tau}\Bigg]d\tau$$

$$+\int\limits_{t_0^i}^{t}\int\limits_{0}^{\tau}\sum_{j=1}^{6}\Bigg[Q_{ij}\big(z+\gamma_i(\tau-t);\upsilon_j^1(z+\gamma_i(\tau-t),\tau-\alpha)\big)e^{-s(\tau-\alpha)}\upsilon_j^2(\alpha)e^{-s\alpha}$$

$$-Q_{ij}\big(z+\gamma_i(\tau-t);\tilde{\upsilon}_j^1(z+\gamma_i(\tau-t),\tau-\alpha)\big)e^{-s(\tau-\alpha)}\tilde{\upsilon}_j^2(\alpha)e^{-s\alpha}\Bigg]d\alpha d\tau\Bigg|$$

$$\le n\big[(\Upsilon_0\tilde{\phi}_0+b_0)\|\upsilon-\tilde{\upsilon}\|_s+2r_0\Upsilon_0\|\upsilon-\tilde{\upsilon}\|_s\tau\big]\int\limits_0^t e^{-s(t-\tau)}\,d\tau$$

$$\le \frac{1}{s}n(\Upsilon_0\tilde{\phi}_0+b_0+2r_0\Upsilon_0 L)\|\upsilon-\tilde{\upsilon}\|_s := \frac{1}{s}\,\beta_1\|\upsilon-\tilde{\upsilon}\|_s.$$

Likewise, we obtain the estimates

$$\big|(\mathcal{A}_i^2 \upsilon - \mathcal{A}_i^2 \tilde{\upsilon})e^{-st}\big| \le \frac{36P_0}{sQ_0}\Big(b_0(2+6\Upsilon_0\tilde{\phi}_0)+\Upsilon_0\tilde{\phi}_0+\Upsilon_0\Gamma_0+2r_0\Upsilon_0 L\Big)\,\|\upsilon-\tilde{\upsilon}\|_s$$

$$:= \frac{1}{s}\,\beta_2\|\upsilon-\tilde{\upsilon}\|_s,$$

$$\big|(\mathcal{A}_i^3 \upsilon - \mathcal{A}_i^3 \tilde{\upsilon})e^{-st}\big| \le \frac{6}{s}\Big(b_0(2+6\Upsilon_0\tilde{\phi}_0)+\Upsilon_0\tilde{\phi}_0+\Upsilon_0\Gamma_0+2r_0\Upsilon_0 L\Big)\,\|\upsilon-\tilde{\upsilon}\|_s$$

$$:= \frac{1}{s}\,\beta_3\|\upsilon-\tilde{\upsilon}\|_s.$$

Hence,

$$\|\mathcal{A}\upsilon-\mathcal{A}\tilde{\upsilon}\|_s = \max\Bigg\{\max_{1\le i\le 6}\sup_{(z,t)\in\Pi_0}\big|(\mathcal{A}_i^1 \upsilon - \mathcal{A}_i^1 \tilde{\upsilon})e^{-st}\big|,\ \max_{1\le i\le 6}\sup_{t\in[0,L]}\big|(\mathcal{A}_i^2 \upsilon - \mathcal{A}_i^2 \tilde{\upsilon})e^{-st}\big|,$$

$$\max_{1\le i\le 6}\sup_{t\in[0,L]}\big|(\mathcal{A}_i^3 \upsilon - \mathcal{A}_i^3 \tilde{\upsilon})e^{-st}\big|\Bigg\} \le \frac{1}{s}\,\beta_0\|\upsilon-\tilde{\upsilon}\|_s,$$

where $\beta_0 := \max(\beta_1, \beta_2, \beta_3)$.

Now, choosing $s > \beta_0$, we conclude that $\mathcal{A}$ contracts the distance between υ and $\tilde{\upsilon}$ by $S(\upsilon^0, \rho)$.

As follows from the estimates above, if s is chosen so that

$$s > s^* := \max\{\alpha_0, \beta_0\}$$

then $\mathcal{A}$ is a contraction on $S(\upsilon^0, \rho)$. In this event, by the Banach Principle (see [40, p. 87–97]), equation (56) has a unique solution in $S(\upsilon^0, \rho)$ for every fixed $L > 0$.

Theorem 2 is proved. □

Knowing $\varphi_i'(t)$ and $\psi_i'(t)$ for $i = 1, 2, 3$, we can find the functions $\varphi_i(t)$ and $\psi_i(t)$:

$$\varphi_i(t) = \varphi_i(0) + \int\limits_0^t \varphi_i'(\tau)\, d\tau, \qquad \psi_i(t) = \psi_i(0) + \int\limits_0^t \psi_i'(\tau)\, d\tau, \qquad i = 1, 2, 3.$$

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