AIP Conference Proceedings

RESEARCH ARTICLE | MARCH 11 2024

The problem of determining the one-dimensional kernel for the system of viscoelasticity in the anisotropic medium

Durdimurod Durdiev; Halim Turdiev; Asliddin Boltaev ✉

AIP Conf. Proc. 3004, 040015 (2024)
https://doi.org/10.1063/5.0199987

The Problem of Determining the One-Dimensional Kernel for the System of Viscoelasticity in the Anisotropic Medium.

Durdimurod Durdiev,[1, 2, a)] Halim Turdiev,[1, 2, b)] and Asliddin Boltaev[1, 2, c)]

[1)] *V.I.Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences, 9, University str., Tashkent 100174, Uzbekistan.*

[2)] *Bukhara State University, 11, M.Ikbol str., Bukhara 200114, Uzbekistan.*

[a)] *durdiev65@mail.ru*

[b)] *hturdiev@mail.ru*

[c)] *Corresponding author:asliddinboltayev@mail.ru*

Abstract. We pose the direct and inverse problem of finding the acoustic wave velocity and pressure, diagonal memory matrix for a reduced canonical system of integro-differential acoustic equations. The problems are replaced by a closed system of Volterra-type integral equations of the second kind with respect to the Fourier transform in the variables x_1 and x_2 of the solution of the unknowns of the direct problem and the inverse problem. To this system, we then apply a reduction method, a mapping in the space of continuous functions with a weighted norm. Thus, we prove global existence and uniqueness theorems to solve the given problems.

INTRODUCTION

This research belongs to the class of inverse problems of nonlinear dynamic viscoelasticity. A viscoelastic medium is a medium with memory (the state of such media at the current moment of time depends on the entire prehistory of the process). The desired value in the problem posed is the kernel of the integral operator that models the memory phenomenon that occurs during the propagation of wave processes in viscoelastic media.

The problem of determining the kernel (depending on time and space variables) of an integral operator is a direction in the theory of inverse problems that appear at the end of the last century [1, 2, 3, 4]. A more detailed analysis of sources in this area is presented in the papers [5, 6, 7, 8], which is one of the latest fundamental works in the field of studying inverse problems for media with memory (or with aftereffect).

Among the first results on inverse problems of linear viscoelasticity, close to the, works [9, 10, 11, 12]. Further development of research is reflected, for example, in [13, 14, 15]. In particular interest to multidimensional inverse problems for determining kernels when the required function depends on two or more variables. The one-dimensional and multidimensional inverse problem for with initial, boundary and additional conditions were studied in [16, 17, 18, 19, 20]. In this paper, based on the method of fixed point spaces, we obtain a local unique solvability of the problem of determining the kernel $K(t)$ in the class of functions that are analytic in the variable t.

The study of inverse problems of determining the kernel or coefficient of an integral operator in hyperbolic integro-differential equations is the object of study by many authors. Among those closest to the present work, we can single out [21, 22, 23, 24, 25, 26, 27]. In these papers, the problems of determining the kernel depending only on the time variable (one-dimensional inverse problem) for the case of distributed [21, 22, 23, 24] and lumped [27] sources of wave excitation are considered. The problems are reduced to solving integral equations of the Volterra type with respect to unknown functions. Further, the principle of contraction mappings (Banach's theorem) is applied to these equations in the corresponding function spaces. Existence and uniqueness theorems are obtained, as well as estimates of the continuous dependence of the solution on given functions.

The inverse problem of determining the convolution kernels of integral terms from a system of first-order integro-differential equations of general form with two independent variables were studied in [28, 29, 30]. The theorem of local existence and global uniqueness is obtained. In the work [31] the method for studying the work [29] was applied to the investigating of the inverse problem of determining the diagonal relaxation matrix from the system of Maxwell's integro-differential equations.

International Scientific and Practical Conference on "Modern Problems of Applied Mathematics and Information Technology (MPAMIT2022)"
AIP Conf. Proc. 3004, 040015-1–040015-12; https://doi.org/10.1063/5.0199987
Published by AIP Publishing. 978-0-7354-4876-6/$30.00

Now, we consider anisotropic media with a matrix of independent elastic module of the tetragonal form [32]:

$$c_{ij}=\begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & -c_{25} & 0\\ c_{12} & c_{11} & c_{13} & -c_{14} & c_{25} & 0\\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0\\ c_{14} & -c_{14} & 0 & c_{44} & 0 & c_{25}\\ -c_{25} & c_{25} & 0 & 0 & c_{44} & c_{14}\\ 0 & 0 & 0 & c_{25} & c_{14} & \frac{c_{11}-c_{12}}{2} \end{pmatrix}.$$

Let us denote by σ_{ij} the projection onto the x_i axis of the stress acting on the area with the normal parallel to the x_j axis, and $\overline{u}_i$ are the projection onto the x_i axis of the vector particle displacement. In viscoelastic anisotropic media, the stress tensor has the following representation [33], [34]:

$$\sigma_{ij}(x,t)=\sum_{k,l=1}^{3} c_{ijkl}\left[S_{kl}+\int_0^t K_{ij}(t-\tau)S_{kl}(x,\tau)\right],\ i,j=1,2,3, \tag{1}$$

$$S_{kl}=\frac{1}{2}\left(\frac{\partial \overline{u}_k}{\partial x_l}+\frac{\partial \overline{u}_l}{\partial x_k}\right),\ x\in\mathbb{R}^3,\ k,l=1,2,3,$$

here $K_{ij}(t)$ are functions responsible for the viscosity of the medium and $K_{ij}=K_{ji},\ i,j=\overline{1,3}$.

The equations of motion of a viscoelastic body particles in the absence of external forces have the form [35]:

$$\rho\frac{\partial^2\overline{u}_i}{\partial t^2}=\sum_{j=1}^{3}\frac{\partial \sigma_{ij}}{\partial x_j},\quad i=\overline{1,3}, \tag{2}$$

where $\rho=\rho(x_3)$ is medium density and $\rho>0$, $\overline{u}(x,t)=(\overline{u}_1(x,t),\overline{u}_2(x,t),\overline{u}_3(x,t))$ is displacement vector.

Note that (1) can be considered as integral Volterra equations of the second kind with respect to the expression $\sum\limits_{k,l=1}^{3} c_{ijkl}S_{kl}$. For each fixed pair (i,j) solving these equations, differentiating with respect to t and introducing the notation $u_i=\frac{\partial}{\partial t}\overline{u}_i$, we get

$$\frac{\partial}{\partial t}\sigma_{ij}(x,t)=\sum_{k,l=1}^{3} c_{ijkl}\left(\frac{\partial u_k}{\partial x_l}+\frac{\partial u_l}{\partial x_k}\right)+r_{ij}(0)\sigma_{ij}(x,t)+\int_0^t r'_{ij}(t-\tau)\sigma_{ij}(x,\tau)\,d\tau, \tag{3}$$

where r_{ij} are the resolvents of the kernels K_{ij} and they are related by the following integral relations [36]:

$$r_{ij}(t)=-K_{ij}(t)-\int_0^t K_{ij}(t-\tau)r_{ij}(\tau)d\tau,\ \ i,j=\overline{1,3}. \tag{4}$$

From the condition $K_{ij}=K_{ji}$ implies the $r_{ij}=r_{ji}$.

