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Determining of a Space Dependent Coefficient of Fractional Diffusion Equation with the Generalized Riemann–Liouville Time Derivative

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Abstract—This work investigates an initial-boundary value and an inverse coefficient problem of determining a space dependent coefficient in the fractional wave equation with the generalized Riemann–Liouville (Hilfer) time derivative. In the beginning, it is considered the initial boundary value problem (direct problem). By the Fourier method, this problem is reduced to equivalent integral equations, which contain Mittag-Leffler type functions in free terms and kernels. Then, using the technique of estimating these functions and the generalized Gronwall inequality, we get a priori estimate for solution via unknown coefficient which will be used to study the inverse problem. The inverse problem is reduced to the equivalent integral equation of Volterra type. To show existence unique solution to this equation the Schauder principle is applied. The local existence and uniqueness results are obtained.

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1. INTRODUCTION

The theory of fractional differential equations and fractional partial differential equations have recently received significant attention in various fields science. The reason is that it can be used to solve practical problems from the fields of science and engineering, such as physics, chemistry, electrodynamics of complex media, polymer rheology and so on ([1–7]). Recently the research of fractional differential equations has made great progress. In the literature, there are several definitions of fractional integrals and derivatives, the most popular definitions are in the sense of the Riemann–Liouville and Caputo derivatives. Hilfer introduced a generalized Riemann–Liouville fractional derivative, the socalled Hilfer fractional derivative. Many authors studied the existence of solutions for fractional differential equations involving Hilfer fractional derivative (see [1, 6], and [8–11]).

Inverse problems for integer and fractional partial differential equations is a rapidly developing area of mathematics. If the classical formulations of boundary value problems for these equations have already been sufficiently well studied and the conditions for their solvability have been obtained, then the situation with inverse problems is more complicated. Often, even the very formulation of such problems requires additional research, including studies of the differential properties of solutions to direct problems. This is especially pronounced in nonlinear problems, such as coefficient inverse problems or problems of determining the kernel in integro-differential equations with an integral convolution operator.

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It was shown in [12] and [13] that the integro-differential equations of heat conduction and the wave equation with the kernel in the integral term of the form of Mittag-Leffler functions are equivalently reduced to fractional diffusion and wave equations with a fractional derivative of order $\alpha \in (0,1]$ in the first and $\alpha \in (1,2]$ in second cases, in the sense of Caputo fractional derivative. Such equations also describe a wide class of diffusion-wave processes occurring in highly porous media with complex components.

Inverse problems of determining the convolution kernel in integro-differential equations, the main part of which coincides with a hyperbolic operator, were studied in [14–20]. In these studies, the authors discussed the unique solvability and stability estimates of the solution, as well as a numerical approach for solving such problems. A wide class of direct and inverse problems of determining the kernel in integro-differential hyperbolic equations with lumped sources of perturbations was studied in the recently published monograph [20] (see also the bibliography there).

Inverse problems for fractional differential wave and diffusion equations have not yet been deeply investigated. In the literature, an order of the time-fractional derivative problems [21–23], source determination linear problems [24–30] and coefficient nonlinear inverse problems [31–40] in the initial boundary value problem for fractional diffusion- wave equations with various types of overdetermination conditions are encountered most often (see also the references in all listed works). The papers [41] and [42] study inverse problems of finding space dependent and time-dependent source terms, respectively, in time-fractional diffusion equation by using eigenfunction expansion of the non-self adjoint spectral problem along the generalized Fourier method. The main results of these studies comprise the existence and uniqueness theorems, as well as a stability estimate for the solution of the problem of determining the coefficient in a time-fractional diffusion and wave equation.

In the paper [43], the inverse problem of determining the time-dependent reaction-diffusion coefficient in the Cauchy problem for the time-fractional diffusion equation with the generalized Riemann–Liouville (Hilfer) operator by a single observation at the point $x = 0$ of the diffusion process was studied. The local existence and global uniqueness results and the conditional stability estimate of solution are proven.

In the domain $\Omega_T = \{(x,t) : 0 < x < l,\ 0 < t \leq T\}$, we consider the problem of determining a function $v(x,t)$ satisfying the equation

$$\left(D^{\alpha,\beta}_{0+,t} u\right)(x,t) - u_{xx} + q(x)u(x,t) = f(x,t), \tag{1}$$

the initial conditions of Cauchy type

$$I^{(1-\alpha)(1-\beta)}_{0+,t} u(x,t)\big|_{t=0} = \varphi(x), \quad x \in [0,l], \tag{2}$$

and the boundary conditions

$$u(0,t) = u(l,t) = 0, \quad 0 \leq t \leq T. \tag{3}$$

Here the generalized Riemann–Liouville (Hilfer) fractional differential operator $D^{\alpha,\beta}_{0+,t}$ of the order $0 < \alpha < 1$ and type $0 \leq \beta \leq 1$ is defined as follows [1, pp. 112–118], [2, pp. 62–65]:

$$D^{\alpha,\beta}_{0+,t} v(\cdot,t) = \left(I^{\beta(1-\alpha)}_{0+,t} \frac{\partial}{\partial t}\left(I^{(1-\beta)(1-\alpha)}_{0+,t} v\right)\right)(\cdot,t),$$

where

$$I^{\sigma}_{0+,t} v(\cdot,t) = \frac{1}{\Gamma(\sigma)} \int_0^t \frac{v(\cdot,\tau)}{(t-\tau)^{1-\sigma}} d\tau, \quad \sigma \in (0,1)$$

is the Riemann–Liouville fractional integral of the function $v(x,t)$ with respect to t [3, pp. 69–72], $\Gamma(\cdot)$ is the Euler's Gamma function.

In [1, pp. 112–118] and [5, pp. 28–37], by R. Hilfer was introduced a generalized form of the Riemann–Liouville fractional derivative of order α and a type $\beta \in [0,1]$, which coincides with the Riemann–Liouville fractional derivative at $\beta = 0$ and with Caputo fractional derivative at $\beta = 1$, and the case $\beta \in (0,1)$ interpolates these both fractional derivatives.

For the given functions $f(x,t)$, $\varphi(x)$, $q(x)$ and numbers $\alpha \in (0,1)$, $\beta \in [0,1]$, the problem of determining the solution to the initial-boundary value problem (1)–(3) we call as the direct problem.

Inverse problem. It is required to determine the function $q(x)$, $x \in [0,l]$ if the nonlocal overdetermination condition is given

$$\int_0^T w(t)u(x,t)dt = h(x), \quad x \in [0,l], \tag{4}$$

where $w(t)$ and $h(x)$ are known functions.

Let $u(x,t)$ be a classical solution to the initial boundary value problem (1)–(3) and $f(x,t)$, $\varphi(x)$ be enough smooth functions. We carry out the next converting of the direct problem (1)–(3).