Then the system of equations (1) and (2) for the velocity u_i and strain $\sigma_{ij}\,(\sigma_{ij}=\sigma_{ji})$ in view of (3) can be written as a system of first-order integro-differential equations. For convenience, let $x_3=z$

$$\left(I\frac{\partial}{\partial t}-B\frac{\partial}{\partial x_1}-C\frac{\partial}{\partial x_2}-D\frac{\partial}{\partial z}-F\right)U(x,t)=\int_0^t R(t-\tau)U(x,\tau)d\tau, \tag{5}$$

where $U=(u_1,u_2,u_3,\sigma_{11},\sigma_{12},\sigma_{13},\sigma_{22},\sigma_{23},\sigma_{33})^*$, $*$ is the transposition sign,

$$B=\begin{pmatrix} \mathbf{O}_{3\times3} & \begin{matrix} \frac{1}{\rho} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{\rho} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{\rho} & 0 \end{matrix}\\ \begin{matrix} c_{11} & c_{14} & -c_{25}\\ c_{12} & c_{25} & 0\\ c_{13} & 0 & 0\\ c_{14} & c_{44} & 0\\ -c_{25} & 0 & c_{44}\\ 0 & c_{25} & c_{14} \end{matrix} & \mathbf{O}_{6\times6} \end{pmatrix},\quad C=\begin{pmatrix} \mathbf{O}_{3\times3} & \begin{matrix} 0 & 0 & 0 & \frac{1}{\rho} & 0 & 0\\ 0 & \frac{1}{\rho} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\rho} \end{matrix}\\ \begin{matrix} c_{14} & c_{12} & 0\\ -c_{14} & c_{11} & 0\\ 0 & c_{13} & 0\\ c_{44} & c_{14} & c_{25}\\ 0 & c_{25} & c_{14}\\ c_{25} & 0 & \frac{c_{11}-c_{12}}{2} \end{matrix} & \mathbf{O}_{6\times6} \end{pmatrix},$$

$$D=\begin{pmatrix} & \mathbf{O}_{3\times3} & & \begin{matrix}0&0&0&0&\frac{1}{\rho}&0\\0&0&0&0&0&\frac{1}{\rho}\\0&0&\frac{1}{\rho}&0&0&0\end{matrix}\\ -c_{25}&0&c_{13}& \\ -c_{25}&0&c_{13}& \\ 0&0&c_{33}& \\ 0&c_{25}&0& \mathbf{O}_{6\times6}\\ c_{44}&c_{14}&0& \\ c_{14}&\frac{c_{12}-c_{11}}{2}&0& \end{pmatrix},$$

$$F=\begin{pmatrix}\mathbf{O}_{3\times3} & \mathbf{O}_{3\times6}\\ \mathbf{O}_{6\times3} & \operatorname{diag}\left(r_{11}(0),r_{22}(0),r_{33}(0),r_{12}(0),r_{13}(0),r_{23}(0)\right)\end{pmatrix},\ R(t)=\begin{pmatrix}\mathbf{O}_{3\times3} & \mathbf{O}_{3\times6}\\ \mathbf{O}_{6\times3} & \operatorname{diag}\left(r'_{11},r'_{22},r'_{33},r'_{12},r'_{13},r'_{23}\right)\end{pmatrix}.$$

The system (5) can be reduced to a symmetric hyperbolic system [37].

Let us reduce the system (5) to the canonical form with respect to the variables t and z. To do this, we write the equation

$$|D-\lambda I|=0, \tag{6}$$

where I is an identity matrix of dimension 9. Equation (6) has roots

$$\lambda_1=\sqrt{\frac{c_{33}}{\rho}},\ \ \lambda_2=\frac{1}{2\sqrt{\rho}}\sqrt{a_1-\sqrt{a_2}},\ \ \lambda_3=\frac{1}{2\sqrt{\rho}}\sqrt{a_1+\sqrt{a_2}},\ \ \lambda_4=\lambda_5=\lambda_6=0, \tag{7}$$

$$\lambda_7=-\frac{1}{2\sqrt{\rho}}\sqrt{a_1+\sqrt{a_2}},\ \ \lambda_8=-\frac{1}{2\sqrt{\rho}}\sqrt{a_1-\sqrt{a_2}},\ \ \lambda_9=\sqrt{\frac{c_{33}}{\rho}}, \tag{8}$$

where $a_1=c_{11}-c_{12}+2c_{44}$, $a_2=(c_{11}-c_{12}-2c_{44})^2+16c_{14}^2$.

Now we choose a nonsingular matrix $T(z,t)$ so that the equality holds

$$T^{-1}DT=\Lambda, \tag{9}$$

where Λ is a diagonal matrix that is composed by the eigenvalues of the matrix D.

The formula (9) implies the equality

$$DT=T\Lambda,$$

which means that the column with index i of the matrix T is an eigenvector of the matrix DT corresponding to the eigenvalue λ_i. Direct calculations show that the matrix T satisfying the above conditions can be chosen as

$$T(z)=\begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0\\
0 & \frac{a_3-\sqrt{a_2}}{4c_{14}} & \frac{a_3+\sqrt{a_2}}{4c_{14}} & 0 & 0 & 0 & \frac{a_3+\sqrt{a_2}}{4c_{14}} & \frac{a_3-\sqrt{a_2}}{4c_{14}} & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
-\frac{c_{13}}{\lambda_1} & \frac{c_{25}}{\lambda_2} & \frac{c_{25}}{\lambda_3} & 1 & 0 & 1 & -\frac{c_{25}}{\lambda_3} & -\frac{c_{25}}{\lambda_2} & \frac{c_{13}}{\lambda_1}\\
-\frac{c_{13}}{\lambda_1} & -\frac{c_{25}}{\lambda_2} & -\frac{c_{25}}{\lambda_3} & 1 & 0 & 1 & \frac{c_{25}}{\lambda_3} & \frac{c_{25}}{\lambda_2} & \frac{c_{13}}{\lambda_1}\\
-\lambda_1\rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1\rho\\
0 & -\frac{c_{25}}{\lambda_2}\cdot\frac{a_3-\sqrt{a_2}}{4c_{14}} & -\frac{c_{25}}{\lambda_3}\cdot\frac{a_3+\sqrt{a_2}}{4c_{14}} & 0 & 0 & 1 & \frac{c_{25}}{\lambda_3}\frac{a_3+\sqrt{a_2}}{4c_{14}} & \frac{c_{25}}{\lambda_2}\cdot\frac{a_3-\sqrt{a_2}}{4c_{14}} & 0\\
0 & -\lambda_2\rho & -\lambda_3\rho & 0 & 0 & 0 & \lambda_3\rho & \lambda_2\rho & 0\\
0 & -\lambda_2\rho\frac{a_3-\sqrt{a_2}}{4c_{14}} & -\lambda_3\rho\frac{a_3+\sqrt{a_2}}{4c_{14}} & 0 & 0 & 0 & \lambda_3\rho\frac{a_3+\sqrt{a_2}}{4c_{14}} & \lambda_2\rho\frac{a_3-\sqrt{a_2}}{4c_{14}} & 0
\end{pmatrix},$$

where $a_3=c_{11}-c_{12}-2c_{44}$.

We introduce a new function in equation (5) using the equality

$$U=T\vartheta.$$

and multiply this equation on the left by the matrix T. Then, for the function ϑ, after obvious transformations, we obtain the equation

$$\left(I\frac{\partial}{\partial t}+\Lambda\frac{\partial}{\partial z}+B_1\frac{\partial}{\partial x_1}+D_1\frac{\partial}{\partial x_2}+F_1\right)\vartheta(x,t)=\int\limits_0^t R(z,t-\tau)\vartheta(x,\tau)d\tau, \qquad (10)$$

where, $B_1=T^{-1}BT=(b_{ij})$, $D_1=T^{-1}CT=(d_{ij})$, $F_1=T^{-1}D\frac{\partial T}{\partial z}+T^{-1}FT=(p_{ij})$, $R_1(t)=T^{-1}RT=(\widetilde{r}_{ij})$.

The (10) system is convenient in the sense that it splits with respect to the derivatives with respect to t and x_3, and is only linked through $\frac{\partial\vartheta}{\partial x}$ and ϑ. The components $\vartheta_i,\ i=1,2,3,7,8,9$ of the vector function ϑ are called Riemannian invariants of the system (5).

SET UP PROBLEMS AND INVESTIGATION OF THE DIRECT PROBLEM

Consider the system of equations (10) in the domain

$$\Omega=\left\{(x,t):(x_1,x_2)\in\mathbb{R}^2,z\in(0,H),t>0\right\},\ H=const.$$

The purpose of this article is to study the direct and inverse problems for the system (10). Moreover, **the direct problem** is an initial-boundary value problem for this system in domain Ω and in the inverse problem, the elements of the matrix R are assumed to be unknown, which are included in the definition of the matrix R_1.

In the direct problem, given matrices B_1, D_1, F_1, and R_1 it is required, in the domain Ω find a vector-function $\vartheta(x,t)$ satisfying equation (10) for the following initial and boundary conditions:

$$\vartheta_i\big|_{t=0}=\varphi_i(x),\ (x_1,x_2)\in\mathbb{R}^2,\ z=[0,H],\ \ i=\overline{1,9}, \qquad (11)$$

$$\vartheta_i\big|_{z=H}=g_i(x_1,x_2,t),\ \ i=\overline{1,3},\quad \vartheta_i\big|_{z=0}=g_i(x_1,x_2,t),\ \ i=\overline{7,9}, \qquad (12)$$

here $\varphi_i(x)$, $i=\overline{1,9}$, $g_i(x_1,x_2,t)$, $i=1,2,3,7,8,9$ are given functions. It is known that [1] the problem (10), (11), (12) is posed well.

The defined of the elements $\widetilde{r}_{ij}(z,t)$, $i,j=\overline{1,9}$ of the matrix $R(z,t)$ includes functions $r_{ij}(t)$, $i,j=\overline{1,9}$, $c_{ij}(z)$ on the module of elasticity and $\rho(z)$ on the density of the medium.

The inverse problem is to determine the nonzero components of the matrix kernel $R(z,t)$, that is $r_{ij}(t),\ i,j=\overline{1,3}$ (where $c_{ij}(z)$ and $\rho(z)$ are given functions) in (10) – (12) if the following conditions are known:

$$\vartheta_i\big|_{z=0}=h_i(x_1,x_2,t),\ \ i=\overline{1,6}, \qquad (13)$$

where $h_i(x_1,x_2,t),\ \ i=\overline{1,6}$, are the given functions. In the inverse problem, the numbers $r_{ij}(0),\ i,j=\overline{1,3}$ are also considered to be given.

Let functions $\varphi(x)$, $g(x_1,x_2,t)$ included in the right-hand side of (10) and the data (11), (12) are compact support in x_1, x_2 for each fixed z,t. From the existence for the system (10) of a compact support domain of dependence and compact support with respect to $x_1,\ x_2$ of the right-hand side (10) and data (11), (12) implies the compact support in $x_1,\ x_2$ solutions to the problem (10)–(12).

Let us study the property of solution to this problem. More precisely, we restrict ourselves to studying the Fourier transform in the variables $x_1,\ x_2$ of the solution. Introduce the notation

$$\widehat{\vartheta}\,(\eta_1,\eta_2,z,t)=\int\limits_{\mathbb{R}^2}\vartheta(x_1,x_2,z,t)e^{i[\eta_1x_1+\eta_2x_2]}dx_1dx_2,$$

where η_1,η_2 are transformation parameters. We fix η_1, η_2 and for convenience, we introduce the notation $\widehat{\vartheta}(\eta_1,\eta_2,z,t)=\widehat{\vartheta}(z,t)$.

In terms of the function $\widehat{\vartheta}$ we write the problem (10)-(12) as

$$\left(\frac{\partial}{\partial t}+\lambda_j\frac{\partial}{\partial z}\right)\widehat{\vartheta}_j(z,t)=\sum_{k=1}^{9}\widehat{p}_{jk}(z)\widehat{\vartheta}_k(z,t)+\int\limits_0^t\sum_{k=1}^{9}\widetilde{r}_{jk}(z,\tau)\widehat{\vartheta}_k(\eta_1,\eta_2,z,t-\tau)d\tau,\quad j=\overline{1,9}, \qquad (14)$$

where $\widehat{p}_{jk}(z) = \widehat{p}_{jk} = -i\eta_1 b_{jk} - i\eta_2 d_{jk} - p_{jk}$.

We will use a similar notations for the Fourier images of functions included in the initial, boundary and additional conditions (11)–(13):

$$\widehat{\vartheta}_i\big|_{t=0} = \widehat{\varphi}_i(z), \quad i = \overline{1,9}, \tag{15}$$

$$\widehat{\vartheta}_i\big|_{z=H} = \widehat{g}_i(t), \;\; i = \overline{1,3}, \quad \widehat{\vartheta}_i\big|_{z=0} = \widehat{g}_i(t), \;\; i = \overline{7,9}, \tag{16}$$

$$\widehat{\vartheta}_i\big|_{z=0} = \widehat{h}_i(t), \;\; i = \overline{1,6}, \tag{17}$$

where $\widehat{\varphi}_i(z),\; i = \overline{1,9},\; \widehat{g}_i(t),\; i = 1,2,3,7,8,9$ are the Fourier images of the corresponding functions from (11), (12) for $\xi = 0$. We also denote by Ω_H the projection of Ω onto the plane z,t.

For the purpose of further research let us introduce the vector function $\omega(z,t) = \frac{\partial\widehat{\vartheta}}{\partial t}(z,t)$. To obtain a problem for a function $\omega(z,t)$ similar to (14) – (17) differentiate equations (14) and the boundary conditions (16) with respect to the variable t, and the condition for $t = 0$ is found using equations (14) and the initial conditions (15). In this case, we get

$$\left(\frac{\partial}{\partial t} + \lambda_i \frac{\partial}{\partial z}\right)\omega_i(z,t) = \sum_{k=1}^{9} \widehat{p}_{ik}(z)\omega_k(z,t) + \sum_{k=1}^{9} \widetilde{r}_{ik}(z,t)\widehat{\varphi}_i(z) + \int_0^t \sum_{k=1}^{9} \widetilde{r}_{ik}(z,\tau)\omega_k(z,t-\tau)d\tau, \quad i = \overline{1,9}, \tag{18}$$

$$\omega_i\big|_{t=0} = -\lambda_i \frac{\partial\widehat{\varphi}_i(z)}{\partial z} + \sum_{j=1}^{9} \widehat{p}_{ji}\widehat{\varphi}_i(z) =: \Phi_i(z), \quad i = \overline{1,9}, \tag{19}$$

$$\omega_i\big|_{z=H} = \frac{d}{dt}\widehat{g}_i(t), \;\; i = \overline{1,3}, \quad \omega_i\big|_{z=0} = \frac{d}{dt}\widehat{g}_i(t), \;\; i = \overline{7,9}. \tag{20}$$

For functions ω_i additional conditions (17) gets

$$\omega_i\big|_{z=0} = \frac{d}{dt}\widehat{h}_i(t), \;\; i = \overline{1,6}. \tag{21}$$

Let us pass from equalities (14)–(17) to the integral relations for the components of the vector $\widehat{\vartheta}$ with integration flux along the corresponding characteristics of the equations of the system (14). We denote

$$\mu_i(z) = \int_0^z \frac{1}{\lambda_i(\beta)} d\beta, \; i = 1,2,3,7,8,9, \;\; \mu_j(z) = 0, \;\; j = 4,5,6.$$

Inverse functions to $\mu_i(z),\; i = \overline{1,9}$, will be denoted by $z = \mu_i^{-1}(t),\; i = \overline{1,9}$. Using the introduced functions, the equations of characteristics passing through the points (z,t) on the plane of variables ξ,τ can be written in the form

$$\tau = t + \mu_i(\xi) - \mu_i(z), \quad i = \overline{1,9}. \tag{22}$$

Consider an arbitrary point $(z,t) \in \Omega_H$ on the plane of variables ξ,τ and draw through it the characteristic of the i th of the system (14) equation tell to intersection in the domain $\tau \le t$. The intersection point is denoted by (z_0^i, t_0^i). Integrating the equations of the system (14) along the corresponding characteristics from the point (z_0^i, t_0^i) to the point (z,t) we find

$$\omega_i(z,t) = \omega_i(z_0^i, t_0^i)$$

$$+ \int_{t_0^i}^{t} \left[\sum_{k=1}^{9} \widehat{p}_{ik}(z)\omega_k(\xi,\tau) + \sum_{k=1}^{9} \widetilde{r}_{ik}(\xi,\tau)\widehat{\varphi}_i(\xi) + \int_0^\tau \sum_{k=1}^{9} \widetilde{r}_{ik}(\xi,\tau-\alpha)\omega_k(\xi,\alpha)d\alpha\right]\Bigg|_{\xi=\mu_i^{-1}[\tau - t + \mu_i(z)]} d\tau, \;\; i = \overline{1,9}. \tag{23}$$

We define in (23) t_0^i. It depends on the coordinates of the point (z,t). It is not difficult to see that $t_0^i(z,t)$ has the form

$$t_0^i(z,t) = \begin{cases} t-\mu_i(z)+\mu_i(H), & t \geq \mu_i(z)-\mu_i(H), \\ 0, & 0 < t < \mu_i(z)-\mu_i(H), \end{cases} \quad i=1,2,3,$$

$$t_0^i(z,t)=0,\ \ i=4,5,6, \quad t_0^i(z,t) = \begin{cases} t-\mu_i(z), & t \geq \mu_i(z), \\ 0, & 0<t<\mu_i(z), \end{cases} \quad i=7,8,9.$$