We consider the weighted spaces of continuous functions [3, pp. 4–5, 162–163]

$$C_\gamma[a,b] := \{g : (a,b) \to R : (t-a)^\gamma g(t) \in C[a,b],\ 0 \le \gamma < 1\},$$

$$C_\gamma^{\alpha,\beta}(\Omega) = \left\{\nu(t) :\ D_{0+,t}^{\alpha,\beta}\nu(t) \in C_\gamma(0,T],\ 0 < \alpha \le 1,\ 0 \le \beta \le 1\right\},$$

$$C_\gamma^{k,\alpha,\beta}(\Omega) = \left\{\mu(x,t) :\ \mu(x,\cdot) \in C^k(0,l);\ t \in [0,T]\right\}$$

and

$$D_{0+,t}^{\alpha,\beta}\mu(\cdot,t) \in C_\gamma(0,T]; \quad x \in [0,l], \quad 0 < \alpha \le 1, \quad 0 \le \beta \le 1\Big\}, \quad k = 0,1,2, \quad C_\gamma^0[a,b] = C_\gamma[a,b],$$

with the norms

$$||f||_{C_\gamma} = ||(t-a)^\gamma f(t)||_C, \quad ||f||_{C_\gamma^n} = \sum_{k=0}^{n-1} ||f^{(k)}||_C + ||f^{(n)}||_{C_\gamma}.$$

Assume that throughout this article, given functions φ_1, φ_2, f, w, and h satisfy the following assumptions:

A1) $\varphi \in C^3[0,l]$, $\varphi^{(4)} \in L_2[0,l]$, $\varphi(0) = \varphi(l) = 0$, $\varphi''(0) = \varphi''(l) = 0$;

A2) $D_{0+,t}^{\alpha,\beta} f(\cdot,t) \in C_\gamma(0,T]$, $f(\cdot,t) \in C^3[0,l]$, $f_{xxxx}(x,t) \in L_2[0,l]$, $f(0,t) = f(l,t) = 0$, $f_{xx}(0,t) = f_{xx}(l,t) = 0$;

A3) $w(t) \in C[0,l]$;

A4) $h(x) \in C^2[0,l]$, $|h(x)| \ge h_0 > 0$, h_0 is a given number;

In the next section, we provide some necessary definitions and well-known assertions.

2. PRELIMINARIES

In this section, we present some useful definitions and results of fractional calculus.

Two parameter Mittag-Leffler function. The two parameter Mittag-Leffler function $E_{\alpha,\beta}(z)$ is defined by the following series

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

where $\alpha, \beta, z \in \mathbb{C}$ with $\Re(\alpha) > 0$, $\Re(\alpha)$ denotes the real part of the complex number α. The Mittag-Leffler function has been studied by many authors who have proposed and studied various generalizations and applications.

Proposition 1. *Let $0 < \alpha < 2$ and $\beta \in \mathbb{R}$ be arbitrary. We suppose that κ is such that $\pi\alpha/2 < \kappa < \min\{\pi, \pi\alpha\}$. Then, there exists a constant $C = C(\alpha,\beta,\kappa) > 0$ such that*

$$|E_{\alpha,\beta}(z)| \le \frac{C}{1+|z|}, \quad \kappa \le |\arg(z)| \le \pi.$$

For the proof, we refer to [3, pp. 40—45], for example.

Lemma 1 [44, p. 189]. *Suppose $b \geq 0$, $\alpha > 0$ and $a(t)$ is nonnegative function locally integrable on $0 \leq t < T$ (some $T \leq +\infty$) and suppose $u(t)$ is nonnegative and locally integrable on $0 \leq t < T$ with*

$$u(t) \leq a(t) + b \int_0^t (t-s)^{\alpha-1} u(s) ds,$$

then

$$u(t) \leq a(t) + b\Gamma(\alpha) \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha} \left(b\Gamma(\alpha)(t-s)^{\alpha}\right) a(s) ds.$$

Lemma 2 [44, p. 189]. *Suppose $b \geq 0$, $\alpha > 0$, $\gamma > 0$, $\alpha + \gamma > 1$ and $a(t)$ is nonnegative function locally integrable on $0 \leq t < T$ and suppose $t^{\gamma-1}u(t)$ is nonnegative and locally integrable on $0 \leq t < T$ with*

$$u(t) \leq a(t) + b \int_0^t (t-s)^{\alpha-1} s^{\gamma-1} u(s) ds,$$

then

$$u(t) \leq a(t) E_{\alpha,\gamma} \left((b\Gamma(\alpha))^{\frac{1}{\alpha+\gamma-1}}\, t \right),$$

where

$$E_{\alpha,\gamma}(t) = \sum_{m=0}^{\infty} c_m t^{m(\alpha+\gamma-1)}, \quad c_0 = 1, \quad \frac{c_{m+1}}{c_m} = \frac{\Gamma(m(\alpha+\gamma-1)+\gamma)}{\Gamma(m(\alpha+\gamma-1)+\alpha+\gamma)}$$

for $m \geq 0$. As $t \to +\infty$ $E_{\alpha,\gamma}(t) = O\left(t^{\frac{1}{2}\frac{\alpha+\gamma-1}{\alpha-\gamma}} \exp\left(\frac{\alpha+\gamma-1}{\alpha} t^{\frac{\alpha+\gamma-1}{\alpha}} \right) \right)$.

3. EXISTENCE AND UNIQUENESS RESULTS FOR DIRECT PROBLEM SOLUTION

In equation (1), transferring the term $q(x)$, $u(x,t)$ to the right side, we introduce the designation $F(x,t) = f(x,t) - q(x)u(x,t)$. Then, we get

$$\left(D_{0+,t}^{\alpha,\beta} u\right)(x,t) - u_{xx} = F(x,t). \tag{5}$$

By applying the Fourier method, the solution $u(x,t)$ of the problem (2), (3) and (5) can be expanded in a uniformly convergent series in term of eigenfunctions of the form

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x) u_n(t), \tag{6}$$

where

$$u_n(t) = \int_0^l u(x,t) X_n(x) dx, \quad X_n(x) = \sqrt{\frac{2}{l}} \sin(\lambda_n x), \quad \lambda_n = \frac{\pi n}{l}, \quad n = 1, 2, \ldots. \tag{7}$$