Then, from the condition that the pair (z_0^i, t_0^i) satisfies equation (22) it follows

$$z_0^i(z,t) = \begin{cases} H, & t \geq \mu_i(z)-\mu_i(H), \\ \mu_i^{-1}\left(\mu_i(z)-t\right), & 0<t<\mu_i(z)-\mu_i(H), \end{cases} \quad i=1,2,3,$$

$$z_0^i(z,t)=z,\ i=4,5,6, \quad z_0^i(z,t) = \begin{cases} 0, & t \geq \mu_i(z), \\ \mu_i^{-1}\left(\mu_i(z)-t\right), & 0<t<\mu_i(z), \end{cases} \quad i=7,8,9.$$

The free terms of the integral equations (23) are defined through the initial and boundary conditions (19) and (20) as follows:

$$\omega_i(z_0^i,t_0^i) = \begin{cases} \frac{\partial}{\partial t}\widehat{g}_i\left(t-\mu_i(z)+\mu_i(H)\right), & t \geq \mu_i(z)-\mu_i(H), \\ \Phi_i\left(\mu_i^{-1}\left(\mu_i(z)-t\right)\right), & 0<t<\mu_i(z)-\mu_i(H), \end{cases} \quad i=1,2,3,$$

$$\omega_i(z_0^i,t_0^i)=\Phi_i(z), i=4,5,6,\ \omega_i(z_0^i,t_0^i) = \begin{cases} \frac{\partial}{\partial t}\widehat{g}_i\left(t-\mu_i(z)\right), & t \geq \mu_i(z), \\ \Phi_i\left(\mu_i^{-1}\left(\mu_i(z)-t\right)\right), & 0<t<\mu_i(z), \end{cases} \quad i=7,8,9.$$

Let the following conditions hold

$$\widehat{\varphi}_i(H)=\widehat{g}_i(0), \ \text{and} \ \left.\frac{\partial \widehat{g}_i(t)}{\partial t}\right|_{t=0} = -\lambda_j \left.\frac{\partial \widehat{\varphi}_i(z)}{\partial z}\right|_{z=H} + \sum_{j=1}^{9} p_{ij}(H)\widehat{\varphi}_j(H), \ \ i=\overline{1,3}, \tag{24}$$

$$\widehat{\varphi}_i(0)=\widehat{g}_i(0), \ \text{and} \ \left.\frac{\partial \widehat{g}_i(t)}{\partial t}\right|_{t=0} = -\lambda_j \left.\frac{\partial \widehat{\varphi}_i(z)}{\partial z}\right|_{z=0} + \sum_{j=1}^{9} p_{ij}(0)\widehat{\varphi}_j(0), \ \ i=\overline{7,9}. \tag{25}$$

It is easy to see that the conditions for matching the initial and boundary data (15), (16), (19), (20) in corner points of the domain Ω_H coincide with the relations (24) and (25). Hence it is clear that at the fulfillment of the same equalities (24) and (25) equations then (23) will have unique continuous solutions $\omega_i(z,t)$, or the same $\frac{\partial}{\partial t}\widehat{\vartheta}_i(z,t)$.

Thus, the following statement holds:

Theorem 1. *Assume functions $\varphi(x)$, $g(x_1,x_2,t)$ have compact supports in x_1, x_2 for each fixed z, t. Let $\rho(z)$, $c_{33}(z)$, $c_{44}(z)$, $c_{66}(z)$, $\widehat{\varphi}(z) \in C^1[0,H]$, $\widehat{g}(t) \in C^1[0,T]$, $\rho(z)>0$, $c_{33}(z)>0$, $c_{44}(z)>0$, $c_{66}(z)>0$, $r_{ij}(t) \in C^1[0,T]$, $i,j=1,2$, and conditions (24),(25) be satisfied. Then there is a unique solution to the problem (18)-(20) in the domain $\overline{\Omega}_{HT}=\{(z,t): 0 \leq z \leq H,\ 0 \leq t \leq T\}$, .*

The problem (18)-(20) in the domain Ω_{HT} is equivalent to a linear integral equation of the second kind of Volterra type with respect to $\omega(z,t)$. As follows from the theory of linear integral equations, it has a unique solutions [38], [39]. So we drop it.

DERIVATION OF EQUIVALENT INTEGRAL EQUATIONS

Consider an arbitrary point $(z,0) \in \Omega_{HT}$ and draw through it the characteristics (22) for $i=\overline{1,6}$, up to the intersection with the boundary of the domain Ω_H. Integrating the first six components of equation (18), we obtain

$$\omega_i(z,0)=\omega_i(0,t_1^i) - \int_0^{t_1^i} \left.\sum_{k=1}^{9}\widehat{p}_{ik}(\xi)\omega_k(\xi,\tau)\right|_{\xi=\mu_i^{-1}[\tau+\mu_i(z)]} d\tau$$

$$-\int_0^{t_1^i}\left.\left[\sum_{k=1}^{9}\widetilde{r}_{ik}(\xi,\tau)\widehat{\varphi}_i(\xi)+\int_0^{\tau}\sum_{k=1}^{9}\widetilde{r}_{ik}(\xi,\tau-\alpha)\omega_k(\xi,\alpha)d\alpha\right]\right|_{\xi=\mu_i^{-1}[\tau+\mu_i(z)]} d\tau, \quad i=\overline{1,6}, \tag{26}$$

where $t_1^i = -\mu_i(z),\ i = 1,2,3,\ t_1^i = 0,\ i = 4,5,6.$

Consider (26) the initial conditions (19), we differentiate (26) with respect to z for $i = 1,2,3$ and for t for $i = 4,5,6$. After simple calculations, taking into account (38)-(42), we pass to integral equations.

$$r_{11}'(t) = \alpha_9 P_4(t) + \alpha_9\alpha_{11}P_1(t) + \alpha_9\int_0^t \alpha_3\left(r_{11}' - r_{33}'\right)(\tau)\frac{d}{dt}\left(\widehat{g}_9 - \widehat{h}_1\right)(t-\tau)d\tau$$

$$+\alpha_9\int_0^t\left[\alpha_4 r_{11}'(\tau)\frac{d}{t}\left(\widehat{g}_8 - \widehat{h}_2\right)(t-\tau) + \alpha_6 r_{11}'(\tau)\frac{d}{t}\left(\widehat{g}_7 - \widehat{h}_3\right)(t-\tau) + r_{11}'(\tau)\frac{d}{t}\left(\widehat{h}_4 + \widehat{h}_5\right)(t-\tau)\right]d\tau$$

$$+\alpha_9\alpha_{11}\int_0^t\left[\frac{\partial}{\partial z}\sum_{j=1}^{9}\widehat{p}_{1j}(\xi)\omega_j(\xi,\tau) + \frac{1}{2\lambda_1}r_{33}'(\tau)\frac{\partial}{\partial z}\left(\widehat{\varphi}_1 - \widehat{\varphi}_9\right)(\xi)\right]\Bigg|_{\xi=\mu_1^{-1}[t-\tau]}d\tau$$

$$+\alpha_9\alpha_{11}\int_0^t\left[\frac{1}{2\lambda_1}r_{33}'(\tau)\frac{d}{dt}\left(\widehat{h}_1 - \widehat{g}_9\right)(t-\tau) + \int_0^\tau r_{33}'(\alpha)\frac{\partial}{\partial z}\left(\omega_1 - \omega_9\right)(\xi,\tau-\alpha)d\alpha\Bigg|_{\xi=\mu_1^{-1}[t-\tau]}\right]d\tau, \qquad (27)$$

$$r_{22}'(t) = \alpha_2 P_5(t) + \alpha_{10}\alpha_{11}P_1(t) + \alpha_{10}\int_0^t \alpha_7\left(r_{22}' - r_{33}'\right)(\tau)\frac{d}{dt}\left(\widehat{g}_9 - \widehat{h}_1\right)(t-\tau)d\tau$$

$$+\alpha_{10}\int_0^t\left[\alpha_4 r_{22}'(\tau)\left(\widehat{h}_2 - \widehat{g}_8\right)(t-\tau) + \alpha_6 r_{22}'(\tau)\left(\widehat{h}_3 - \widehat{g}_7\right)(t-\tau) + r_{22}'(\tau)\widehat{h}_5(t-\tau)\right]d\tau$$

$$+\alpha_{10}\alpha_{11}\int_0^t\left[\frac{\partial}{\partial z}\sum_{j=1}^{9}\widehat{p}_{1j}(\xi)\omega_j(\xi,\tau) + \frac{1}{2\lambda_1}r_{33}'(\tau)\frac{\partial}{\partial z}\left(\widehat{\varphi}_1 - \widehat{\varphi}_9\right)(\xi)\right]\Bigg|_{\xi=\mu_1^{-1}[t-\tau]}d\tau$$

$$+\alpha_{10}\alpha_{11}\int_0^t\left[\frac{1}{2\lambda_1}r_{33}'(\tau)\frac{d}{dt}\left(\widehat{h}_1 - \widehat{g}_9\right)(t-\tau) + \int_0^\tau r_{33}'(\alpha)\frac{\partial}{\partial z}\left(\omega_1 - \omega_9\right)(\xi,\tau-\alpha)d\alpha\Bigg|_{\xi=\mu_1^{-1}[t-\tau]}\right]d\tau, \qquad (28)$$

$$r_{33}'(t) = \alpha_{11}P_1(t) + \alpha_{11}\int_0^t\left[\frac{\partial}{\partial z}\sum_{j=1}^{9}\widehat{p}_{1j}(\xi)\omega_j(\xi,\tau) + \frac{1}{2\lambda_1}r_{33}'(\tau)\frac{\partial}{\partial z}\left(\widehat{\varphi}_1 - \widehat{\varphi}_9\right)(\xi)\right]\Bigg|_{\xi=\mu_1^{-1}[t-\tau]}d\tau$$

$$+\alpha_{11}\int_0^t\left[\frac{1}{2\lambda_1}r_{33}'(\tau)\frac{d}{dt}\left(\widehat{h}_1 - \widehat{g}_9\right)(t-\tau) + \int_0^\tau r_{33}'(\alpha)\frac{\partial}{\partial z}\left(\omega_1 - \omega_9\right)(\xi,\tau-\alpha)d\alpha\Bigg|_{\xi=\mu_1^{-1}[t-\tau]}\right]d\tau, \qquad (29)$$

$$r_{12}'(t) = \alpha_{12}P_6(t) + \alpha_{14}\left(P_2 + P_3\right)(t) + \alpha_{13}\alpha_{14}\int_0^t\frac{\partial}{\partial z}\sum_{j=1}^{9}\widehat{p}_{1j}(\xi)\omega_j(\xi,\tau)\Bigg|_{\xi=\mu_2^{-1}[t-\tau]}d\tau$$

$$+\alpha_{12}\int_0^t \Big[\left(\alpha_7 r'_{13}-\alpha_5 r'_{12}\right)(\tau)\left(\widehat{h}_2-\widehat{g}_8\right)(t-\tau)+\widehat{g}_6(t-\tau)+\left(\alpha_8 r'_{13}-\alpha_6 r'_{12}\right)(\tau)\left(\widehat{h}_3-\widehat{g}_7\right)(t-\tau)\Big]\,d\tau$$

$$+\int_0^t \left(r'_{23}-r'_{13}\right)(\tau)\left[\alpha_{13}\alpha_{15}\left(\widehat{\varphi}_2-\widehat{\varphi}_8\right)(\xi)+\alpha_{13}\alpha_{16}\left(\widehat{\varphi}_3-\widehat{\varphi}_7\right)(\xi)\right]\Big|_{\xi=\mu_2^{-1}[t-\tau]}\,d\tau$$

$$+\int_0^t\int_0^\tau \left(r'_{23}-r'_{13}\right)(\alpha)\left[\alpha_{13}\alpha_{15}\frac{\partial}{\partial z}\left(\omega_2-\omega_8\right)(\xi,\tau-\alpha)+\alpha_{13}\alpha_{16}\frac{\partial}{\partial z}\left(\omega_3-\omega_7\right)(\xi,\tau-\alpha)\right]\Big|_{\xi=\mu_2^{-1}[t-\tau]}\,d\tau$$

$$+\int_0^t \left(r'_{23}-r'_{13}\right)(\tau)\left[\alpha_{13}\alpha_{15}\left(\widehat{\varphi}_2-\widehat{\varphi}_8\right)(\xi)+\alpha_{13}\alpha_{16}\left(\widehat{\varphi}_3-\widehat{\varphi}_7\right)(\xi)\right]\Big|_{\xi=\mu_2^{-1}[t-\tau]}\,d\tau, \tag{30}$$

$$r'_{13}(t)=\alpha_{11}\left(P_2+P_3\right)(t)+\alpha_{11}\int_0^t \frac{\partial}{\partial z}\sum_{j=1}^{9}\widehat{p}_{1j}(\xi)\omega_j(\xi,\tau)\Big|_{\xi=\mu_2^{-1}[t-\tau]}\,d\tau$$

$$+\int_0^t \left(r'_{23}-r'_{13}\right)(\tau)\left[\alpha_{15}\left(\widehat{\varphi}_2-\widehat{\varphi}_8\right)(\xi)+\alpha_{16}\left(\widehat{\varphi}_3-\widehat{\varphi}_7\right)(\xi)\right]\Big|_{\xi=\mu_2^{-1}[t-\tau]}\,d\tau$$

$$+\int_0^t\int_0^\tau \left(r'_{23}-r'_{13}\right)(\alpha)\left[\alpha_{15}\frac{\partial}{\partial z}\left(\omega_2-\omega_8\right)(\xi,\tau-\alpha)+\alpha_{16}\frac{\partial}{\partial z}\left(\omega_3-\omega_7\right)(\xi,\tau-\alpha)\right]\Big|_{\xi=\mu_2^{-1}[t-\tau]}\,d\tau$$

$$+\int_0^t \left(r'_{23}-r'_{13}\right)(\tau)\left[\alpha_{15}\left(\widehat{\varphi}_2-\widehat{\varphi}_8\right)(\xi)+\alpha_{16}\left(\widehat{\varphi}_3-\widehat{\varphi}_7\right)(\xi)\right]\Big|_{\xi=\mu_2^{-1}[t-\tau]}\,d\tau, \tag{31}$$

$$r'_{23}(t)=\alpha_{17}\left(P_2-P_3\right)(t)+\alpha_{17}\int_0^t \frac{\partial}{\partial z}\sum_{j=1}^{9}\widehat{p}_{1j}(\xi)\omega_j(\xi,\tau)\Big|_{\xi=\mu_2^{-1}[t-\tau]}\,d\tau$$

$$+\int_0^t \left(r'_{23}-r'_{13}\right)(\tau)\left[\alpha_{18}\left(\widehat{\varphi}_2-\widehat{\varphi}_8\right)(\xi)+\alpha_{19}\left(\widehat{\varphi}_3-\widehat{\varphi}_7\right)(\xi)\right]\Big|_{\xi=\mu_2^{-1}[t-\tau]}\,d\tau$$

$$+\int_0^t\int_0^\tau \left(r'_{23}-r'_{13}\right)(\alpha)\left[\alpha_{18}\frac{\partial}{\partial z}\left(\omega_2-\omega_8\right)(\xi,\tau-\alpha)+\alpha_{19}\frac{\partial}{\partial z}\left(\omega_3-\omega_7\right)(\xi,\tau-\alpha)\right]\Big|_{\xi=\mu_2^{-1}[t-\tau]}\,d\tau$$

$$+\int_0^t \left(r'_{23}-r'_{13}\right)(\tau)\left[\alpha_{18}\left(\widehat{\varphi}_2-\widehat{\varphi}_8\right)(\xi)+\alpha_{19}\left(\widehat{\varphi}_3-\widehat{\varphi}_7\right)(\xi)\right]\Big|_{\xi=\mu_2^{-1}[t-\tau]}\,d\tau, \tag{32}$$

where $P_i(t) = \frac{d^2}{dt^2}\widehat{h}_i(t_1^i) - \frac{\partial}{\partial z}\Phi_i(0) - \frac{1}{\lambda_i(0)}\sum\limits_{j=1}^{9}\widehat{p}_{ij}(0)\omega_j(0,t),\ i=\overline{1,6}$ and the coefficients are determined by the following equalities:

$$\alpha_1 = \frac{a_3+\sqrt{a_2}}{4\sqrt{a_2}},\ \alpha_2 = \frac{\lambda_3\left(a_3-\sqrt{a_2}\right)}{4\lambda_2\sqrt{a_2}},\ \alpha_3 = -\frac{c_{13}}{\lambda_1},\ \alpha_4 = \frac{c_{25}}{\lambda_2},\ \alpha_5 = \frac{c_{25}\left(a_3-\sqrt{a_2}\right)}{4c_{14}\lambda_2},$$

$$\alpha_6 = \frac{c_{25}}{\lambda_3},\quad \alpha_7 = -\frac{c_{25}\left(a_3^2-a_2\right)}{2c_{14}\sqrt{a_2}\lambda_3}\left[\frac{\lambda_2}{\lambda_3}-\frac{\lambda_3}{\lambda_2}\right],\quad \alpha_8 = -\frac{c_{25}\left(a_3^2-a_2\right)}{2c_{14}\sqrt{a_2}\lambda_3}\left[\frac{\lambda_2}{\lambda_3}+\frac{\lambda_3}{\lambda_2}\right],$$