The coefficients $u_n(t)$ for $n \geq 1$ are to be found by making use of the orthogonality of the eigenfunctions $X_n(x)$. Recall that the scalar product in $L_2[0,l]$ is defined by $(f,g) = \int_0^l f(x)g(x)dx$. Let us note the express coefficients of $F(x,t)$ and $\psi(x)$ in the eigenfunctions (7) for $n \geq 1$ respectively by

$$(F(x,t), X_n(x)) = F_n(t), \quad (\varphi(x), X_n(x)) = \varphi_n.$$

In view of (2) and (5), we obtain for each n Cauchy type problems

$$\begin{cases} \left(D^{\alpha,\beta}_{0+,t}u_n\right)(t) + \lambda_n^2 u_n = F_n(t), \\ I^{(1-\alpha)(1-\beta)}_{0+,t} u_n(t)\big|_{t=0} = \varphi_n. \end{cases} \tag{8}$$

Based on [6, pp. 61–114], we find that solutions of (8) are given by formulas

$$u_n(t) = t^{(\beta-1)(1-\alpha)} E_{\alpha,1+(\beta-1)(1-\alpha)}\left(-\lambda_n^2 t^\alpha\right)\varphi_n + \int\limits_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n^2(t-\tau)^\alpha) F_n(\tau)d\tau. \tag{9}$$

Substituting (9) into (6), taking into account designation $F(x,t) = f(x,t) - q(x)u(x,t)$, we get an integral equation which is equivalent to problem (2), (3), and (5)

$$u(x,t) = \sum_{n=1}^{\infty}\left[t^{(\beta-1)(1-\alpha)} E_{\alpha,1+(\beta-1)(1-\alpha)}\left(-\lambda_n^2 t^\alpha\right)\varphi_n \right.$$
$$\left. + \int\limits_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n^2(t-\tau)^\alpha) f_n(\tau)d\tau \right] X_n(x)$$
$$- \int\limits_0^t \int\limits_0^l \sum_{n=1}^{\infty} E_{\alpha,\alpha}(-\lambda_n^2(t-\tau)^\alpha) X_n(\xi) X_n(x) q(\xi) u(\xi,\tau) d\xi d\tau. \tag{10}$$

We write the integral equation (10) in the following form

$$u(x,t) = \Phi(x,t) - \int\limits_0^t \int\limits_0^l G_{\alpha,\alpha}(x,\xi,t-\tau) q(\xi) u(\xi,\tau) d\xi d\tau, \tag{11}$$

where

$$\Phi(x,t) = \int\limits_0^l G_{\alpha,1+(\beta-1)(1-\alpha)}(x,\xi,t)\varphi(\xi)d\xi + \int\limits_0^t \int\limits_0^l G_{\alpha,\alpha}(x,\xi,t-\tau) f(\xi,\tau) d\xi d\tau,$$
$$G_{\alpha,\lambda}(x,\xi,t-\tau) = \sum_{n=1}^{\infty} (t-\tau)^{\lambda-1} E_{\alpha,\lambda}(-\lambda_n^2(t-\tau)^\alpha) X_n(\xi) X_n(x), \tag{12}$$

$G_{\alpha,\lambda}(x,\xi,t-\tau)$ is the Green's function of the first initial-boundary problem for equation $\left(D^{\alpha,\beta}_{0+,t}v\right)(x,t) - v_{xx}(x,t) = 0$.

It is true the following assertion.

Theorem 1. *Let $q(x) \in C[0,l]$, A1) and A2) be satisfied, then there exists a unique solution of the integral equation (11) such that $u(x,t) \in C^{2,\alpha,\beta}_{\gamma}(\Omega_T)$.*

Proof. To show the existence of unique solution to equation (11) we use the method of successive approximations, representing the solution in the form

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t), \tag{13}$$

where

$$u_0(x,t) = \Phi(x,t),$$

$$u_k(x,t) = -\int_0^t\int_0^l G_{\alpha,\alpha}(x,\xi,t-\tau)q(\xi)u_{k-1}(\xi,\tau)d\xi d\tau. \tag{14}$$

Let

$$||q||_{C[0,l]} = \max_{x\in[0,l]} |q(x)|, \quad f_0 = ||t^\gamma f(x,t)||_{C^{2,0}(\overline{\Omega}_T)}, \quad \varphi_0 = ||\varphi||_{C^2[0,l]},$$

where γ satisfies condition $(1-\beta)(1-\alpha) < \gamma < 1$. This solution (14) is bounded in $C_\gamma^{2,\alpha,\beta}(\Omega_T)$ in view of A1)–A2). Multiplying $u_0(x,t)$ by t^γ and estimating, we get

$$t^\gamma|u_0(x,t)| = t^\gamma \left| \int_0^l G_{\alpha,1+(\beta-1)(1-\alpha)}(x,\xi,t)\varphi(\xi)d\xi + \int_0^t\int_0^l G_{\alpha,\alpha}(x,\xi,t-\tau)f(\xi,\tau)d\xi d\tau \right|$$
$$\le T^{\gamma+(\beta-1)(1-\alpha)}\varphi_0 + C_1 f_0 T^\alpha B(\alpha,1-\gamma),$$

where $C_1 = const$. Similarly way from (14) for $k = 1,2$, we obtain

$$t^\gamma|u_1(x,t)| = t^\gamma \left| \int_0^t\int_0^l G_\alpha(x,\xi,t-\tau)q(\xi)u_0(\xi,\tau)d\xi d\tau \right|$$
$$\le C_1\left(T^{\gamma+(\beta-1)(1-\alpha)}\varphi_0 + C_1 f_0 T^\alpha B(\alpha,1-\gamma)\right) t^\gamma \left| \int_0^t (t-\tau)^{\alpha-1}\tau^{-\gamma}d\tau \right|$$
$$\le \left(T^{\gamma+(\beta-1)(1-\alpha)}\varphi_0 + C_1 f_0 T^\alpha B(\alpha,1-\gamma)\right) \frac{C_1\Gamma(\alpha)\Gamma(1-\gamma)}{\Gamma(\alpha+1-\gamma)} t^\alpha,$$

$$t^\gamma|u_2(x,t)| = t^\gamma \left| -\int_0^t\int_0^l G_\alpha(x,\xi,t-\tau)q(\xi)u_1(\xi,\tau)d\xi d\tau \right|$$
$$\le \left(T^{\gamma+(\beta-1)(1-\alpha)}\varphi_0 + C_1 f_0 T^\alpha B(\alpha,1-\gamma)\right) \frac{C_1^2||q||^2_{C[0,l]}\Gamma^2(\alpha)\Gamma(1-\gamma)}{\Gamma(2\alpha+1-\gamma)} t^{2\alpha}.$$

Thus, we have

$$t^\gamma|u_k(x,t)| = t^\gamma \left| -\int_0^t\int_0^l G(x,\xi,t-\tau)q(\xi)u_{k-1}(\xi,\tau)d\xi d\tau \right|$$
$$\le \left(T^{\gamma+(\beta-1)(1-\alpha)}\varphi_0 + C_1 f_0 T^\alpha B(\alpha,1-\gamma)\right) \frac{C_1^k||q||^k_{C[0,l]}\Gamma^k(\alpha)\Gamma(1-\gamma)}{\Gamma(k\alpha+1-\gamma)} t^{k\alpha}.$$