$$\alpha_9 = \left[\alpha_3(\widehat{\varphi}_1-\widehat{\varphi}_9)(0)+\alpha_4(\widehat{\varphi}_2-\widehat{\varphi}_8)(0)+\alpha_6(\widehat{\varphi}_3-\widehat{\varphi}_7)(0)+(\widehat{\varphi}_4+\widehat{\varphi}_6)(0)\right]^{-1},$$

$$\alpha_{10} = \left[\alpha_3(\widehat{\varphi}_1-\widehat{\varphi}_9)(0)+\alpha_4(\widehat{\varphi}_2-\widehat{\varphi}_8)(0)+\alpha_6(\widehat{\varphi}_3-\widehat{\varphi}_7)(0)+\widehat{\varphi}_6(0)\right]^{-1},\alpha_{11} = \left[(\widehat{\varphi}_9-\widehat{\varphi}_1)(0)\right]^{-1},$$

$$\alpha_{12} = \left[\alpha_5(\widehat{\varphi}_8-\widehat{\varphi}_2)(0)+\alpha_6(\widehat{\varphi}_3-\widehat{\varphi}_7)(0)+\widehat{\varphi}_6(0)\right]^{-1},\alpha_{13} = \left[\alpha_7(\widehat{\varphi}_2-\widehat{\varphi}_8)(0)+\alpha_8(\widehat{\varphi}_3-\widehat{\varphi}_7)(0)\right]\alpha_{12},$$

$$\alpha_i = \frac{\lambda_2}{2(\alpha_1(\widehat{\varphi}_2-\widehat{\varphi}_8)(0)+\alpha_2(\widehat{\varphi}_3-\widehat{\varphi}_7)(0))}+(-1)^i\frac{\lambda_3}{2(\alpha_2(\widehat{\varphi}_2-\widehat{\varphi}_8)(0)+\alpha_1(\widehat{\varphi}_3-\widehat{\varphi}_7)(0))},\quad i=14,17,$$

$$\alpha_i = \frac{\lambda_2\alpha_1}{2(\alpha_1(\widehat{\varphi}_2-\widehat{\varphi}_8)(0)+\alpha_2(\widehat{\varphi}_3-\widehat{\varphi}_7)(0))}+(-1)^{i+1}\frac{\lambda_3\alpha_2}{2(\alpha_2(\widehat{\varphi}_2-\widehat{\varphi}_8)(0)+\alpha_1(\widehat{\varphi}_3-\widehat{\varphi}_7)(0))},\quad i=15,18,$$

$$\alpha_i = \frac{\lambda_2\alpha_2}{2(\alpha_1(\widehat{\varphi}_2-\widehat{\varphi}_8)(0)+\alpha_2(\widehat{\varphi}_3-\widehat{\varphi}_7)(0))}+(-1)^i\frac{\lambda_3\alpha_1}{2(\alpha_2(\widehat{\varphi}_2-\widehat{\varphi}_8)(0)+\alpha_1(\widehat{\varphi}_3-\widehat{\varphi}_7)(0))},\quad i=16,19.$$

In order for our $r'_{ij}(t),\ i,j=\overline{1,3}$ kernels to exist, these relations must be fulfilled:

$$\lambda_i\neq 0,\ a_i\neq 0,\ i=1,2,3,\ \alpha_3(\widehat{\varphi}_1-\widehat{\varphi}_9)(0)+\alpha_4(\widehat{\varphi}_2-\widehat{\varphi}_8)(0)+\alpha_6(\widehat{\varphi}_3-\widehat{\varphi}_7)(0)+(\widehat{\varphi}_4+\widehat{\varphi}_6)(0)\neq 0, \tag{33}$$

$$\alpha_3(\widehat{\varphi}_1-\widehat{\varphi}_9)(0)+\alpha_4(\widehat{\varphi}_2-\widehat{\varphi}_8)(0)+\alpha_6(\widehat{\varphi}_3-\widehat{\varphi}_7)(0)+\widehat{\varphi}_6(0)\neq 0,\ \alpha_2(\widehat{\varphi}_2-\widehat{\varphi}_8)(0)+\alpha_1(\widehat{\varphi}_3-\widehat{\varphi}_7)(0), \tag{34}$$

$$\alpha_5(\widehat{\varphi}_8-\widehat{\varphi}_2)(0)+\alpha_6(\widehat{\varphi}_3-\widehat{\varphi}_7)(0)+\widehat{\varphi}_6(0)\neq 0,\ \alpha_1(\widehat{\varphi}_2-\widehat{\varphi}_8)(0)+\alpha_2(\widehat{\varphi}_3-\widehat{\varphi}_7)(0)\neq 0,\ (\widehat{\varphi}_9-\widehat{\varphi}_1)(0)\neq 0. \tag{35}$$

Equation (27) – (32) contains unknown functions $\frac{\partial\omega_j}{\partial z},\ j=\overline{1,9}$. For them we will receive integral equations from (23) by differentiating them with respect to the variable z. Moreover, we have

$$\frac{\partial}{\partial z}\omega_i(z,t) = \frac{\partial}{\partial z}\omega_i(z_0^i,t_0^i) - \frac{\partial}{\partial z}t_0^i\left[\sum_{k=1}^{9}\widehat{p}_{ik}(z_0^i)\omega_k(z_0^i,t_0^i)+\sum_{k=1}^{9}\widetilde{r}_{ik}(z_0^i,t_0^i)\widehat{\varphi}_i(z_0^i)\right]$$

$$+\int_{t_0^i}^{t}\frac{\partial}{\partial z}\left[\sum_{k=1}^{9}\widehat{p}_{ik}(\xi)\omega_k(\xi,\tau)+\sum_{k=1}^{9}\widetilde{r}_{ik}(\xi,\tau)\widehat{\varphi}_i(\xi)+\int_0^{\tau}\sum_{k=1}^{9}\widetilde{r}_{ik}(\xi,\tau-\alpha)\omega_k(\xi,\alpha)d\alpha\right]\Bigg|_{\xi=\mu_i^{-1}[\tau-t+\mu_i(z)]}d\tau$$

$$+\frac{\partial}{\partial z}t_0^i\int_0^{t_0^i}\sum_{k=1}^{9}\widetilde{r}_{ik}(\xi,t_0^i-\tau)\omega_k(\xi,\tau)\Bigg|_{\xi=\mu_i^{-1}\left[t_0^i-t+\mu_i(z)\right]}d\tau,\quad i=\overline{1,9}. \tag{36}$$

We require the fulfillment of the matching conditions

$$-\lambda_j\frac{\partial\widehat{\varphi}_i(z)}{\partial z}\Bigg|_{z=0}+\sum_{j=1}^{9}\widehat{p}_{ij}\widehat{\varphi}_j(0) = \frac{d}{dt}\widehat{h}_i\Bigg|_{t=0},\quad i=\overline{1,6}. \tag{37}$$

MAIN RESULT AND ITS PROOF

The main result of this work is the following theorem:

Theorem 2: *Let the conditions of Theorem 1 be satisfied, besides function $h(x_1,x_2,t)$ have compact support in x_1, x_2 for each fixed t, $\varphi_i(z) \in C^2[0,\ H]$, $i=\overline{1,9}$, $g_i(t) \in C^2[0,\ H]$, $i=1,2,3,7,8,9$, $h_i(t) \in C^2[0,H]$, $i=\overline{1,6}$, equality (33), (34) and matching condition (37) hold. Then for any $H>0$ on the segment $\left[0,H\right]$ there is a unique solution to the inverse problems* $(10)-(13)$.

Proof. We introduce the following notation for the unknowns:

$$\upsilon_i^1(z,t)=\omega_i(z,t), \quad i=\overline{1,9},\ \upsilon_1^2(t)=r'_{11}(t),\ \upsilon_2^2(t)=r'_{12}(t),\ \upsilon_3^2(t)=r'_{13}(t), \tag{38}$$

$$\upsilon_4^2(t)=r'_{22}(t),\ \ \upsilon_5^2(t)=r'_{23}(t),\ \upsilon_6^2(t)=r'_{33}(t),\ \upsilon_i^3(z,t)=\frac{\partial}{\partial z}\omega_i(z,t), \quad i=\overline{4,6}, \tag{39}$$

$$\upsilon_i^3(z,t)=\frac{\partial}{\partial z}\omega_i(z,t)-\frac{r'_{33}(t_0^i)}{2}\left(\widehat{\varphi}_1(z_0^i)-\widehat{\varphi}_9(z_0^i)\right)\frac{\partial}{\partial z}t_0^i, \quad i=1,9, \tag{40}$$

$$\upsilon_i^3(z,t)=\frac{\partial}{\partial z}\omega_i(z,t)-\frac{r'_{23}(t_0^i)}{2}\left(\widehat{\varphi}_2(z_0^i)-\widehat{\varphi}_8(z_0^i)\right)\frac{\partial}{\partial z}t_0^i, \quad i=2,7, \tag{41}$$

$$\upsilon_i^3(z,t)=\frac{\partial}{\partial z}\omega_i(z,t)-\frac{r'_{13}(t_0^i)}{2}\left(\widehat{\varphi}_3(z_0^i)-\widehat{\varphi}_7(z_0^i)\right)\frac{\partial}{\partial z}t_0^i, \quad i=3,8. \tag{42}$$

The system of equations (23), (27)–(32) and (36) form a complete system of equalities for unknown functions in the domain $D_0 := \left\{(z,t) : 0 \le z \le H,\ 0 \le t \le H\right\}$. According to the introduced notation for the vector function $\upsilon(x,t)=\left(\upsilon_i^1(x,t),\upsilon_j^2(t),\upsilon_i^3(x,t)\right)$, $i=\overline{1,9},\ j=\overline{1,6}$, we write this system in the operator form

$$\upsilon = A\upsilon, \tag{43}$$

where the operator $A=(A_i^1,\ A_j^2,\ A_i^3),\ i=\overline{1,9}, \ \ j=\overline{1,6}$, the components of the operator A are determined by the right-hand sides of equations (23), (27)–(32) and (36), respectively.

Let $C_s(D_0),\ (s\ge 0)$ be the Banach space of continuous functions with the ordinary norm, denoted by $\|\cdot\|_s$,

$$\|\upsilon\|_s=\max\left\{\max_{1\le i\le 9,\ (z,t)\in D_0}\left|\upsilon_i^1(z,t)e^{-st}\right|,\right.$$

$$\left.\max_{1\le i\le 6,\ t\in\left[0,H\right]}\left|\upsilon_i^2(t)e^{-st}\right|,\ \max_{1\le i\le 9,\ (z,t)\in D_0}\left|\upsilon_i^3(z,t)e^{-st}\right|\right\}.$$

Obviously, C_s with $s=0$ is the usual space of continuous functions with the ordinary norm, denoted by $\|\cdot\|$ in what follows, because

$$e^{-sH}\|\upsilon\| \le \|\upsilon\|_s \le \|\upsilon\|.$$

The norms $\|\upsilon\|_s$ and $\|\upsilon\|$ are equivalent for any $H\in(0,\infty)$, where $s\in(0,1)$ and we choose that number later.

Next, consider the set of functions $S(\upsilon^0,r)\subset C_s(D_0)$, satisfying the inequality

$$\left\|\upsilon-\upsilon^0\right\|_s \le r, \tag{44}$$

where r is a known number, the vector function $\upsilon^0(z,t)=\left(\upsilon_i^{01}(z,t),\ \ i=\overline{1,9}, \ \ \upsilon_i^{02}(t),\ i=\overline{1,6}, \ \ \upsilon_i^{03}(z,t),\ i=\overline{1,9}\right)$, defined by the free terms of the operator equation (43). It is easy to see that for $\upsilon\in S(\upsilon^0,r)$ the estimate $\|\upsilon\|_s \le \|\upsilon^0\|_s+r\le\|\upsilon^0\|+r:=r_0$. Thus, r_0 is known.

Note that the operator A maps the space $C_s(D_0)$ into itself. Let us show that for a suitable choice of s (recall that $H > 0$ is an arbitrary fixed number) it is on the set $S(\upsilon^0, r)$ a contraction operator. First, let us make sure that the operator A takes the set $S(\upsilon^0, r)$ into itself, that is, it follows from the condition $\upsilon(z,t) \in S(\upsilon^0, r)$ that $A\upsilon \in S(\upsilon^0, r)$, if s satisfies some constraints. In fact, for any $(z,t) \in D_0$ and $\upsilon \in S(\upsilon^0, r)$ the following inequalities hold:

$$\left\|A\upsilon - \upsilon^0\right\|_s \leq \frac{r_0}{s}\beta_i, \;\; i = \overline{1,24},$$

where, $\varphi_0 := \max_{i=\overline{1,9}} \left\|\widehat{\varphi}_i\right\|_{C^2[0,H]}, \;\; g_0 := \max_{i=1,2,3,7,8,9} \left\|\widehat{g}_i\right\|_{C^2[0,H]}, \;\; h_0 := \max_{i=\overline{1,6}} \left\|h_i\right\|_{C^2[0,H]}, \alpha_0 := \max_{i=\overline{1,19}} \left\|\alpha_i\right\|_{C[0,H]},$

$$p_0 := \max_{i,j=\overline{1,9}} \left\|\widehat{p}_{ij}\right\|_{C^1[0,H]}, \; M_0 = \max\left\{\alpha_0; \left\|\frac{1}{2\lambda_i}\right\|; \left\|\frac{\partial r_1^i}{\partial z}\right\|\right\}$$

and

$$\beta_i := 9M_0 + 9\varphi_0 + 9M_0r_0, \; i = \overline{1,9}, \; \beta_{10} := 5M_0^2(g_0+h_0) + M_0^3(g_0+h_0) + 18M_0^3 + 2M_0^3\varphi_0 + 2M_0^2r_0,$$
$$\beta_{11} := 5M_0^2(g_0+h_0) + 18M_0^3 + 2M_0^3\varphi_0 + M_0^3(g_0+h_0) + 2r_0,$$
$$\beta_{12} := M_0^2(g_0+h_0) + 18M_0^2 + 2M_0^2\varphi_0 + 2M_0r_0, \; \beta_{13} := 18M_0^3 + 2M_0^2(h_0+g_0) + 2M_0^2 + 12M_0^2\varphi_0 + 8M_0^2r_0,$$
$$\beta_i := 18M_0^2 + 16M_0\varphi_0 + 8M_0r_0, \;\; i = 14,15, \; \beta_i := 18M_0 + 18M_0\varphi_0 + 18M_0r_0, \; i = \overline{16,24}.$$

Choosing $s > (1/r)\beta_0$, $\left(\beta_0 = \max\left\{\beta_i, \; i = \overline{1,24}\right\}\right)$ we get that the operator A maps the set $S\left(\upsilon^0, r\right)$ into itself.

Now, let υ and $\widetilde{\upsilon}$ be two arbitrary elements in $S(\upsilon^0, r)$. Using the obvious inequality

$$\left|\upsilon_i^k\upsilon_i^l - \widetilde{\upsilon}_i^k\widetilde{\upsilon}_i^l\right|e^{-st} \leq \left|\upsilon_i^k - \widetilde{\upsilon}_i^k\right|\left|\upsilon_i^l\right|e^{-st} + \left|\widetilde{\upsilon}_i^k\right|\left|\upsilon_i^l - \widetilde{\upsilon}_i^l\right|e^{-st} \leq 2r_0\|\upsilon - \widetilde{\upsilon}\|_s, \quad (z,t) \in D_0,$$

after some easy estimations, we find that for $(z,t) \in D_0$,

$$\|A\upsilon - A\widetilde{\upsilon}\|_s \leq \frac{\|\upsilon - \widetilde{\upsilon}\|}{s}\gamma_i, \quad i = \overline{1,24},$$

where,

$$\gamma_i := 9M_0 + 9\varphi_0 + 18M_0r_0, \; i = \overline{1,9}, \; \gamma_{10} := 5M_0^2(g_0+h_0) + M_0^3(g_0+h_0) + 18M_0^3 + 2M_0^3\varphi_0 + 4M_0^2r_0,$$
$$\gamma_{11} := 5M_0^2(g_0+h_0) + 18M_0^3 + 2M_0^3\varphi_0 + M_0^3(g_0+h_0) + 4r_0,$$
$$\gamma_{12} := M_0^2(g_0+h_0) + 18M_0^2 + 2M_0^2\varphi_0 + 8M_0r_0, \; \gamma_{13} := 18M_0^3 + 2M_0^2(h_0+g_0) + 2M_0^2 + 12M_0^2\varphi_0 + 16M_0^2r_0,$$
$$\gamma_i := 18M_0^2 + 16M_0\varphi_0 + 16M_0r_0, \;\; i = 14,15, \; \gamma_i := 18M_0 + 18M_0\varphi_0 + 54M_0r_0, \; i = \overline{16,24}.$$

Choosing now $s > \gamma_0$, $\left(\gamma_0 = \max\left\{\gamma_i, \; i = \overline{1,24}\right\}\right)$ we get, that the operator A compresses the distance between the elements $\upsilon, \widetilde{\upsilon}$ to $S\left(\upsilon^0, r\right)$.