It follows from the above estimates that the series $t^\gamma u(x,t) = \sum_{k=0}^\infty t^\gamma u_k(x,t)$, converges uniformly in $\overline{\Omega}_T$, since it can be majorized in $\overline{\Omega}_T$ by the convergent numerical series

$$\left(T^{\gamma+(\beta-1)(1-\alpha)}\varphi_0 + C_1 f_0 T^\alpha B(\alpha,1-\gamma)\right)\Gamma(1-\gamma)\sum_{k=0}^\infty \left(||q||_{C[0,l]}\Gamma(\alpha)\right)^k \frac{T^{k\alpha}}{\Gamma(k\alpha+1-\gamma)}.$$

This means the following estimate for the solution of the integral equation (14) takes place

$$t^\gamma|u(x,t)| \le \left(T^{\gamma+(\beta-1)(1-\alpha)}\varphi_0 + C_1 f_0 T^\alpha B(\alpha,1-\gamma)\right)\Gamma(1-\gamma)$$
$$\times \sum_{k=0}^\infty \left(||q||_{C[0,l]}\Gamma(\alpha)\right)^k \frac{T^{k\alpha}}{\Gamma(k\alpha+1-\gamma)}$$

$$= \left(T^{\gamma+(\beta-1)(1-\alpha)}\varphi_0 + C_1 f_0 T^\alpha B(\alpha, 1-\gamma)\right)\Gamma(1-\gamma)\, E_{\alpha,1-\gamma}\left(C_1 ||q||_{C[0,l]}\Gamma(\alpha)T^\alpha\right), \qquad (15)$$

where $E_{\alpha,\gamma}(\cdot)$ is the Mittag-Leffler function of a nonnegative real argument defined in Preliminaries.

Since, when equation (11) has a unique solution, $\Phi(x,t)$ and $G_{\alpha,\alpha}(x,\xi,t-\tau)$ are continuous functions of their arguments by the conditions of the theorem. According to the general theory of integral equations, this implies that the same property will be possessed the function $t^\gamma u(x,t)$ in $\overline{\Omega}_T$. As usual, this function is a solution of integral equation (11).

Lemma 3. *If the conditions A1)−A2) are fulfilled, then there are equalities*

$$\varphi_n = \frac{1}{\lambda_n^4}\varphi_n^{(4)}, \quad f_n(t) = \frac{1}{\lambda_n^4} f_n^{(4)}(t), \qquad (16)$$

where

$$\varphi_n^{(4)} = \sqrt{\frac{2}{l}}\int_0^l \varphi^{(4)}(x)\cos(\lambda_n x)dx, \quad f_n^{(4)}(t) = \sqrt{\frac{2}{l}}\int_0^l f_{xxxx}^{(4)}(x,t)\cos(\lambda_n x)dx,$$

with the following estimates

$$\sum_{n=1}^{\infty}|\varphi_{n,i}^{(4)}|^2 = ||\varphi_i^{(4)}||_{L_2[0,l]}, \quad \sum_{n=1}^{\infty}|f_n^{(4)}(t)|^2 = ||f^{(4)}(t)||_{L_2[0,l]\times C[0,T]}. \qquad (17)$$

Let us derive an estimate for the norm of the difference between the solution of the original integral equation (11) and the solution of this equation with perturbed functions $\widetilde{q}$, $\widetilde{\psi}_n$, and $\widetilde{f}_n$. Let $\widetilde{u}(x,t)$ be solution of the integral equation (11) corresponding to the functions $\widetilde{q}$, $\widetilde{\varphi}_n$, and $\widetilde{f}_n$; i.e.,

$$\widetilde{u}(x,t) = \widetilde{\Phi}(x,t) - \int_0^t\int_0^l G_{\alpha,\alpha}(x,\xi,t-\tau)\widetilde{q}(\xi)\widetilde{u}(\xi,\tau)d\xi d\tau, \qquad (18)$$

where

$$\widetilde{\Phi}(x,t) = \int_0^l G_{\alpha,1+(\beta-1)(1-\alpha)}(x,\xi,t)\widetilde{\varphi}(\xi)d\xi + \int_0^t\int_0^l G_{\alpha,\alpha}(x,\xi,t-\tau)\widetilde{f}(\xi,\tau)d\xi d\tau. \qquad (19)$$

Composing the difference $u(x,t)-\widetilde{u}(x,t)$ with the help of the equations (11), (18) and introducing the notations $\widehat{u}(x,t) = u(x,t)-\widetilde{u}(x,t)$, $\widehat{q}(x) = q(x)-\widetilde{q}(x)$, $\widehat{\varphi}(x) = \varphi(x)-\widetilde{\varphi}(x)$, $\widehat{f}(x,t) = f(x,t) - \widetilde{f}(x,t)$, we obtain the integral equation

$$\widehat{u}(x,t) = \widehat{\Phi}(x,t) - \int_0^t\int_0^l G_{\alpha,\alpha}(x,\xi,t-\tau)\Big[\widetilde{q}(\xi)\widehat{u}(\xi,\tau) + \widehat{q}(\xi)u(\xi,\tau)\Big]d\xi d\tau, \qquad (20)$$

where

$$\widehat{\Phi}(x,t) = \int_0^l G_{\alpha,1+(\beta-1)(1-\alpha)}(x,\xi,t)\widehat{\varphi}(\xi)d\xi + \int_0^t\int_0^l G_{\alpha,\alpha}(x,\xi,t-\tau)\widehat{f}(\xi,\tau)d\xi d\tau. \qquad (21)$$

From the equalities (21) it follows the estimate for $(x,t)\in\Omega_T$:

$$t^\gamma\left|\widehat{\Phi}(x,t)\right| = t^\gamma\left|\int_0^l G_{\alpha,1+(\beta-1)(1-\alpha)}(x,\xi,t)\widehat{\varphi}(\xi)d\xi + \int_0^t\int_0^l G_{\alpha,\alpha}(x,\xi,t-\tau)\widehat{f}(\xi,\tau)d\xi d\tau\right|$$
$$\le T^{\gamma+(\beta-1)(1-\alpha)}||\widehat{\varphi}||_{C[0,l]} + C_1||\widehat{f}||_\gamma T^\alpha B(\alpha,1-\gamma).$$

From which is derived the following linear integral inequality for $t^\gamma\,|\widehat{u}(x,t)|$

$$t^\gamma\,|\widehat{u}(x,t)| \le T^{\gamma+(\beta-1)(1-\alpha)}||\widehat{\varphi}||_{C[0,l]} + C_1||\widehat{f}||_\gamma T^\alpha B(\alpha,1-\gamma)$$

$$+\left(T^{\gamma+(\beta-1)(1-\alpha)}\varphi_0+C_1f_0T^{\gamma+\alpha}B(\alpha,1-\gamma)\right)\Gamma\left(1-\gamma\right)E_{\alpha,1-\gamma}\left(C_1||q||_{C[0,l]}\Gamma(\alpha)T^{\alpha}\right)||\widehat{q}||_{C[0,l]}$$

$$+||\widehat{q}||_{C[0,l]}t^{\gamma}\int\limits_0^t(t-\tau)^{\alpha-1}\left|\widehat{u}(\xi,\tau)\right|d\tau, \tag{22}$$

where $B(\cdot,\cdot)$ is the Euler's Beta function and to estimate the second term on the right side of (20) it was used the equality $\int\limits_0^t(t-\tau)^{\alpha-1}\tau^{-\gamma}d\tau=t^{\alpha-\gamma}B(\alpha,1-\gamma)$.

Let

$$\theta=\theta\big(\alpha,\,\beta,\,T,\,||\widehat{q}||_{C[0,l]},\,||\widehat{\varphi}||_{C[0,l]},\,||\widehat{f}||_{\gamma}\big)$$

$$=\max\left\{1,\,T^{\gamma+(\beta-1)(1-\alpha)},T^{\alpha}B(\alpha,1-\gamma),\,\left(T^{\gamma+(\beta-1)(1-\alpha)}\varphi_0+C_1f_0T^{\gamma+\alpha}\Gamma\left(1-\gamma\right)\right)\right.$$

$$\left.\times\,B(\alpha,1-\gamma)E_{\alpha,1-\gamma}\left(C_1||q||_{C[0,l]}\Gamma(\alpha)T^{\alpha}\right)\right\}.$$

Applying the successive approximation method to inequality (22) with the help of the scheme

$$t^{\gamma}\left|\widehat{u}(x,t)\right|_0\leq\theta\left(||\widehat{\varphi}||+||\widehat{f}||_{\gamma}+||\widehat{q}||_{C[0,l]}\right).$$

Similarly way from (22) for $k=1,2$, we obtain

$$t^{\gamma}\left|\widehat{u}(x,t)\right|_1\leq t^{\gamma}\left|\int\limits_0^t\int\limits_0^l G_{\alpha,\alpha}(x,\xi,t-\tau)|\widetilde{q}(\xi)||\widehat{u}(\xi,\tau)|_0\right|$$

$$\leq\theta\left(||\widehat{\varphi}||+||\widehat{f}||_{\gamma}+||\widehat{q}||_{C[0,l]}\right)\frac{C_1||\widetilde{q}||_{C[0,l]}\Gamma(1-\gamma)\Gamma(\alpha)}{\Gamma(\alpha+1-\gamma)}t^{\alpha},$$

$$t^{\gamma}\left|\widehat{u}(x,t)\right|_2\leq t^{\gamma}\left|\int\limits_0^t\int\limits_0^l G_{\alpha,\alpha}(x,\xi,t-\tau)|\widetilde{q}(\xi)||\widehat{u}(\xi,\tau)|_1\right|$$

$$\leq\theta\left(||\widehat{\varphi}||+||\widehat{f}||_{\gamma}+||\widehat{q}||_{C[0,l]}\right)\frac{C_1^2||\widetilde{q}||^2_{C[0,l]}\Gamma(1-\gamma)\Gamma^2(\alpha)}{\Gamma(2\alpha+1-\gamma)}t^{2\alpha}.$$

Thus,we get

$$t^{\gamma}\left|\widehat{u}(x,t)\right|_k\leq\theta\left(||\widehat{\varphi}||+||\widehat{f}||_{\gamma}+||\widehat{q}||_{C[0,l]}\right)\frac{C_1^k||\widetilde{q}||^k_{C[0,l]}\Gamma^k(\alpha)\Gamma(1-\gamma)}{\Gamma(k\alpha+1-\gamma)}t^{k\alpha},$$

we obtain the estimate

$$t^{\gamma}\left|\widehat{u}(x,t)\right|\leq\theta\left(||\widehat{\varphi}||+||\widehat{f}||_{\gamma}+||\widehat{q}||_{C[0,l]}\right)\Gamma\left(1-\gamma\right)$$

$$\times\sum_{k=0}^{\infty}\left(C_1||\widetilde{q}||_{C[0,l]}\Gamma(\alpha)\right)^k\frac{T^{k\alpha}}{\Gamma\left(k\alpha+1-\gamma\right)}\leq\theta\left(||\widehat{\varphi}||+||\widehat{f}||_{\gamma}+||\widehat{q}||_{C[0,l]}\right)$$

$$\times\,\Gamma\left(1-\gamma\right)E_{\alpha,1-\gamma}\left(C_1||\widetilde{q}||_{C[0,l]}||\Gamma(\alpha)T^{\alpha}\right),\;(x,t)\in\overline{\Omega}_T, \tag{23}$$

which will be used in the next section. Indeed the expression (23) is the stability estimate for the solution $u(x,t)$ to the initial boundary value problem (1)–(3) in the class $C_{\gamma}^{2,\alpha,\beta}(\overline{\Omega}_T)$. The uniqueness for this solution follows from (23). □

4. INVESTIGATION OF INVERSE PROBLEM

Let us consider the inverse problem (1)–(4) and obtain an operator equation for the coefficient $q(x)$. Thus, let $q(x)$ be an arbitrary function from $C[0,l]$. Let us multiply equation (1) by $w(t)$ and integrate over the closed interval $[0,T]$. Taking into account conditions (2)–(4), A3) and A4), we obtain the next relation

$$q(x)=\frac{1}{h(x)}\left\{\int_0^T w(t)f(x,t)dt+h''(x)-\int_0^T w(t)\left(D^{\alpha,\beta}_{0+,t}u\right)(x,t)dt\right\}. \tag{24}$$