As follows from the performed estimates, if the number s is chosen from conditions $s > s^* := max\{\beta_0, \gamma_0\}$, then the operator A is contracting on $S\left(\upsilon^0, r\right)$. In this case, according to the Banach principle [40] equation (43) has the only solution in $S\left(\upsilon^0, r\right)$ for any fixed $H > 0$. Theorem 2 is proved.

By the found functions $r'_{ij}(t), \; i,j = \overline{1,3}$ the functions $r_{ij}(t), \; i,j = \overline{1,3}$ are found by the formulas

$$r_{ij}(t) = r_{ij}(0) + \int_0^t r'_{ij}(\tau)d\tau, \quad i,j = \overline{1,3}.$$

Note that by the function $r_{ij}(t), \; i,j = \overline{1,3}$ the functions $K_{ij}(t), \; i,j = \overline{1,3}$ are defined as solutions of integral equations (4).

CONCLUSION

In this work, inverse problem was considered for determining the kernel $R(t)$ included in the equation (10) with by using additional condition (13) of the solution of problem with the initial and boundary conditions (11), (12). Sufficient conditions for given functions are obtained, under which the inverse problem has unique solutions for a sufficiently small interval.

REFERENCES

1. V. Romanov, *Inverse problems of mathematical physics* (Utrecht, The Netherlands, 1987).
2. V. Yakhno, *Inverse problems for differential equations of elasticity* (Springer, Berlin, 1988).
3. A. Lorenzi, "An identification problem related to a nonlinear hyperbolic integrodifferential equation," Nonlinear Anal., Theory, Methods Appl. **22:1**, 21–44 (1994).
4. V. Isakov, *Inverse Problems for Partial Differential Equations* (Novosibirsk: Nauka, (in Russian), 1998).
5. M. Klibanov, "Carleman estimates and inverse problems in the last two decades," Surveys on Solutions Methods for Inverse Problems , 119–149 (2000).
6. V. Romanov, "Inverse problems for differential equations with memory," Eurasian Journal of Mathematical and Computer Applications **2:4**, 51–80 (2014).
7. A. Lorenzi and V. Romanov, "Stability estimates for an inverse problem related to viscoelastic media," J. Inv. Ill - Posed Problems **14:1**, 360–370 (2006).
8. J. Janno and V. Wolfersdorf, "An inverse problem for identification of a time- and space-dependent memory kernel in viscoelasticity," Inverse Problems **17:1**, 13–24 (2001).
9. V. Romanov, "Stability estimates for the solution in the problem of determining the kernel of the viscoelasticity equation," Journal of Applied and Industrial Mathematics **6:3**, 360–370 (2012).
10. V. Romanov, "A two-dimensional inverse problem for the viscoelasticity equation," Siberian Math. J. **53:6**, 1128 (2012).
11. D. Durdiev and Zh. Totieva, "The problem of determining the one-dimensional kernel of viscoelasticity equation with a source of explosive type," Journal of Inverse and Ill-Posed Problems **28:1**, 43–52 (2020).
12. U. Durdiev, "An inverse problem for the system of viscoelasticity equations in homogeneous anisotropic media," Journal of Applied and Industrial Mathematics **13:4**, 623–628 (2019).
13. D. Durdiev and A. Rakhmonov, "Inverse problem for a system of integro-differential equations for sh waves in a visco-elastic porous medium: Global solvability," Theoret. and Math. Phys. **195:3**, 923–937 (2018).
14. D. Durdiev and Zh. Totieva, "The problem of determining the one-dimensional matrix kernel of the system of viscoelasticity equations," Mathematical Methods in the Applied Sciences **41:17**, 8019–8032 (2018).
15. Zh. Totieva and D. Durdiev, "The problem of finding the one-dimensional kernel of the thermoviscoelasticity equation," Mathematical Methods in the Applied Sciences **103:1-2**, 118–132 (2018).
16. D. Durdiev and Zh. Zhumaev, "One-dimensional inverse problems of finding the kernel of the integro-differential heat equation in a bounded domain," Ukrainskyi Matematychnyi Zhurnal **73**, 1492–1506 (2021).
17. D. Durdiev and Zh. Zhumaev, "Memory kernel reconstruction problems in the integro-differential equation of rigid heat conductor," Mathematical Methods in the Applied Sciences **45**, 8374–8388 (2022).
18. U. Durdiev and Zh. Totieva, "A problem of determining a special spatial part of 3d memory kernel in an integro-differential hyperbolic equation," Math. Methods Appl. Scie. **42**, 7440–7451 (2019).
19. D. Durdiev and Zh. Zhumaev, "Problem of determining a multidimensional thermal memory in a heat conductivity equation," Methods of Functional Analysis and Topology **25**, 219–226 (2019).
20. D. Durdiev and Zh. Zhumaev, "Problem of determining the thermal memory of a conducting medium," Differential Equations **56**, 785–796 (2020).
21. V. Romanov, "Problem of kernel recovering for the viscoelasticity equation," Doklady Mathematics **86:2**, 608–610 (2012).
22. D. Durdiev and A. Rahmonov, "A 2d kernel determination problem in a visco-elastic porous medium with a weakly horizontally inhomogeneity," Mathematical Methods in the Applied Sciences **43:15**, 8776–8796 (2020).
23. D. Durdiev and A. Rakhmonov, "The problem of determining the 2d-kernel in a system of integro-differential equations of a viscoelastic porous medium," J. Appl. Industr. Math. **14:2**, 281–295 (2020).
24. D. Durdiev and Zh. Safarov, "Inverse problem of determining the one-dimensional kernel of the viscoelasticity equation in a bounded domain," Math. Notes **97:6**, 867–877 (2015).
25. U. Durdiev, "Problem of determining the reaction coefficient in a fractional diffusion equation," Differential Equations, **59**, 1195–1204 (2021).
26. U. Durdiev, "Inverse problem of determining an unknown coefficient in the beam vibration equation," Differential Equations **58**, 36–43 (2022).
27. Zh. Safarov and D. Durdiev, "Inverse problem for an integro-differential equation of acoustics," Differential Equations **54:1**, 134–142 (2018).
28. Zh. Totieva, "The problem of determining the matrix kernel of the anisotropic viscoelasticity equations system," Vladikavkaz Math. Jour. **21:2**, 58–66 (2019).
29. D. Durdiev and Kh. Turdiev, "The problem of finding the kernels in the system of integro-differential maxwell's equations," J. Appl. Industr. Math. **15:2**, 190–211 (2021).
30. A. Boltaev and D. Durdiev, "Inverse problem for viscoelastic system in a vertically layered medium," Vladikavkaz Math. Jour **24**, 30–47 (2022).
31. D. Durdiev and Kh. Turdiev, "Inverse problem for a first-order hyperbolic system with memory," Differential Equations **56**, 1634–1643 (2020).
32. E. Delesan and D. Ruaye, *Elastic waves in solids* (Moscow:Nauka, 1982).
33. Mura. Toshio, *Micromechanics of defects in solids, Second, Revised Edition* (IL, USA, Northwestern University, Evanston, 1987).
34. L. Landau and E. Lifshitz, *Electrodynamics of continuous media* (New York, Pergamon Press, 1984).
35. L. Galin, *Contact problems of the theory of elasticity and viscoelasticity* (Moscow:Nauka, 1980).
36. Zh. Totieva, "Linearized two-dimensional inverse problem of determining the kernel of the viscoelasticity equation," Vladikavkaz Math. Jour. **23:2**, 87–103 (2021).
37. S. Godunov, *Equations of Mathematical Physics* (Moscow:Nauka, 1979).
38. A. Kilbas, *Integral Equations: Course of Lectures* (Minsk, Belarusian State University, 2005).
39. A. Bukhgeym, *Volterra equations and inverse problems* (DeGruyter, 1999).
40. A. Kolmogorov and S. Fomin, *Elements of the Theory of Functions and Functional Analysis* (Moscow:Nauka, 1989).