Equation (24) contain the unknown functions $D^{\alpha,\beta}_{0+,t}u$, we will find it. For this, we introduce a new function $D^{\alpha,\beta}_{0+,t}u(x,t)=v(x,t)$, $(x,t)\in\Omega_T$ in direct problem (1)–(3) and we get the following problem

$$\left(D^{\alpha,\beta}_{0+,t}v\right)(x,t)-v_{xx}+q(x)v(x,t)=\left(D^{\alpha,\beta}_{0+,t}f\right)(x,t), \tag{25}$$

with initial condition

$$I^{(1-\alpha)(1-\beta)}_{0+,t}v(x,t)\big|_{t=0}=I^{(1-\alpha)(1-\beta)}_{0+,t}f(x,t)\big|_{t=0}+\varphi''(x)-q(x)\varphi(x):=\psi(x),\quad x\in[0,l], \tag{26}$$

and the boundary conditions

$$v(0,t)=v(l,t)=0,\quad t\in[0,T]. \tag{27}$$

Solution of problem (25)–(27) will investigate following view by the similar method to the problem (1)–(3)

$$v(x,t)=\Phi(x,t)-\int_0^t\int_0^l G_{\alpha,\alpha}(x,\xi,t-\tau)q(\xi)v(\xi,\tau)d\xi d\tau, \tag{28}$$

where

$$\Phi(x,t)=\int_0^l G_{\alpha,1+(\beta-1)(1-\alpha)}(x,\xi,t)\left(I^{(1-\alpha)(1-\beta)}_{0+,t}f(\xi,t)\big|_{t=0}+\varphi''(\xi)-q(\xi)\varphi(\xi)\right)d\xi$$
$$+\int_0^t\int_0^l G_{\alpha,\alpha}(x,\xi,t-\tau)\left(D^{\alpha,\beta}_{0+,\tau}f\right)(\xi,\tau)d\xi d\tau.$$

The existence of a solution to the integral equation (28) is proven by the method of successive approximation, as in the case of the existence of a solution to the integral equation (11).

If we use the view of the function $v(x,t)$ in the Volterra integral equation of the second type (28), we obtain an integral equation for $D^{\alpha,\beta}_{0+,t}u(x,t)$

$$D^{\alpha,\beta}_{0+,t}u(x,t)=\Upsilon(x,t)-\int_0^l G_{\alpha,1+(\beta-1)(1-\alpha)}(x,\xi,t)q(\xi)\varphi(\xi)d\xi$$
$$-\int_0^t\int_0^l G_{\alpha,\alpha}(x,\xi,t-\tau)q(\xi)D^{\alpha,\beta}_{0+,\tau}u(\xi,\tau)d\xi d\tau, \tag{29}$$

where

$$\Upsilon(x,t)=\int_0^l G_{\alpha,1+(\beta-1)(1-\alpha)}(x,\xi,t)\left(I^{(1-\alpha)(1-\beta)}_{0+,t}f(\xi,t)\big|_{t=0}+\varphi''(\xi)\right)d\xi$$

$$+\int_0^t\int_0^l G_{\alpha,\alpha}(x,\xi,t-\tau)\left(D^{\alpha,\beta}_{0+,\tau}f\right)(\xi,\tau)d\xi d\tau.$$

Equations (11), (24), and (29) show that the values of $u(x,t)$, $\left(D^{\alpha,\beta}_{0+,t}u\right)(x,t)$, and $q(x)$ for $(x,t)\in\Omega_T$ are expressed in terms of the integrals of some combinations of these functions over segments lying in Ω_T.

Write (11), (24), and (29) as a closed system of integral equations. To this end, introduce the vector-functions $\upsilon(x,t)=(\upsilon_1(x,t),\ \upsilon_2(x,t),\ \upsilon_3(x,t))$, by defining their components by the equalities

$$\upsilon_1(x,t)=u(x,t),\quad \upsilon_2(x,t):=\upsilon_2(x)=q(x),$$

$$\upsilon_3(x,t)=\left(D^{\alpha,\beta}_{0+,t}u\right)(x,t)-\int_0^l G_{\alpha,1+(\beta-1)(1-\alpha)}(x,\xi,t)q(\xi)\varphi(\xi)d\xi.$$

Then, system (11), (24), and (29) takes the operator form

$$\upsilon=\mathcal{A}\upsilon, \tag{30}$$

where A is the operator $\mathcal{A}=(\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3)$, that is defined in accordance with the right-hand sides of (11), (24), and (29) by the equalities

$$\mathcal{A}_1\upsilon(x,t)=\upsilon_{01}(x,t)-\int_0^t\int_0^l G_{\alpha,\alpha}(x,\xi,t-\tau)\upsilon_2(\xi)\upsilon_1(\xi,\tau)d\xi d\tau, \tag{31}$$

$$\mathcal{A}_2\upsilon(x)=\upsilon_{02}(x,t)-\frac{1}{h(x)}\int_0^T w(t)\Big(\upsilon_3(x,t)-\int_0^l G_{\alpha,1+(\beta-1)(1-\alpha)}(x,\xi,t)\upsilon_2(\xi)\varphi(\xi)d\xi\Big)dt, \tag{32}$$

$$\upsilon_3(x,t)=\upsilon_{03}(x,t)$$
$$-\int_0^t\int_0^l G_{\alpha,\alpha}(x,\xi,t-\tau)\upsilon_2(\xi)\Big(\upsilon_3(\xi,\tau)-\int_0^l G_{\alpha,1+(\beta-1)(1-\alpha)}(\xi,\zeta,\tau)\upsilon_2(\zeta)\varphi(\zeta)d\zeta\Big)d\xi d\tau. \tag{33}$$

In these formulas, we used the notations

$$\upsilon_{01}(x,t)=\Phi(x,t),\quad \upsilon_{02}(x,t)=\frac{1}{h(x)}\left\{\int_0^T w(t)f(x,t)dt+h''(x)\right\},\quad \upsilon_{03}(x,t)=\Upsilon(x,t).$$

Theorem 2. *Let A1)–A4) are satisfied. Then, there exist sufficiently small number $T\in(0,T^*)$, there exists a unique solution to inverse problem (1)–(4) of the class $q(x)\in C[0,l]$.*

Proof. For simplicity, we denote

$$H:=||h||_{C^2[0,l]},\quad w_0:=||w||_{C[0,T]}.$$

Consider the functional space of vector functions $\upsilon(x,t)\in C(\overline{\Omega}_T)$ with the norm given by the relation

$$||\upsilon||=\max\left\{\max_{(x,t)\in\overline{\Omega}_T}t^\gamma|\upsilon_1(x,t)|,\ \max_{x\in[0,l]}|\upsilon_2(x)|,\ \max_{(x,t)\in\overline{\Omega}_T}t^\gamma|\upsilon_3(x,t)|\right\},$$

besides

$$||\upsilon_0||=\max\left\{\max_{(x,t)\in\overline{\Omega}_T}t^\gamma|\upsilon_{01}(x,t)|,\ \max_{x\in[0,l]}|\upsilon_{02}(x)|,\ \max_{(x,t)\in\overline{\Omega}_T}t^\gamma|\upsilon_{03}(x,t)|\right\},$$

where

$$\max_{(x,t)\in\overline{\Omega}_T} t^\gamma |\upsilon_{01}(x,t)|$$

$$= \max_{(x,t)\in\overline{\Omega}_T} t^\gamma \left| \int_0^l G_{\alpha,1+(\beta-1)(1-\alpha)}(x,\xi,t)\varphi(\xi)d\xi + \int_0^t \int_0^l G_{\alpha,\alpha}(x,\xi,t-\tau)f(\xi,\tau)d\xi d\tau \right|$$

$$\leq T^{\gamma+(\beta-1)(1-\alpha)}\varphi_0 + C_1 f_0 T^\alpha B(\alpha,1-\gamma),$$

$$\max_{x\in[0,l]} |\upsilon_{02}(x)| = \max_{x\in[0,l]} \left| \frac{1}{h(x)} \left\{ \int_0^T w(t)f(x,t)dt + h''(x) \right\} \right| \leq \frac{1}{h_0}\Big(Tw_0 f_0 + H_0\Big),$$

$$\max_{(x,t)\in\overline{\Omega}_T} t^\gamma |\upsilon_{03}(x,t)| = \max_{(x,t)\in\overline{\Omega}_T} t^\gamma \Big|\Upsilon(x,t)\Big|$$

$$\leq \max_{(x,t)\in\overline{\Omega}_T} t^\gamma \left| \int_0^l G_{\alpha,1+(\beta-1)(1-\alpha)}(x,\xi,t) \left(I_{0+,t}^{(1-\alpha)(1-\beta)} f(\xi,t)\big|_{t=0} + \varphi''(\xi) \right) d\xi \right.$$

$$\left. + \int_0^t \int_0^l G_{\alpha,\alpha}(x,\xi,t-\tau) D_{0+,\tau}^{\alpha,\beta} f(\xi,\tau)d\xi d\tau \right|$$

$$\leq T^{\gamma+(\beta-1)(1-\alpha)}\Big(||f_0|| + ||\varphi||_{C[0,l]}\Big) + C_1 T^\alpha B(\alpha,1-\gamma)f_0.$$

In this space, by $B(\upsilon_0, r)$ we denote the ball with center υ_0 and radius r, i.e., $B(\upsilon_0,r) := \{\upsilon : ||\upsilon - \upsilon_0|| \leq r\}$. Obviously, $||\upsilon|| \leq ||\upsilon_0|| + r := r_0$.

Let us show that A is a contraction operator in the ball $B(\upsilon_0, r)$ provided that T and l are sufficiently small numbers.

Let us verify the first condition of a fixed point argument. Let $\upsilon \in B(\upsilon_0, r)$; then $||\upsilon|| \leq \upsilon_0 + r$. In addition, for $(x,t) \in \Omega_T$ using equality $\int_0^l G_{\alpha,\alpha}(x,\xi,t)d\xi \leq C_1 t^\alpha$, we have estimates

$$||\mathcal{A}_1\upsilon - \upsilon_{01}||_\gamma = \max_{(x,t)\in\overline{\Omega}_T} \left\{ t^\gamma \left| -\int_0^t \int_0^l G_{\alpha,\alpha}(x,\xi,t-\tau)\upsilon_2(\xi)\upsilon_1(\xi,\tau)d\xi d\tau \right| \right\} \leq \frac{r_0^2 C_1}{\alpha+1} T^{\alpha+1},$$

$$||\mathcal{A}_2\upsilon - \upsilon_{02}|| = \max_{(x,t)\in\overline{\Omega}_T} \left\{ \left| -\frac{1}{h(x)} \int_0^T w(t)\Big(\upsilon_3(x,t) - \int_0^l G_{\alpha,1+(\beta-1)(1-\alpha)}(x,\xi,t)\upsilon_2(\xi)\varphi(\xi)d\xi\Big)dt \right| \right\}$$

$$\leq \frac{w_0}{h_0}\left(1 + \frac{||\varphi||_{C[0,l]}}{\alpha+1} C_1 T^\alpha\right) r_0 T,$$

$$||\mathcal{A}_3\upsilon - \upsilon_{03}||_\gamma = \max_{(x,t)\in\overline{\Omega}_T} \left\{ t^\gamma \left| -\int_0^t \int_0^l G_{\alpha,\alpha}(x,\xi,t-\tau)\upsilon_2(\xi)\Big(\upsilon_3(\xi,\tau) \right.\right.$$

$$\left.\left. - \int_0^l G_{\alpha,1+(\beta-1)(1-\alpha)}(\xi,\zeta,\tau)\upsilon_2(\zeta)\varphi(\zeta)d\zeta\Big)d\xi d\tau \right| \right\}$$

$$\leq \left(\frac{T}{\alpha+1} + ||\varphi||_{C[0,l]} T^\alpha B(\alpha+1,\alpha+1)C_1 \right) C_1 r_0^2 T^\alpha.$$

These together with (30) and (31)–(33) imply the estimates

$$||\mathcal{A}\upsilon-\upsilon_0||=\max\left\{\max_{(x,t)\in\overline{\Omega}_T}t^{\gamma}|\mathcal{A}_1\upsilon(x,t)-\upsilon_{01}(x,t)|,\quad \max_{x\in[0,l]}|\mathcal{A}_2\upsilon(x)-\upsilon_{02}(x)|,\right.$$

$$\left.\max_{(x,t)\in\overline{\Omega}_T}t^{\gamma}|\mathcal{A}_3\upsilon(x,t)-\upsilon_{03}(x,t)|\right\}\leq\max\left\{\frac{r_0C_1}{\alpha+1}T^{\alpha},\right.$$

$$\left.\frac{w_0}{h_0}\left(1+\frac{||\varphi||_{C[0,l]}}{\alpha+1}C_1T^{\alpha}\right)T,\quad\left(\frac{T}{\alpha+1}+||\varphi||_{C[0,l]}T^{\alpha}B(\alpha+1,\alpha+1)C_1\right)C_1r_0T^{\alpha}.\right.$$

Therefore, if by T_1 we denote the positive root of the equation (for T)

$$\max\left\{\frac{r_0C_1}{\alpha+1}T^{\alpha},\quad\frac{w_0}{h_0}\left(1+\frac{||\varphi||_{C[0,l]}}{\alpha+1}C_1T^{\alpha}\right)T,\right.$$

$$\left.\left(\frac{T}{\alpha+1}+||\varphi||_{C[0,l]}T^{\alpha}B(\alpha+1,\alpha+1)C_1\right)C_1r_0T^{\alpha}\right\}=r,$$

then $||\mathcal{A}\upsilon-\upsilon_0||\leq r$ for $T\leq T_1$; i.e., $\mathcal{A}\upsilon\in B(\upsilon_0,r)$.

Now let us take any functions υ, $\widetilde{\upsilon}\in B(\upsilon_0,r)$ and estimate the norm of the difference $||\mathcal{A}\upsilon-||\mathcal{A}\widetilde{\upsilon}||$. Using the obvious inequality

$$|\upsilon_1\upsilon_2-\widetilde{\upsilon}_1\widetilde{\upsilon}_2|\leq|\upsilon_1-\widetilde{\upsilon}_1||\widetilde{\upsilon}_2|+|\upsilon_2-\widetilde{\upsilon}_2||\widetilde{\upsilon}_1|\leq 2r_0||\upsilon-\widetilde{\upsilon}||$$

and estimates for integrals similar to the ones given above, we obtain

$$||\mathcal{A}_1\upsilon-\mathcal{A}_1\widetilde{\upsilon}||_{\gamma}=\max_{(x,t)\in\overline{\Omega}_T}\left\{t^{\gamma}\left|-\int_0^t\int_0^l G_{\alpha,\alpha}(x,\xi,t-\tau)\right.\right.$$

$$\left.\left.\times\Big(\upsilon_2(\xi)\upsilon_1(\xi,\tau)-\widetilde{\upsilon}_2(\xi)\widetilde{\upsilon}_1(\xi,\tau)\Big)d\xi d\tau\right|\right\}\leq\frac{2r_0^2C_1}{\alpha+1}T^{\alpha+1}||\upsilon-\widetilde{\upsilon}||,$$

$$||\mathcal{A}_2\upsilon-\mathcal{A}_2\widetilde{\upsilon}||=\left|-\frac{1}{h(x)}\int_0^T w(t)\Big(\upsilon_3(x,t)-\int_0^l G_{\alpha,1+(\beta-1)(1-\alpha)}(x,\xi,t)\upsilon_2(\xi)\varphi(\xi)d\xi\Big)dt\right.$$

$$\left.+\frac{1}{h(x)}\int_0^T w(t)\Big(\widetilde{\upsilon}_3(x,t)-\int_0^l G_{\alpha,1+(\beta-1)(1-\alpha)}(x,\xi,t)\widetilde{\upsilon}_2(\xi)\varphi(\xi)d\xi\Big)dt\right|$$

$$\leq\frac{w_0}{h_0}\left(1+2\frac{||\varphi||_{C[0,l]}}{\alpha+1}C_1T^{\alpha}\right)r_0T||\upsilon-\widetilde{\upsilon}||,$$

$$||\mathcal{A}_3\upsilon-\mathcal{A}_3\widetilde{\upsilon}||_{\gamma}=\max_{(x,t)\in\overline{\Omega}_T}\left\{t^{\gamma}\left|-\int_0^t\int_0^l G_{\alpha,\alpha}(x,\xi,t-\tau)\upsilon_2(\xi)\Big(\upsilon_3(\xi,\tau)\right.\right.$$

$$-\int_0^l G_{\alpha,1+(\beta-1)(1-\alpha)}(\xi,\zeta,\tau)\upsilon_2(\zeta)\varphi(\zeta)d\zeta\Big)d\xi d\tau$$

$$+\int_0^t\int_0^l G_{\alpha,\alpha}(x,\xi,t-\tau)\widetilde{\upsilon}_2(\xi)\Big(\widetilde{\upsilon}_3(\xi,\tau)$$

$$-\int_0^l G_{\alpha,1+(\beta-1)(1-\alpha)}(\xi,\zeta,\tau)\widetilde{\upsilon}_2(\zeta)\varphi(\zeta)d\zeta\Big)d\xi d\tau\Bigg|\Bigg\}$$

$$\leq 2\left(\frac{T}{\alpha+1}+2||\varphi||_{C[0,l]}T^{\alpha}B(\alpha+1,\alpha+1)C_1\right)C_1 r_0^2 T^{\alpha}||\upsilon-\widetilde{\upsilon}||.$$

It follows that

$$||\mathcal{A}\upsilon-\mathcal{A}\widetilde{\upsilon}||_{\gamma}=\max\Bigg\{\max_{(x,t)\in\overline{\Omega}_T} t^{\gamma}|\mathcal{A}_1\upsilon(x,t)-\mathcal{A}_1\widetilde{\upsilon}(x,t)|,\quad \max_{x\in[0,l]}|\mathcal{A}_2\upsilon(x)-\mathcal{A}_2\widetilde{\upsilon}(x)|,$$

$$\max_{(x,t)\in\overline{\Omega}_T} t^{\gamma}|\mathcal{A}_3\upsilon(x,t)-\mathcal{A}_3\widetilde{\upsilon}(x,t)|\Bigg\}\leq\max\left\{\frac{2r_0C_1}{\alpha+1}T^{\alpha},\quad \frac{w_0}{h_0}\left(1+2\frac{||\varphi||_{C[0,l]}}{\alpha+1}C_1T^{\alpha}\right)r_0T,\right.$$

$$\left.2\left(\frac{T}{\alpha+1}+2||\varphi||_{C[0,l]}T^{\alpha}B(\alpha+1,\alpha+1)C_1\right)C_1r_0^2T^{\alpha}\right\}||\upsilon-\widetilde{\upsilon}||.$$

Therefore, if T_2 is the positive root of the equation (for T)

$$\max\left\{\frac{2r_0C_1}{\alpha+1}T^{\alpha},\ \frac{w_0}{h_0}\left(1+2\frac{||\varphi||_{C[0,l]}}{\alpha+1}C_1T^{\alpha}\right)r_0T,\right.$$

$$\left.2\left(\frac{T}{\alpha+1}+2||\varphi||_{C[0,l]}T^{\alpha}B(\alpha+1,\alpha+1)C_1\right)C_1r_0^2T^{\alpha}\right\}=1,$$

then for $T\leq T_2$ the operator $\mathcal{A}$ contracts the distance between the elements υ and $\widetilde{\upsilon}$. Under the inequality $T<T^*=\min\{T_1,\,T_2\}$, the operator $\mathcal{A}$ is a contraction operator on the set $B(\upsilon_0,r)$. Consequently, by the Banach principle [45, pp. 87–97], equation (30) has a unique solution on this set. The proof of the theorem is complete. □

5. CONCLUSIONS

In this work, the solvability of a nonlinear inverse problem for a fractional wave equation with the generalized Riemann–Liouville (Hilfer) time derivative with initial-boundary conditions and integral type overdetermination conditions was studied. Firstly we investigated solvability direct problem. Then, the (1)–(3) problem replaced by an equivalent problem of Volterra type integral equations of the second kind. Existence and uniqueness results of direct problem were proven. The inverse problem was considered of determining $q(x)$ included in the equation (1) with additional condition (4) of the solution of equation (1) with the initial and boundary conditions (2) and (3). The inverse problem is reduced to an equivalent Volterra-type integral equation. Contraction principle of Banach was used to show that this equation has a unique solution. Local solubility and uniqueness results are obtained.

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